Duke University
Math 218: Matrices and Vector Spaces

Exam 2 Practice

THE FOLLOWING INSTRUCTIONS WILL BE ON THE TEST:

- Do not open this test booklet until you are directed to do so.
- You will have 50 minutes to complete the exam. If you finish early go back and check your work.
- This exam is closed book, but you may use a one-sided sheet of notes in your own handwriting.
- You may use a calculator for arithmetical purposes only.
- Throughout the exam, show your work so that your reasoning is clear. Otherwise no credit will be given. Circle your answers.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.

1. Consider the matrix $A = \begin{pmatrix} 3 & -3 \\ 3 & 1 \\ 3 & 5 \\ 3 & 1 \end{pmatrix}$. Note that $\vec{v} = [2, 4, 6, 8]^T \notin C(A)$.

   (a) Use Gram-Schmidt to find the QR decomposition of A. You should get $Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & 0 \end{pmatrix}$.

   (b) Find the least squares solution $\hat{x}$ of $A\vec{x} = \vec{v}$, using your QR decomposition above or otherwise.

(c) Find the projection $\vec{p}$ of $\vec{v}$ onto $C(A)$. (Hint: you only need to multiply one matrix by one vector!)

(d) Consider the matrix $[A|I_4]$, A with the 4×4 identity matrix adjoined. If I carry out Gram-Schmidt on this matrix, the first two columns will be v_1 and v_2 as above. Describe the other four columns. Be sure to refer to (an)other fundamental space(s) of A, and explain your answers.

2. Suppose that V and W are subspaces of $\mathbb{R}^n$ with $dim(V) + dim(W) > n$. Show that $dim(V \cap W) > 0$. You may not use any formulas that have not been covered in class.

3. (a) Construct a matrix A with the vectors $[1, 0, 1]^T$ and $[0, 1, 1]^T$ in its row space and $[1, 0, 1]^T$ in its left nullspace.

(b) Find bases for each of the column space and nullspace of the matrix A in part (a).

4. Determine if each of the following statements is true or false. If it is true, explain why. If it is false, give a counterexample.

   (a) If $C(A) = C(A^T)$, then $A = A^T$ (i.e. if a matrix has the same row and column spaces, it is symmetric.)

   (b) If Q is a matrix with orthonormal columns, then $QQ^T = I$.

5. The columns of the matrix $\frac{1}{3}\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}$ form an orthonormal basis (or frame) of $\mathbb{R}^3$. Find the coordinates of $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ in this frame.