And Once Again, The Past...

1. The derivative of a function \( f(x) \) is

2. A function is continuous at a point \( x = a \) if

Differentiability

**Question**  If we zoom in far enough near any point of the graph of the function \( f(x) = x^2 \), the graph looks like a ________ _______

**Question** What about \( f(x) = |x| \) near 0? Do we get the same thing? What is going on?

A function \( f \) is *differentiable* at a point \( a \) if the \( \frac{f(a+h) - f(a)}{h} \) exists and is finite at \( x = a \). In other words, \( f(x) \) is differentiable at \( x = a \) if

\[
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}
\]

exists and is finite.

**Questions**

1. Show from the definition that the following functions are not differentiable at \( x = 0 \):

   (a) \( f(x) = |x| \)

   (b) \( f(x) = x^{\frac{1}{3}} \)
(c) \( f(x) = \begin{cases} 
-1 & \text{if } x > 0 \\
1 & \text{if } x \leq 0
\end{cases} \)

2. What can cause a function not to be differentiable at a point \( x = a \)? Draw an example of each.

- If the function

- If the function

- If the function

**Continuity and Differentiability**

**Theorem**  If \( f(x) \) is \( \underline{\text{ }} \) at a point \( x = a \), then \( f(x) \) is \( \underline{\text{ }} \) at \( x = a \).

**Questions**

1. Fill in the following blanks with the words “continuous” and “differentiable”:

- If \( f(x) \) is not \( \underline{\text{ }} \) at \( x = a \), it is not \( \underline{\text{ }} \) at \( x = a \).

- If \( f(x) \) is \( \underline{\text{ }} \) at \( x = a \), it is \( \underline{\text{ }} \) at \( x = a \).

- It is possible for \( f(x) \) to be \( \underline{\text{ }} \) at \( x = a \), but not \( \underline{\text{ }} \) at \( x = a \).
2. (a) Find \( f'(0) \) or explain why it does not exist if
\[
f(x) = \begin{cases} 
4 - x^2 & \text{if } x > 0 \\
x^2 - 4 & \text{if } x \leq 0 
\end{cases}
\]

(b) Find \( f'(0) \) or explain why it does not exist if
\[
f(x) = \begin{cases} 
4 - x^2 & \text{if } x > 0 \\
x^2 + 4 & \text{if } x \leq 0 
\end{cases}
\]

3. Find \( f'(0) \) or explain why it does not exist if \( f(x) = (x + |x|)^2 + 1 \).
   (Hint: Write \( |x| \) as a piecewise function!)
4. Find $a$ and $b$ such that the following function is differentiable everywhere:

$$f(x) = \begin{cases} 
  ax^3 & \text{if } x \leq 2 \\
  x^2 + b & \text{if } x > 2 
\end{cases}$$

(Hint: First make $f(x)$ continuous, then differentiate...)