Preliminaries: Prior Limits and Limitations

Questions

- Compute $\lim_{t\to 0} \frac{2t+5}{t+7}$

- Using tools we have now, can you compute $\lim_{t\to 0} \frac{\ln(t+1)}{t}$? Why or why not?

L'Hôpital's rule

We are trying to compute expressions like

$$\lim_{x\to a} \frac{f(x)}{g(x)}.$$

We encounter a problem if $f(a) = g(a) =$ _____

Suppose that this is indeed the case. Then:

$$\begin{aligned}
\lim_{x\to a} \frac{f(x)}{g(x)} &= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} && \text{(since _____}\\
&= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}} && \text{(divide top and bottom by _____)}\\
&= \frac{\lim_{x\to a} \frac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \frac{g(x)-g(a)}{x-a}} && \text{(limit of a _____ is the _____ of the limits)}\\
&= \frac{f'(a)}{g'(a)} && \text{(this is the definition of the _____)}
\end{aligned}$$

Turns out we can do this if $f(a) = g(a) = \infty$ as well. This result is called *L'Hôpital's rule*.

Theorem - L'Hôpital's Rule: If f and g are differentiable, and either

$$\text{(a) } f(a) = g(a) = 0 \quad \textbf{or} \quad \text{(b) } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. (Here, $a = \pm\infty$ is allowed, too.)

Caution: You can ONLY use L'Hopital's Rule if either (a) or (b) is true. Do not attempt to use L'Hopital's Rule if neither (a) nor (b) is true. Also, keep in mind that the derivative is taken with respect to whatever is varying in the limit.

Examples

Compute the following limits, using L'Hôpital's Rule where needed. Indicate any time you use it.

- $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

- $\lim_{x\to 2} \frac{7x-5}{5-x}$

- $\lim_{x\to\infty} \frac{x^3}{e^{.01x}}$

- $\lim_{x\to\infty} xe^{-x}$ (What, no fraction?)

- $\lim_{x\to 0^+} x\ln(x)$ (Hint: $x = \frac{1}{1/x}$, so $x\ln(x) = \ldots$)

Function Dominance

Definition: The function g *dominates* f as $x \to \infty$ if and only if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Examples

For each of the following pairs of functions:

- Graph the two functions on the same axes.
- Using Wolfram Alpha, find out approximately where the two functions intersect.
- Which one of the two do you think dominates? Prove your hypothesis using limits.
  - $f(x) = e^x$, $g(x) = 6x$
  - $f(x) = \ln(x)$, $g(x) = 0.25x$

A Note of Caution

It is *not always true* that if two functions intersect and one is then bigger than the other, then the bigger one dominates:
Graph $f(x) = x^2 + 2x + 3$ and $g(x) = x^2$. Just looking at the graphs, which one looks dominant? Check your hypothesis using the definition. You may be surprised!

Equivlance: We call a pair of functions neither of which dominates the other *equivalent under dominance.*

Family Dominance

Looking back at the previous page, we see that the exponent e^x dominates the power function $6x$, and the power function $0.25x$ dominates the logarithmic function $\ln x$. Indeed, it is generally true that:

> **Family Dominance:** Exponentials dominate polynomials/power functions, which in turn dominate logarithmic functions.

This is true even in extreme examples, as the following example (and the similar homework question) show:

Example Consider the functions $f(x) = (\ln(x))^{1,000}$, and $g(x) = \sqrt{x}$.

$$f(25) \approx 5 \times 10^{507};\ g(25) = 5.$$

Which does this make you think dominates? Check your hypothesis. (Try not to take $1,000$ derivatives here. Instead, figure out after a couple of them what will happen if you continue.)

Homework Problems

In Questions 1-9, check whether l'Hôpital's rule applies. If so, use it to evaluate the limit. If not, use another way.

1. $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x^2 - 9}$

2. $\lim_{x \to \infty} \dfrac{x^7 + 3x^4 - x^2 + 4}{x^3 - 2x^2 - 4x + 1}$

3. $\lim_{t \to 1} \dfrac{(t-1)^4}{(t^4 - 1)}$

4. $\lim_{x \to 0} \dfrac{-2e^{-2x} - 2x + 2}{x^2 + x}$

5. $\lim_{x \to 2} \dfrac{x^2 - 1}{2x^2 + 1}$

6. $\lim_{x \to \infty} \dfrac{e^x}{\ln 1/x}$

7. $\lim_{x\to\infty} \frac{x^3}{e^{.01x}}$

8. $\lim_{x\to 7} \frac{x^2 - \sqrt[4]{3x-5} - 5x - 12}{x^2 + \ln(x) - 1}$

9. $\lim_{x\to 1} \frac{\sqrt{2x - x^4} - \sqrt[3]{x}}{1 - x^{4/3}}$

10. We want to find $\lim_{t\to\infty} \dfrac{t}{\sqrt{t^2-1}}$. We have $\frac{\infty}{\infty}$, so LH applies.

    (a) Do LH once and simplify your answer.

    (b) Do LH again and simplify.

    (c) If you did these two correctly, you should have found we get right back to where we started. So this isn't going to work! However, recall that $\frac{1}{\frac{a}{b}} = \frac{b}{a}$. Use this together with your equality in part (a) to find the limit.

11. Consider the functions $f(x) = x^{10,000}$, and $g(x) = e^{0.00001x}$. Note that

$$f(2) = 2^{10,000} \approx 2 \times 10^{3010} \text{ (2 with 3,010 zeros after it); } g(2) = e^{0.00002} \approx 1.00002.$$

Which does this make you think dominates? Check your hypothesis. (Try not to take $10,000$ derivatives here. Instead, figure out after a couple of them what will happen if you continue.)