Rational Functions

A rational function is a function of the form

\[ f(x) = \frac{p(x)}{q(x)} , \]

where \( p(x) \) and \( q(x) \) are polynomials.

Examples  The following are rational functions:

\[ \frac{1}{x} \quad \frac{x^2 - 1}{x^4 + x^3} \quad \frac{x^7 - 8}{x^2 + 2x - 7} \]

Questions

1. Suppose \( f(x) = \frac{p(x)}{q(x)} \) is a rational function. Then

(a) Suppose \( p(a) = 0 \) for some number \( a \). Then \( f(a) = \) ___, so the graph of \( f(x) \) intercepts the ___-axes at \( x = a \).

(b) Suppose \( q(b) = 0 \) for some number \( b \). Then the graph of \( f(x) \) has a __________ at \( x = b \).

2. Suppose \( r(x) = \frac{x^3 - 1}{x^2 + 2x + 1} \).

(a) Does \( r(x) \) intercept the \( x \)-axis? Where?

(b) Does it have any vertical asymptotes? Where?

(c) Where is \( r(x) \) positive? Negative?

(d) What happens as \( x \to \infty \)? What about as \( x \to -\infty \)?
(e) You should now have enough information to sketch a graph of \( r(x) \). Do so on the axes below. Be sure to label all intercepts and asymptotes before you start drawing the graph!

<table>
<thead>
<tr>
<th>( x )</th>
<th>( (x-1)(x+2) )</th>
<th>( (2x-1)(x+3) )</th>
<th>Total Sign</th>
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<tr>
<td>( x &lt; )</td>
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A Helpful Method

Note that the sign (positive or negative) of a rational function can change only when it hits a zero or a vertical asymptote. This can help make sketching such functions much easier.

3. Consider the function \( r(x) = \frac{(x-1)(x+2)}{(2x-1)(x+3)} \).

(a) \( r(x) \) has zeros at \( x = \) ______ and \( x = \) ______.

(b) \( r(x) \) has vertical asymptotes at \( x = \) _____ and \( x = \) _____.

(c) Fill in the following table:

(d) What happens as \( x \to \infty \)? What about as \( x \to -\infty \)?
(e) Now draw the graph of $r(x)$.

4. By repeating the steps from the previous question, sketch the graph of function

$$f(x) = \frac{(x + 1)(x + 2)(x + 3)}{x(x - 1)^2}.$$
Extra Homework Problems

1. Match each of the following rational functions with the correct graph.

\[
\begin{align*}
\frac{x^2 - 1}{x^2 + 1} & \quad (A) \\
\frac{x - 1}{(x + 1)(x - 2)} & \quad (B) \\
\frac{(x + 1)(x - 2)}{x - 1} & \quad (C)
\end{align*}
\]

2. Find all of the intercepts and asymptotes of the following functions.

(a) \( r(x) = \frac{x}{x^2 - 4} \)

(b) \( r(x) = \frac{x + 3}{x - 3} \)

(c) \( r(x) = \frac{(x + 1)(x - 1)}{x + 2} \)

3. Sketch the graph of \( r(x) = \frac{(2x + 2)(x - 2)}{(x - 1)(x + 2)} \). Be sure to include and label all of the intercepts and asymptotes.
Extra Homework Answers

1. B, A, C

2.  (a) HA at $y = 0$, VA's at $x = \pm 2$.
    (b) HA at $y = 1$, VA at $x = 3$.
    (c) No HA. VA at $x = -2$.

3. Graph. Check using a graphing tool.