Preliminaries

A *rational function* is a function of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomials.

Examples The following are rational functions:

$$\frac{1}{x}, \frac{x^2-1}{x^4+x^3}, \frac{x^7-8}{x^2+2x-7}.$$

Suppose $f(x) = \frac{p(x)}{q(x)}$ is a rational function. Then

- Suppose $p(a) = 0$ for some number a. Then $f(a) =$ _____, so the graph of $f(x)$ intercepts the _____-axes at $x = a$.

- Suppose $q(b) = 0$ for some number b. Then $f(b)$ is _____ at $x = b$. So the graph of $f(x)$ has a _____ at $x = b$.

- We can find the y intercept by computing _____.

Graphs of Rational Functions

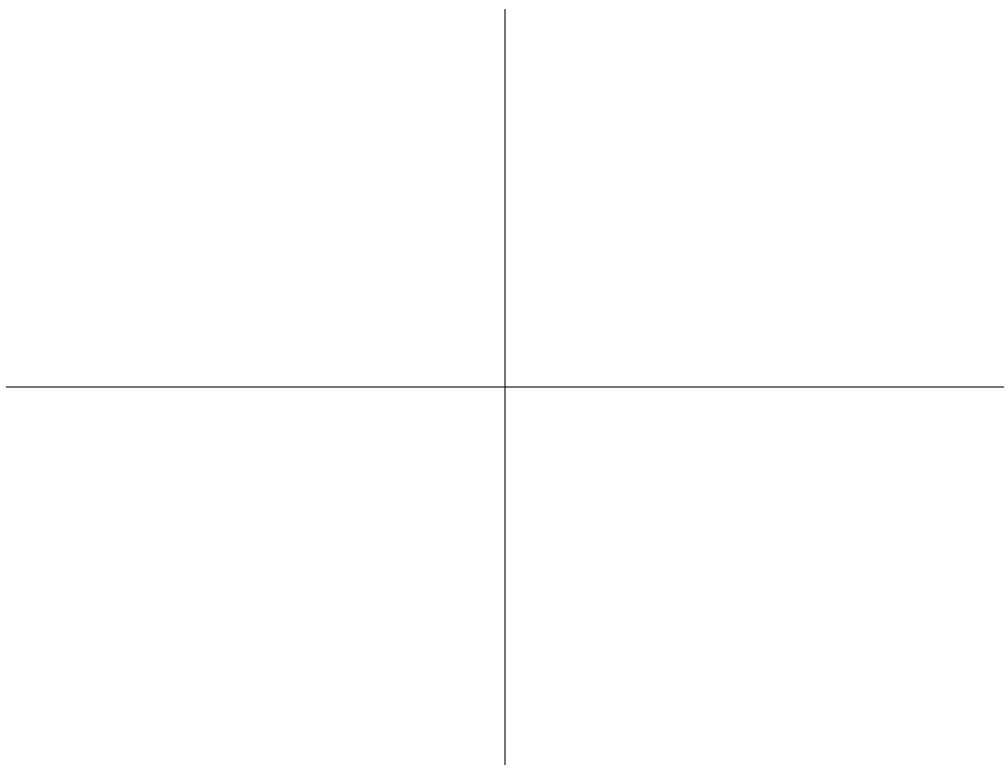
In order to draw the graphs of rational functions, we need their intercepts and vertical asymptotes, as above. In addition, we need to know where they are positive and where they are negative. Lastly, we need to understand their behavior as $x \to \infty$ and $x \to -\infty$. Is there a horizontal asymptote, or does the function go to $\pm\infty$? We will develop tools to answer these questions by example.

Example 1

Consider $r(x) = \frac{x-1}{x+1}$.

- $r(x)$ has an x-intercept where _____ $= 0$. That is, when $x =$ _____.
- $r(x)$ has a vertical asymptote where _____ $= 0$. That is, when $x =$ _____.
- To compute the y intercept, we plug in $x = 0$: $r(0) =$ _____ $=$ ___.
- $x - 1$ is positive when _____ and negative when _____. $x + 1$ is positive when _____ and negative when _____.
- Therefore, $\frac{x-1}{x+1}$ is negative when _____ and positive otherwise.
- When x is very large (in the positive *or* negative direction) $x + 1$ and $x - 1$ are both very close to x. So as $x \to \pm\infty$, $\frac{x-1}{x+1} \approx$ _____ $=$ _____.

Add all this information to the axes below and draw the graph. Label all intercepts and asymptotes.



Example 2

Suppose $r(x) = \frac{x^3-1}{x^2+2x+1}$.

- Does $r(x)$ intercept the x-axis? Where?

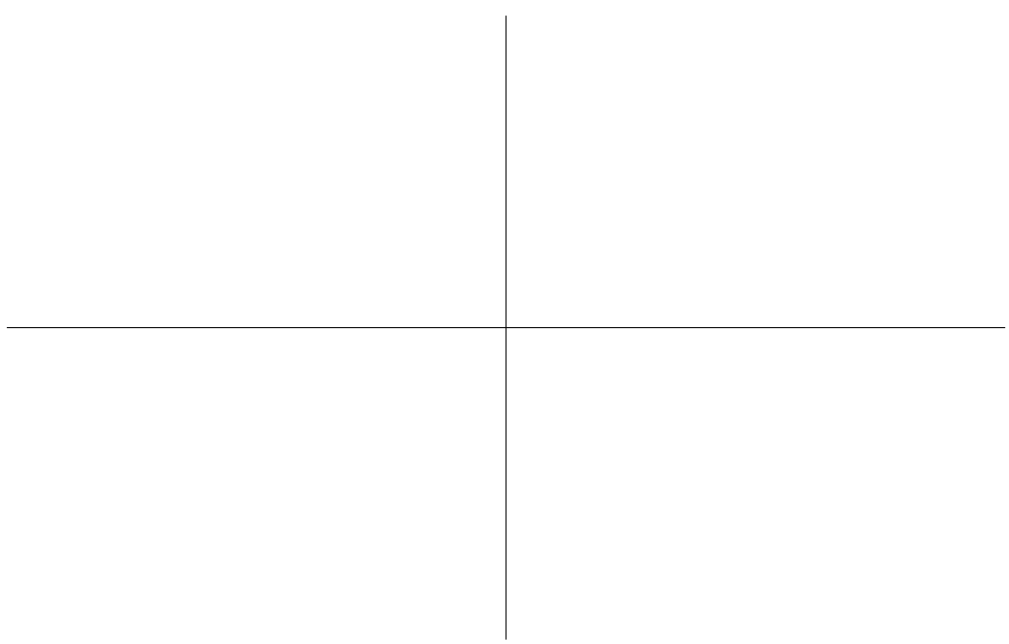
- Does it have any vertical asymptotes? Where?

- What is the y intercept?

- Where is $r(x)$ positive? Negative?

- What happens as $x \to \infty$? What about as $x \to -\infty$?

- You should now have enough information to sketch a graph of $r(x)$. Do so on the axes below. Be sure to label all intercepts and asymptotes *before you start drawing the graph!*



Example 3

Consider $r(x) = \frac{(2x-1)(x+1)}{(x-1)(x+2)}$.

- $r(x)$ has an x-intercept where _____ $= 0$. That is, when $x =$ _____ or $x =$ _____.
- $r(x)$ has a vertical asymptote where _____ $= 0$. That is, when $x =$ _____ or $x =$ _____.
- To compute the y intercept, we plug in $x = 0$: $r(0) =$ _____ $=$ ___.

The rest of the analysis is a little more complex in this case. To understand where the function is positive/negative, we use the following table. Note that the factors of the numerators and denominator are the columns, and the rows are values of x bounded by the zeros and VA's of the function we found above. Fill the table in with $+$'s and $-$'s.

	$(2x-1)$	$(x+1)$	$(x-1)$	$(x+2)$	Total Sign
$x < -2$					
$-2 < x < -1$					
$-1 < x < 0.5$					
$0.5 < x < 1$					
$1 < x$					

To get the horizontal asympote, we first FOIL the numerator and denominator:

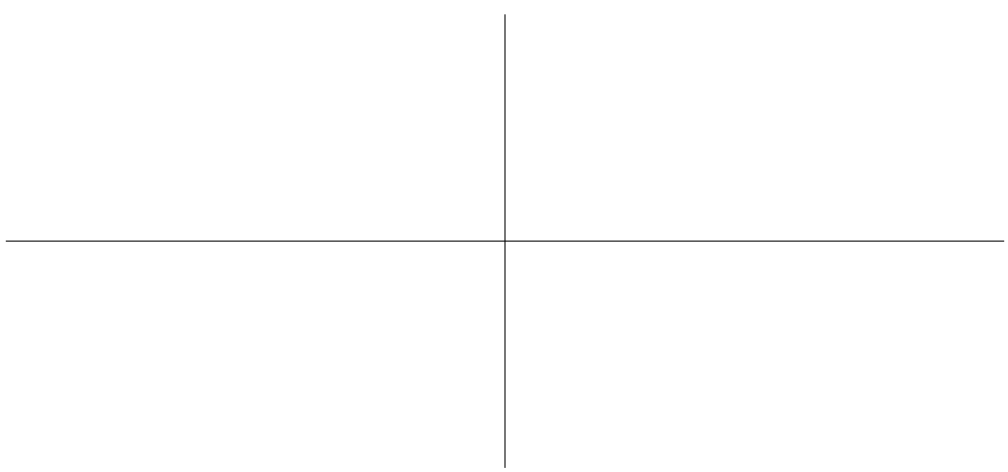
$$r(x) = \frac{(2x-1)(x+1)}{(x-1)(x+2)} = \underline{\hspace{3cm}}.$$

As x get very large (in the positive or negative direction), the x^2 terms in the numerator and denominator are vastly larger than the rest of the terms. So when x is large:

$$r(x) = \frac{(2x-1)(x+1)}{(x-1)(x+2)} = \underline{\hspace{3cm}} \approx \underline{\hspace{1.5cm}} = \underline{\hspace{1.5cm}}.$$

This is the horizontal asymptote!

Add all this information to the axes below and draw the graph. Label all intercepts and asymptotes.



Questions

1. Consider the function $r(x) = \frac{(x-1)(x+2)}{(2x-1)(x+3)}$.

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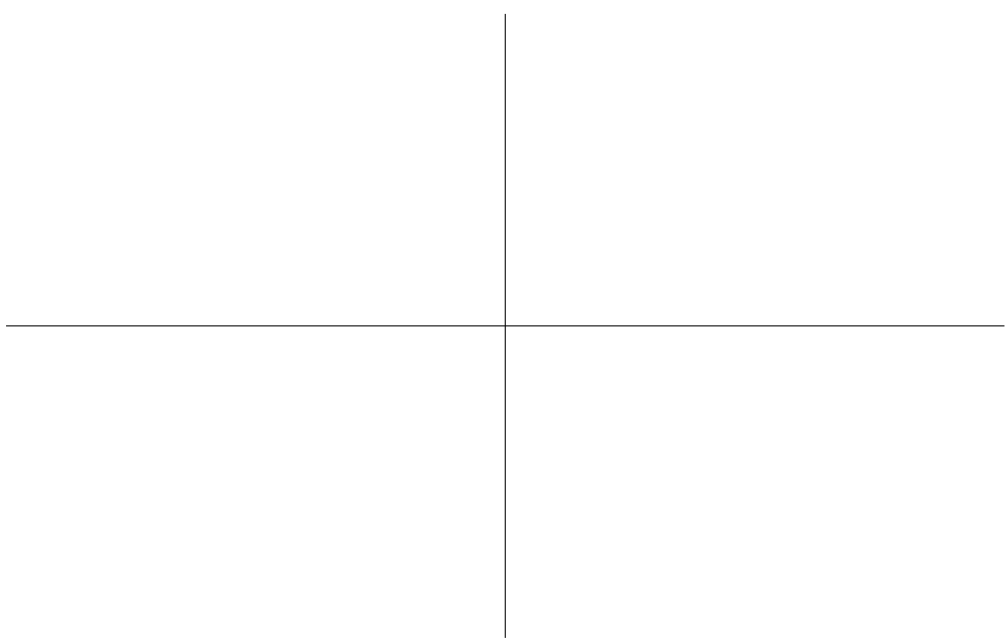
- $r(x)$ has zeros at $x =$ _____ and $x =$ _____.
- $r(x)$ has vertical asymptotes at $x =$ _____ and $x =$ _____.
- What is the y-intercept of $r(x)$?

- Fill in the following table. Recall that the columns should be the factors in the numerator and denominator, and the rows values of x bounded by the zeros and the VA's.

		Total Sign
$x <$ _____		
_____ $< x <$ _____		
_____ $< x <$ _____		
_____ $< x <$ _____		
_____ $< x$		

- What happens as $x \to \pm\infty$?

- Add all this information to the axes below and draw the graph. Label all intercepts and asymptotes.



Note: Despite what you were told in high school, functions *can* cross horizontal asymptotes!

2. By repeating the steps from the previous question, sketch the graph of function

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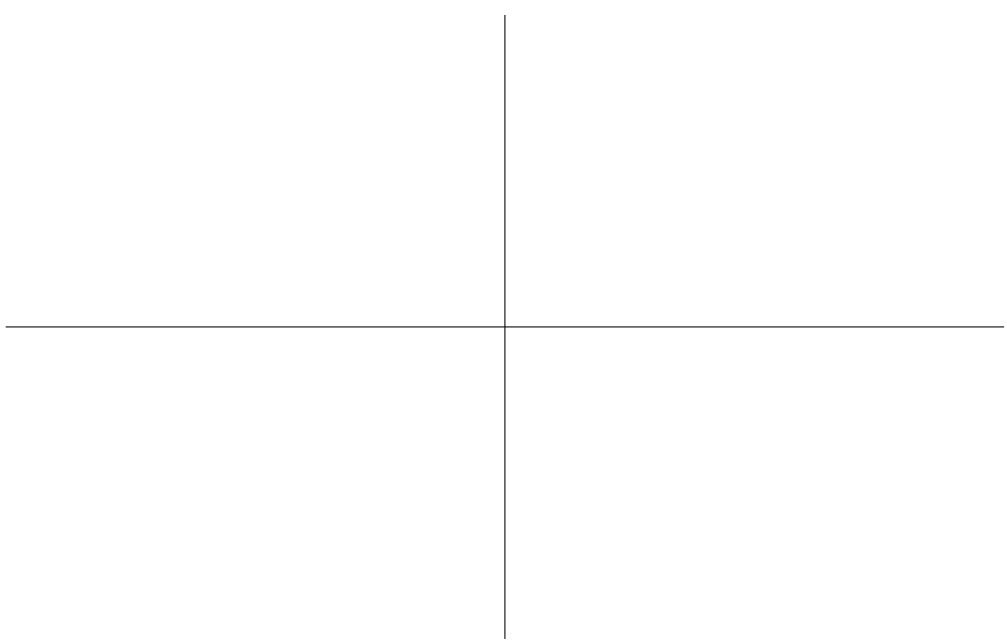
$$f(x) = \frac{(x+1)(x+2)(x+3)}{x(x-1)^2}.$$



3. Match each of the following rational functions with the correct graph.

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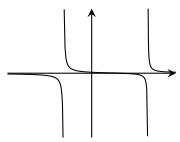
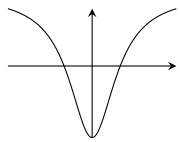
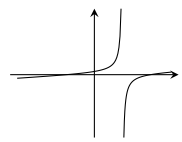
$$\frac{x^2-1}{x^2+1} \qquad \frac{x-1}{(x+1)(x-2)} \qquad \frac{(x+1)(x-2)}{x-1}$$

(A) **(B)** **(C)**







_____ _____ _____

4. Find all of the intercepts and asymptotes of the following functions.

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(a) $r(x) = \dfrac{x}{x^2-4}$

(b) $r(x) = \dfrac{x+3}{x-3}$

(c) $r(x) = \dfrac{(x+1)(x-1)}{x+2}$