

Let's look at a question we already know the answer to and see where our current knowledge gets us stuck:

Questions

1. Compute $\lim_{t \rightarrow 0} \frac{2t + 5}{t + 7}$

2. Using tools we have now, can you compute $\lim_{t \rightarrow 0} \frac{\ln(t + 1)}{t}$? Why or why not?

L'Hôpital's rule

We are trying to compute expressions like

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}.$$

We encounter a problem if $f(a) = g(a) = \underline{\hspace{2cm}}$!

Suppose that this is indeed the case. Then:

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{g(x) - g(a)} && \text{(since } \underline{\hspace{2cm}} \text{)} \\ &= \lim_{x \rightarrow a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} && \text{(divide top and bottom by } \underline{\hspace{2cm}} \text{)} \\ &= \frac{\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}}{\lim_{x \rightarrow a} \frac{g(x) - g(a)}{x - a}} && \text{(limit of a } \underline{\hspace{2cm}} \text{ is the } \underline{\hspace{2cm}} \text{ of the limits)} \\ &= \frac{f'(a)}{g'(a)} && \text{(this is the definition of the } \underline{\hspace{2cm}} \text{)} \end{aligned}$$

Turns out we can do this if $f(a) = g(a) = \infty$ as well. This result is called *L'Hôpital's rule*.

Theorem - L'Hôpital's Rule: If f and g are differentiable, and either

$$(a) f(a) = g(a) = 0 \quad \text{or} \quad (b) \lim_{x \rightarrow a} f(x) = \infty \text{ and } \lim_{x \rightarrow a} g(x) = \infty$$

Then

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right exists. (Here, $a = \pm\infty$ is allowed, too.)

Caution: You can ONLY use L'Hôpital's Rule if either (a) or (b) is true. Do not attempt to use L'Hôpital's Rule if neither (a) nor (b) is true. Also, keep in mind that the derivative is taken with respect to whatever is varying in the limit.

Question

3. Compute the following limits, using L'Hôpital's Rule where needed. Indicate any time you use it.

(a) $\lim_{x \rightarrow 1} \frac{\ln(x)}{x - 1}$

(b) $\lim_{x \rightarrow 2} \frac{7x - 5}{5 - x}$

(c) $\lim_{x \rightarrow \infty} \frac{x^3}{e^{0.01x}}$

(d) $\lim_{x \rightarrow \infty} x e^{-x}$ (What, no fraction?)

(e) $\lim_{x \rightarrow 0^+} x \ln(x)$ (Hint: $x = \frac{1}{1/x}$, so $x \ln(x) = \dots$)

Function Dominance

Definition: The function g *dominates* f as $x \rightarrow \infty$ if and only if

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0.$$

Questions

4. For each of the following pairs of functions:

- Graph the two functions on the same axes.
- Using Wolfram Alpha, find out approximately where the two functions intersect.
- Which one of the two do you think dominates? Prove your hypothesis using limits.

(a) $f(x) = e^x$, $g(x) = 6x$

(b) $f(x) = \ln(x)$, $g(x) = 0.25x$

5. Which of the following two functions do you think dominates? Check your hypothesis. You may be surprised!

$$f(x) = x^2 + 2x + 3, g(x) = x^2$$

6. Consider the functions $f(x) = x^{10,000}$, and $g(x) = e^{0.00001x}$. Note that

$$f(2) = 2^{10,000} \approx 2 \times 10^{3010} \text{ (2 with 3,010 zeros after it); } g(2) = e^{0.00002} \approx 1.00002.$$

Which do you think dominates? Check your hypothesis. (Try not to take 10,000 derivatives here. Instead, figure out after a couple of them what will happen if you continue.)

7. Do the same for $f(x) = (\ln(x))^{1,000}$, and $g(x) = \sqrt{x}$.

$$f(25) \approx 5 \times 10^{507}; g(25) = 5.$$

Question

8. Use the questions above to fill in the blanks in the following hypothesis with the phrases 'power and polynomial functions', 'logarithmic functions', and 'exponential functions':

In general, _____ dominate _____,
which in turn dominate _____.

Homework Problems

Evaluate each of these limits. If appropriate, use l'Hôpital's rule.

$$1. \lim_{x \rightarrow 3} \frac{x^2 - x - 6}{x^2 - 9}$$

$$2. \lim_{x \rightarrow \infty} \frac{x^7 + 3x^4 - x^2 + 4}{x^3 - 2x^2 - 4x + 1}$$

$$3. \lim_{t \rightarrow 1} \frac{(t-1)^4}{(t^4 - 1)}$$

$$4. \lim_{x \rightarrow 0} \frac{-2e^{-2x} - 2x + 2}{x^2 + x}$$

$$5. \lim_{x \rightarrow 2} \frac{x^2 - 1}{2x^2 + 1}$$

$$6. \lim_{x \rightarrow \infty} \frac{e^x}{\ln 1/x}$$

$$7. \lim_{x \rightarrow \infty} \frac{x^3}{e^{.01x}}$$

$$8. \lim_{x \rightarrow 7} \frac{x^2 - \sqrt[4]{3x-5} - 5x - 12}{x^2 + \ln(x) - 1}$$

$$9. \lim_{x \rightarrow 1} \frac{\sqrt{2x-x^4} - \sqrt[3]{x}}{1-x^{4/3}}$$

$$10. \lim_{t \rightarrow \infty} \frac{t}{\sqrt{t^2 - 1}}$$