Linear Functions

A function is *linear* if its slope at every point is the same.

Equivalently, any linear function can be written in the form

\[ y = f(x) = \text{__________,} \]

where \( m \) is the ________ and \( c \) is the ________.

Questions

1. Suppose \( f(x) \) is a linear function. Find its equation given that \( f(2) = 4 \) and \( f(4) = -2 \).

2. Suppose \( g(x) \) is a linear function. Find its equation given that its slope is 6 and that \( g(5) = 9 \).

Fill in the blank: The slope, \( m \), of a linear function is \( \frac{\Delta y}{\Delta x} \). If \( \Delta x = 1 \), then \( \Delta y = \text{___} \).

This leads to the following:

**An Important Characterization of the Slope of a Linear Function:**

Slope is the amount the ________ variable increases by every time the ________ variable increases by ____.

**Question**  One group calculated in the Cancer Mortality lab that if \( I \) is the index of exposure, and \( M \) is cancer mortality (in cancer deaths per 20,000 people per year), then a good model for the data is

\[ M = 11.2I + 112. \]

If a mayor of a town manages to decrease the index of exposure in his town, whose population is 115,000 people, by exactly 1.7 units, how many cancer deaths does the model predict will be prevented over the next five years?
**Question  Revisting the Apple Orchard** (Worksheet 1-3 Question 8) In a certain orchard, the following facts are known:

1. if 24 trees are planted per acre, each tree will yield 600 apples and
2. for each additional tree planted per acre, the yield decreases by 12 apples per tree.

If we plant \( x \) trees per acre (with \( x > 24 \)), what would be the expected yield?

Before answering this (again), consider why the second bullet point above tells you this question involves a linear function. What do each of the two bullet points tell you about that function?

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**Parallel and Perpendicular Lines**

1. Use the picture below, the given information, what you have learned in high school geometry and the hints given below the picture to show that if two lines are parallel then their slopes are equal. In the diagram below, \( CB \perp AB \) and \( EF \perp DE \), Line \( L_1 \) is parallel to line \( L_2 \), the slope of \( L_1 \) is \( m_1 \) and the slope of \( L_2 \) is \( m_2 \).

![Diagram](image)

(a) What do you know about triangles \( ABC \) and \( DEF \) and why?

(b) What does your previous answer tell you about the relationships among \( AB, BC, DE \) and \( EF \)?
(c) Write $m_1$ and $m_2$ in terms of $AB$, $BC$, $DE$ and $EF$.

2. Given two linear functions $f(x) = m_1x + c_1$ and $g(x) = m_2x + c_2$, how can you tell if the two lines are parallel?

3. Suppose that the line $l$ is parallel to $y = \frac{1}{2}x - 2$ and passes though the point $(1, 2)$. Find its equation.

4. Use the picture below, the given information, what you have learned in high school geometry and the hints given below the picture to show that if two lines are perpendicular ($\perp$) then the product of their slopes is $-1$. In the picture below, the slope of $L_1$ is $m_1$, the slope of $L_2$ is $m_2$, $CD \perp AC$, and the length of $AC = 1$.

(a) $L_1 \perp L_2 \iff \angle DAB$ is a right angle $\iff (AB)^2 + (\quad )^2 = (\quad )^2$.

(b) If the coordinates of $A$ are $(x_0, y_0)$ then the coordinates of $C$ are $(\quad , \quad )$, the coordinates of $D$ are $(x_0+1, \quad )$ and the coordinates of $B$ are $(\quad , \quad )$.

(c) By writing $m_1$ and $m_2$ in terms of $CD$, $BC$, and $AC$, and using Pythagoras explain why

$\bullet$ $(AB)^2 = 1 + m_2^2$
\( (AD)^2 = 1 + m_1^2 \)

\( (BD)^2 = (m_1 - m_2)^2 \)

(d) So in terms of \(x_0, y_0, m_1\) and \(m_2\), \(L_1 \perp L_2\) if and only if

\[
\text{_________________________} = \text{__________}. 
\]

(e) Simplify the expression above to show that \(m_1 \cdot m_2 = -1\).

5. Find an equation for the line \(l\) that is perpendicular to the line \(y = \frac{1}{3}x + 2\) and passes through the point \((1, 2)\).

### Proportionality

- We say that \(y\) is **directly proportional** to \(x\) if there is a non-zero constant \(k\) such that
  \[ y = kx; \]
- We say that \(y\) is **inversely proportional** to \(x\) if there is a non-zero constant \(k\) such that
  \[ y = \frac{k}{x}. \]

In either case, \(k\) is called the **constant of proportionality**.
Extra Homework Problems

1. Given the line \( s(t) = 3t - 4 \), answer the following.
   (a) Find an equation for the line through the point \((1, 3)\) that is parallel to the graph of \( s(t) \).
   (b) Find an equation for the line through \((1, 3)\) that is perpendicular to the graph of \( s(t) \).
   (c) Let \( g(x) = s(2x) \). First, give a formula for \( g(x) \), then find an equation of the line through the point \((0, 0)\) that is parallel to the graph of \( g(x) \).

2. Given that a triangle \( ABC \) in the coordinate plane has its vertices at the points \((1, 4)\), \((4, 8)\), and \((5, 1)\), show that triangle \( ABC \) is a right triangle.

3. Consider the piecewise defined function
   \[
   h(t) = \begin{cases} 
   t^2 - 1, & t \leq -2, \\
   t + 5, & -2 < t < 1, \\
   3 - t^2, & t \geq 1. 
   \end{cases}
   \]
   (a) Sketch a graph of \( h(t) \).
   (b) Evaluate \( h(-3) \), \( h(-2) \), \( h(0) \), \( h(1) \), and \( h(3) \).
   (c) Let \( j(z) = 2h(z) \). Evaluate \( j(-3) \), \( j(-2) \), \( j(0) \), \( j(1) \), and \( j(3) \).
   (d) Let \( l(z) = h(z + 1) \). Evaluate \( l(-3) \), \( l(-2) \), \( l(0) \), \( l(1) \), and \( l(3) \).

4. Explain why, for any given straight line, you can choose any two points to compute the slope, i.e. that no matter which two points you pick you always get the same slope. The picture below and what you learned in high school geometry should help you solve this problem.

5. The following law of gravitation was suggested by Ismael Bullialdus in 1645: “The gravitational attraction force \((F)\) between two point masses is directly proportional to the product of their masses and inversely proportional to the square of their separation distance \((d)\).”
Extra Homework Answers

1. (a) $y = 3t$.
   (b) $y = \frac{1}{3}t + \frac{10}{3}$.
   (c) $g(x) = 6x - 4$, $y = 6x$.

2. Not numeric – ask a teacher or help room assistant.

3. (a) Graph – ask a teacher or help room assistant.
   (b) $h(-3) = 8$, $h(-2) = 3$, $h(0) = 5$, $h(1) = 2$, $h(3) = -6$.
   (c) $j(-3) = 16$, $j(-2) = 6$, $j(0) = 10$, $j(1) = 4$, $j(3) = -12$.
   (d) $l(-3) = 3$, $l(-2) = 4$, $l(0) = 2$, $l(1) = -1$, $l(3) = -13$.

4. Not numeric – ask a teacher or help room assistant.

5. $F = \frac{km_1m_2}{d^2}$. 