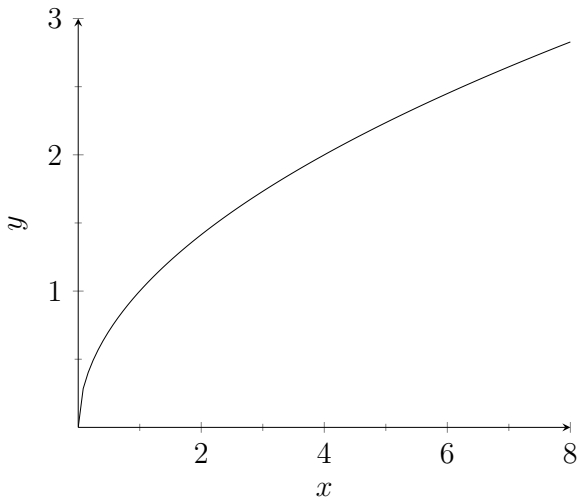
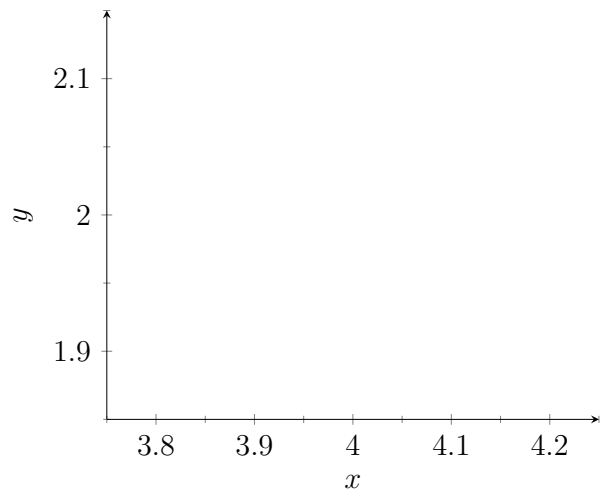
Preliminaries: Tangent Lines

The tangent line to a function $f(x)$ at $x = a$ has slope _____. It passes through the point $(a, _____)$. So its equation is

$$_____ = _____$$

Question Let $f(x) = \sqrt{x}$. It's graph is shown on the left for x in the domain $[0, 8]$:





Find the equation for the tangent line to $f(x)$ at the point $x = 4$ and draw it on top of the graph of $f(x)$. Try to be as precise as possible. Draw a very small box around the point $(4, 2)$.

Use Geogebra (or other software) to draw $f(x)$ and the tangent line you found with domain $3.75 \leq x \leq 4.25$ and range $1.85 \leq y \leq 2.15$. Copy that onto the axes on the right. What do you observe?

Your observation is true because $f(x)$ is _____ at $x = 2$ (see worksheet on differentiability, page 1).

Now suppose you only have a very simple calculator with $+$, $-$, $\times$, and $\div$ operators. By observing your graphs and using your computations above, find a good approximation of $\sqrt{4.1}$.

Use your calculator to get the value of $\sqrt{4.1}$ to six decimal places. How far off was your approximation above?

Linear Approximation

Linear Approximation
What you just carried out above is called a *linear approximation* of the value of $\sqrt{4.1}$. Linear approximation uses the fact that for differentiable functions, the tangent line to the function at a point $x = a$ lies very near $f(x)$ itself *if we don't stray far from* $x = a$.

Let's see what happens if we try to use the same tangent line you found on the previous page to find $\sqrt{6}$:

- By looking at the graphs you draw on the previous page, do you expect that your approximation will be as good as your approximation of $\sqrt{4.1}$? Why or why not?

- Use the same line as above to approximate $\sqrt{6}$.

- Now use your calculator again to get the value of $\sqrt{6}$ to four decimal places. How far off was your approximation?

Definition When we refer to '***the linear approximation of*** $f(x)$ ***at a point*** $x = a$', we mean the tangent line to the graph of $f(x)$ at that point.
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Questions

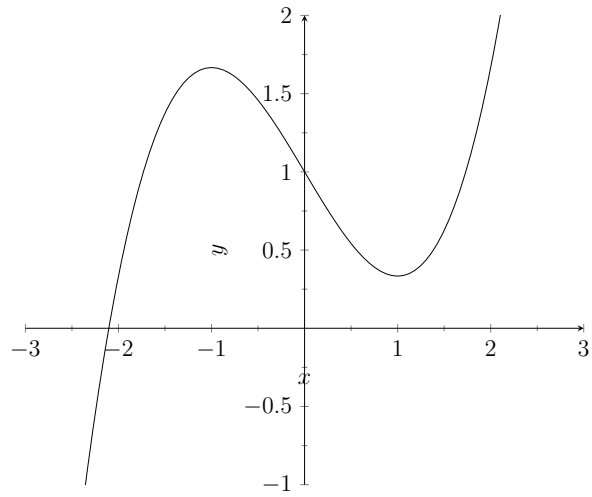
1. Let $y = \ln(x)$. Find the linear approximation at $x = 1$. Then draw the curve and the
□Regrade tangent line, and use the latter to estimate $\ln(1.1)$. How far off is your approximation? Do the same for $\ln(1.5)$. Are your estimates overestimates of the true values of $\ln(x)$ at 1.1 and 1.5 or underestimates?

2. Use linear approximations to estimate $e^{0.1}$ and $e^{0.5}$. (Hint: where would it make sense
□Regrade to base your linear approximation?) Are your estimates of the values of e^x at 0.1 and 0.5 overestimates or underestimates? Draw a graph (and a tangent line) to explain your answers.

You should have found that one of your two linear approximations above *overestimates* while the other *underestimates*. What is the difference between the shapes of the two graphs that might explain this?

3. Consider the function graphed below

☐Regrade



(a) Draw the tangent lines to curve at the point $x = -1.5$ and $x = 1.25$.

(b) At $x = -1.5$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(c) At $x = 1.25$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(d) Under what circumstances are (b) and (c) true? Why?

4. Suppose that $f(x)$ is differentiable function, that $f(4) = 3$ and $f'(4) = 6$.

☐Regrade

(a) Write down the linear approximation to $f(x)$ at $x = 4$.

(b) Use your approximation to estimate $f(3.9)$.

(c) If $f''(x) < 0$ on the interval $[3.9, 4]$, is your estimate an overestimate or an underestimate?