From Way Back When....

- The derivative of a function $f(x)$ at a point $a$ is
  \[ f'(a) = \text{________________________} \, . \]

- The derivative function of a function $f(x)$ is
  \[ f'(x) = \text{________________________} \, . \]

Tangent Lines and Linear Approximations

**Question**  Let $f(x) = \sqrt{x}$.

1. Draw the graph of $f(x)$ for $x$ in the domain [0, 8] on the axes below:

![Graph of $f(x) = \sqrt{x}$](image)

2. Find the equation for the tangent line to $f(x)$ at the point $x = 4$ and draw it on top of the graph of $f(x)$. Try to be as precise as possible.
3. Draw a very small box around the point (4, 2).

(a) What do you notice? Can you explain your observation?

(b) Suppose you only have a very simple calculator with +, −, ×, and ÷ operators. Use your work above to find a good approximation of $\sqrt{4.1}$.

(c) Use your calculator to get the value of $\sqrt{4.1}$ to six decimal places. How far off was your approximation above?

4. Let’s see what happens if we try to use the same idea to find $\sqrt{6}$:

(a) By looking at the graphs you draw on the previous page, do you expect that your approximation will be as good as your approximation of $\sqrt{4.1}$? Why or why not?

(b) Use the same method as above to approximate $\sqrt{6}$.

(c) Now use your calculator again to get the value of $\sqrt{6}$ to four decimal places. How far off was your approximation?
Definition  When we refer to ‘the linear approximation of \( f(x) \) at a point \( x = a \)’, we mean the tangent line to the graph of \( f(x) \) at that point.

Questions

1. Let \( y = \ln(x) \). Find the linear approximation at \( x = 1 \). Then draw the curve and the tangent line, and use the latter to estimate \( \ln(1.1) \). How far off is your approximation? Do the same for \( \ln(1.5) \). Are your estimates overestimates of the true values of \( \ln(x) \) at 1.1 and 1.5 or underestimates?

2. Use linear approximations to estimate \( e^{0.1} \) and \( e^{0.5} \). (Hint: where would it make sense to base your linear approximation?) Are your estimates of the values of \( e^x \) at 0.1 and 0.5 overestimates or underestimates? Draw a graph (and a tangent line) to explain your answers.
3. Consider the function graphed below

(a) Draw the tangent lines to curve at the point $x = -1.5$ and $x = 1.25$.

(b) At $x = -1.5$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(c) At $x = 1.25$, the linear approximation will overestimate/underestimate the value of the function at nearby points. (Cross out the wrong answer.)

(d) Under what circumstances are (b) and (c) true? Why?

4. Suppose that $f(x)$ is differentiable function, that $f(4) = 3$ and $f'(4) = 6$.

(a) Write down the linear approximation to $f(x)$ at $x = 4$.

(b) Use your approximation to estimate $f(3.9)$.

(c) If $f''(x) < 0$ on the interval $[3.9, 4]$, is your estimate an overestimate or an underestimate?