From Way Back When....

- The derivative of a function $f(x)$ at a point $a$ is
  
  $$f'(a) = \underline{\text{___________}}.$$

- The derivative function of a function $f(x)$ is
  
  $$f'(x) = \underline{\text{___________}}.$$

Tangent Lines and Linear Approximations

**Question**  Let $f(x) = \sqrt{x}$.

1. Draw the graph of $f(x)$ for $x$ in the domain $[0, 8]$ on the axes below:

   ![Graph of $f(x) = \sqrt{x}$]

2. Find the equation for the tangent line to $f(x)$ at the point $x = 4$ and draw it on top of the graph of $f(x)$. Try to be as precise as possible.
3. Draw a very small box around the point (4, 2).

(a) What do you notice? Can you explain your observation?

(b) Suppose you only have a very simple calculator with +, −, ×, and ÷ operators. Use your work above to find a good approximation of $\sqrt{4.1}$.

(c) Use your calculator to get the value of $\sqrt{4.1}$ to six decimal places. How far off was your approximation above?

4. Let’s see what happens if we try to use the same idea to find $\sqrt{6}$:

(a) By looking at the graphs you draw on the previous page, do you expect that your approximation will be as good as your approximation of $\sqrt{4.1}$? Why or why not?

(b) Use the same method as above to approximate $\sqrt{6}$.

(c) Now use your calculator again to get the value of $\sqrt{6}$ to four decimal places. How far off was your approximation?
**Definition**  When we refer to ‘the linear approximation of $f(x)$ at a point $x = a$’, we mean the tangent line to the graph of $f(x)$ at that point.

**Questions**

1. Let $y = \ln(x)$. Find the linear approximation at $x = 1$. Then draw the curve and the tangent line, and use the latter to estimate $\ln(1.1)$. How far off is your approximation? Do the same for $\ln(1.5)$. Are your estimates overestimates of the true values of $\ln(x)$ at 1.1 and 1.5 or underestimates?

2. Use linear approximations to estimate $e^{0.1}$ and $e^{0.5}$. (Hint: where would it make sense to base your linear approximation?) Are your estimates of the values of $e^x$ at 0.1 and 0.5 overestimates or underestimates? Draw a graph (and a tangent line) to explain your answers.
3. Consider the function graphed below

(a) Draw the tangent lines to curve at the point $x = -1.5$ and $x = 1.25$.

(b) At $x = -1.5$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(c) At $x = 1.25$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(d) Under what circumstances are (b) and (c) true? Why?

4. Suppose that $f(x)$ is differentiable function, that $f(4) = 3$ and $f'(4) = 6$.

(a) Write down the linear approximation to $f(x)$ at $x = 4$.

(b) Use your approximation to estimate $f(3.9)$.

(c) If $f''(x) < 0$ on the interval $[3.9, 4]$, is your estimate an overestimate or an underestimate?