Previous facts about f, f', and f''

- if $f' > 0$ on an interval, then f is __________________ on that interval.
- if $f' < 0$ on an interval, then f is __________________ on that interval.
- if $f'' > 0$ on an interval, then f is __________________ on that interval.
- if $f'' < 0$ on an interval, then f is __________________ on that interval.

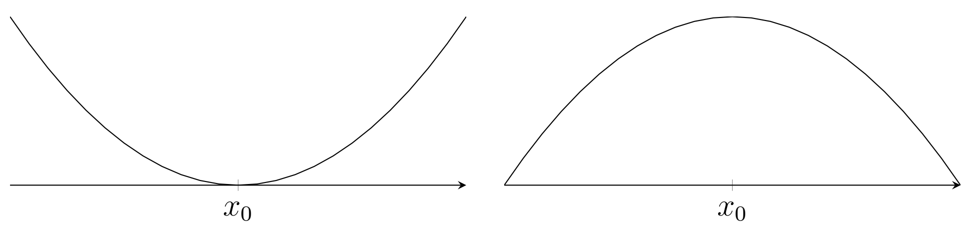
The First Derivative Test

Let a be a po int in the domain of a function f. Then

- f has a **local maximum** at a if $f(a)$ ___ $f(x)$ for all x values near a.
- f has a **local minimum** at a if $f(a)$ ___ $f(x)$ for all x values near a.
- a is called a **critical point** of f if $f'(a) =$ ___ or $f'(a)$ is ____________.

Note: we use the term *extrema* to refer to both maxima and minima.

Question Consider the graphs below:



What happens to f' around x_0 in each of the graphs?

The First Derivative Test
Let x_0 be a critical point of a continuous function f.

- If ______________________________________, then f has a local max at x_0.
- If ______________________________________, then f has a local min at x_0.

Questions

1. Find and classify all of the local extrema of $f(x) = x^3 - 3x + 4$. (i.e. decide whether they are local maxima, minima, or neither.)

2. True or False? : If $f'(x_0) = 0$, then f has a local max or min at x_0. Explain.

The Second Derivative Test

Consider again the plots on the previous page. How does the concavity of a function of at a critical point relate to local extrema?

The Second Derivative Test:

- If $f'(x_0) = 0$ and $f''(x_0) > 0$ then f has a local ____________________ at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) < 0$ then f has a local ____________________ at x_0.
- If $f'(x_0) = 0$ and $f''(x_0) = 0$ then the test gives no information.

Questions

3. Use the second derivative test to classify the critical points of $f(x) = x^3 - 3x + 4$.

4. Classify the critical point $x = 0$ for $y = x^3$, $y = x^4$, and $y = -x^4$.

Definition A function f has an **inflection point** at a if the concavity of f changes at a. At an inflection point x_0,

$$f''(x_0) = \underline{\quad\quad} \quad \text{or} \quad f''(x_0) \text{ is } \underline{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad}$$

WARNING!!! Not every point where $f''(x) = 0$ is an inflection point (just as not every point where $f' = 0$ is a local max or local min.). Example?

Questions

5. Consider $f(x) = (x^2 - 4)^7$. What are the roots of this function? Find and classify all local extrema. Also find the inflection points. Then draw the graph.

6. Consider $y = x \ln x$ for $x > 0$. Find any critical points, and classify them. Also find the roots and any inflection points. Draw the graph. What is your best guess as to what happens as we approach 0? We will prove this next week.

7. Find and classify all the critical points of $f(x) = \sqrt[3]{x}$. Find roots and IPs too, and draw the graph, labeling all critical points and IPs.

8. Find and classify all the critical points of $g(x) = |x|(x^2 - 9)$. Find all roots and IPs too, and draw the graph, labeling all those points. (Hints: Write it as a piecewise function! Be careful with the derivative at $x = 0$!)

9. Find and classify all critical points of $f(x) = xe^{-\frac{x^2}{2}}$. Draw the graph, labeling all critical points and IPs.

10. Consider $f(x) = x^4 + ax^2 + b$.

    (a) Under what conditions on a and b will this function have one critical point? Will it be a local maximum, local minimum or neither?

    (b) Under what conditions on a and b will this function have exactly three critical points? What are they and which are local maxima and local minima?

    (c) Is it ever possible for this function to have two critical points? No critical points? More than three critical points? Explain.

    (d) Draw examples of the graph of this functions with $a = 0$, $a = 3$, and $a = -3$ to illustrate your analysis above. Label the critical points.

Challenge Question Consider the general cubic:

$$f(x) = x^3 + ax^2 + bx + c.$$

(a) By taking derivatives and using the quadratic formula, show that:

- if $a^2 = 3b$, then $f(x)$ has exactly one critical point at $x = -\frac{1}{3}a$, and it is an inflection point;
- if $a^2 < 3b$, then $f(x)$ has no critical points, but does have one inflection point;
- if $a^2 > 3b$, then $f(x)$ has two critical points: one max, and one min, and an inflection point exactly in the middle between the two.

(b) Draw examples of each case.