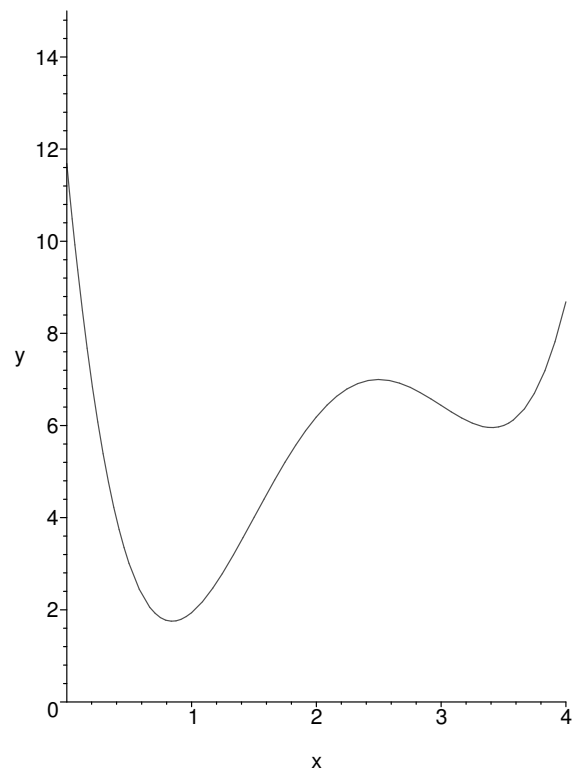
Review from Earlier This Week

- **First Derivative Test:**
  Let $f(x)$ be a function and let a be in the domain of f. Then:
  - If _____, then f has a local max at a.
  - If _____, then f has a local min at a.
- **Second Derivative Test:**

Preliminaries: Local vs. Global Extrema

During the week, we talked about local max and mins of functions. Today we are going to look at global max and mins. A **global maximum** of a function f is the greatest value of f over a specified domain. A **global minimum** is the least value of f over the domain. The global min and the global max define lower and upper bounds, respectively, for the function. In particular, global min $\leq f(x) \leq$ global max for all x in the domain.

Example Consider the function below.



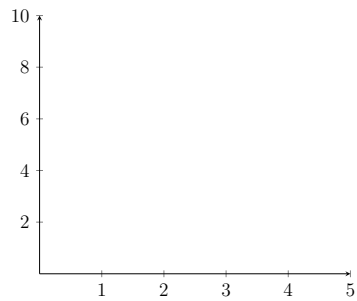
- Identify the local and global mins and maxes.

- Using the graph above to guide you, what are the possible locations for the global maximums and minimums?

When Do Global Maxima and Minima Exist?

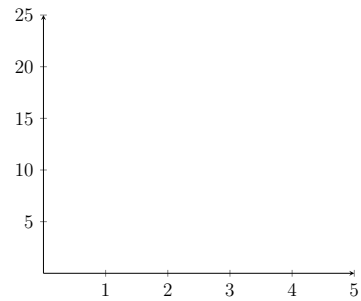
Question

Draw the graph of $g(x) = \frac{1}{x}$ on the axes below.



Does g have a global maximum over the interval $0 < x \leq 1$? Explain.

Draw the graph of $h(x) = x^2$ on the axes below.



Does h have a global maximum over the interval $0 \leq x < \infty$? Explain.

Extreme Value Theorem: If f is continuous on a closed finite interval $a \leq x \leq b$, then $f(x)$ has both a global maximum and a global minimum on that interval.

Questions

Why do $g(x)$ and $h(x)$ fail this theorem?

Explain why $f(x) = x$ does not have a global min on $-2 < x < 2$, but $g(x) = x^2$ does.

Finding Global Minima and Maxima

As we saw above, global maxima and minima can only occur at critical points of the function, or at endpoints of a closed domain. So if a continuous function is only defined on a closed interval, all we need to do is check those:

Procedure for finding the global max and min of a function f

- Find the critical points of f.
- Find the values of f at the critical points and at the endpoints.
- Determine the global max and min from these values.

Continuous Functions on Finite Intervals

Example Find the global max and min of $f(x) = x(x-1)$ on the interval $0 \leq x \leq 3$.

Questions

1. For the following functions, find the global maximum and minimum on the specified domain, if they exist.

□Regrade

(a) $f(x) = x^3 - 7x + 6$ on the interval $-4 \leq x \leq 2$

(b) $g(x) = \ln(1 + x^2)$ for $-1 \leq x \leq 2$

2. When you cough, your windpipe contracts. The speed, v with which air comes out depends on the radius, r, of your windpipe. If R is the normal (rest) radius of your windpipe, then $r \leq R$, the speed is given by:

□Regrade

$$v(r) = a(R - r)r^2 \qquad \text{where } a \text{ is some positive constant}$$

(a) Why is $0 \leq r \leq R$ the domain of $v(r)$?

(b) What value of r maximizes the speed? Your answer may be in terms of R.

Open or Infinite Domains

If the domain is not closed or finite, we need to be careful. In the following question, the domain doesn't include $x = 0$. Best way to deal with this is to include it in the table anyway. If it turns out the global max or min would be there had it been included, what would your conclusion be? See the last question on page 2!

3. $h(x) = \frac{x^3}{3} - x^2 + x$ on $0 < x \leq 2$

□Regrade

When the domain is not finite, we can't include (at least one of its) endpoints in our table. In such cases, we need a different approach. Or rather, we need a combination of approaches. For example, if a function is always increasing, it can't have a global max. Or, if a function is always positive and decreases from a global max toward zero, but never gets there, it may not have a global min. The following questions explore this.

4. Consider $f(t) = te^{-t}$ for $t \geq 0$. Draw the function, then find the global max and min and label them on your graph.

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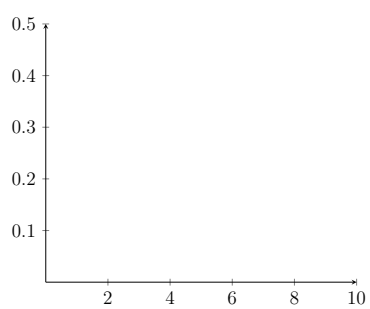


5. When an electric current passes through two resistors with resistance r_1 and r_2 con-
□Regrade nected in parallel, the combined resistance, R can be calculated from the equation

$$\frac{1}{R} = \frac{1}{r_1} + \frac{1}{r_2}$$

where R, r_1, and r_2 are positive. Assume r_2 is constant.

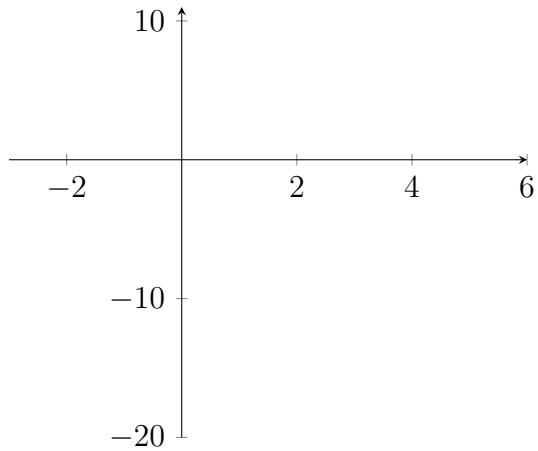
(a) Show that $R = \frac{r_1 r_2}{r_1 + r_2}$

(b) Does R have a global max on $0 \leq r_1 < \infty$?

(c) Does the combined resistance go to ∞ as $r_1 \to \infty$? If not, what does it approach?

Bonus: Draw the circuit and explain your last answer!

Continuity and Global Maxima and Minima

Lastly, if a function is discontinuous at a point, we need to look at what happens to it nearby.

6. Graph $p(x) = \ln x^2 - (x-1)^2 + 8$ on $-3 \leq x \leq 6$. Find the global max and global ☐Regrade min of the function and label them, if they exist. Take care to look at $x = 0$ carefully!