Rational Functions

A rational function is a function of the form

\[ f(x) = \frac{p(x)}{q(x)}, \]

where \( p(x) \) and \( q(x) \) are polynomials.

Examples  The following are rational functions:

\[ \frac{1}{x}, \frac{x^2 - 1}{x^4 + x^3}, \frac{x^7 - 8}{x^2 + 2x - 7}. \]

Questions

1. Suppose \( f(x) = \frac{p(x)}{q(x)} \) is a rational function. Then

   (a) Suppose \( p(a) = 0 \) for some number \( a \). Then \( f(a) = \), so the graph of \( f(x) \) intercepts the ___-axes at \( x = a \).

   (b) Suppose \( q(b) = 0 \) for some number \( b \). Then the graph of \( f(x) \) has a _______ at \( x = b \).

2. Suppose \( r(x) = \frac{x^3 - 1}{x^2 + 2x + 1} \).

   (a) Does \( r(x) \) intercept the \( x \)-axis? Where?

   (b) Does it have any vertical asymptotes? Where?

   (c) Where is \( r(x) \) positive? Negative?

   (d) What happens as \( x \to \infty \)? What about as \( x \to -\infty \)?
(e) You should now have enough information to sketch a graph of \( r(x) \). Do so on the axes below. Be sure to label all intercepts and asymptotes \textit{before you start drawing the graph!}

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A Helpful Method

Note that the sign (positive or negative) of a rational function can change only when it hits a zero or a vertical asymptote. This can help make sketching such functions much easier.

3. Consider the function \( r(x) = \frac{(x-1)(x+2)}{(2x-1)(x+3)} \).

(a) \( r(x) \) has zeros at \( x = \) _____ and \( x = \) _____.

(b) \( r(x) \) has vertical asymptotes at \( x = \) _____ and \( x = \) _____.

(c) Fill in the following table:

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<tr>
<th>( x ) Sign</th>
<th>( x-1 )</th>
<th>( x+2 )</th>
<th>( 2x-1 )</th>
<th>( x+3 )</th>
<th>Total Sign</th>
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<tbody>
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<td>( x &lt; ) ___</td>
<td>(x-1)</td>
<td>(x+2)</td>
<td>(2x-1)</td>
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<td>___ &lt; ( x )</td>
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</table>

(d) What happens as \( x \to \infty \)? What about as \( x \to -\infty \)?
(e) Now draw the graph of \( r(x) \).

4. By repeating the steps from the previous question, sketch the graph of function

\[
f(x) = \frac{(x + 1)(x + 2)(x + 3)}{x(x - 1)^2}.
\]
Extra Homework Problems

1. Match each of the following rational functions with the correct graph.

\[
\begin{align*}
\frac{x^2 - 1}{x^2 + 1} & \quad \text{(A)} \\
\frac{x - 1}{(x + 1)(x - 2)} & \quad \text{(B)} \\
\frac{(x + 1)(x - 2)}{x - 1} & \quad \text{(C)}
\end{align*}
\]

2. Find all of the intercepts and asymptotes of the following functions.

\(\text{a) } r(x) = \frac{x}{x^2 - 4} \)

\(\text{b) } r(x) = \frac{x + 3}{x - 3} \)

\(\text{c) } r(x) = \frac{(x + 1)(x - 1)}{x + 2} \)

3. Sketch the graph of \(r(x) = \frac{(2x + 2)(x - 2)}{(x - 1)(x + 2)}\). Be sure to include and label all of the intercepts and asymptotes.