Exam 1

- Do not open this test booklet until you are directed to do so.
- You will have 50 minutes to complete the exam. If you finish early go back and check your work.
- This exam is closed book. However, you can use a one-sided letter sized cheat sheet in your own handwriting.
- You may use a calculator for arithmetic purposes only.
- Throughout the exam, show your work so that your reasoning is clear. Otherwise no credit will be given. Circle your answers.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
</tr>
</tbody>
</table>

Name: ______________________
Question 1 (15 points) Let \( A = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 1 & 6 \\ 2 & 4 & 14 \end{bmatrix} \).

(a) By using Gauss-Jordan reduction, find \( A^{-1} \). You are not required to do maximal partial pivoting for this question, but be sure to write down every step of the row reduction.

Solution:

\[
[A|I] = \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 1 & 1 & 6 & 0 & 1 & 0 \\ 2 & 4 & 14 & 0 & 0 & 1 \end{pmatrix} \]

\[
\xrightarrow{r_2 \rightarrow r_1} \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 2 & 4 & 14 & 0 & 0 & 1 \end{pmatrix} \]

\[
\xrightarrow{r_3 - 2r_1} \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & -2 & 0 & 1 \end{pmatrix} \]

\[
\xrightarrow{r_3/2} \begin{pmatrix} 1 & 2 & 6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{pmatrix} \]

\[
\xrightarrow{r_2/\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 7 & 0 & -3 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{pmatrix} \]

\[
\xrightarrow{r_1 - 2r_2} \begin{pmatrix} 1 & 0 & 0 & 5 & 2 & -3 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1/2 \end{pmatrix} = [I|A^{-1}] \]

(b) Let \( \vec{c}_1 \) and \( \vec{c}_2 \) be the first two columns of \( A \). Let \( \theta \) the angle between these vectors. Find \( \cos \theta \).

Solution:

\[
\cos \theta = \frac{\vec{c}_1 \cdot \vec{c}_2}{||\vec{c}_1|| ||\vec{c}_2||} = \frac{1 \cdot 2 + 1 \cdot 1 + 2 \cdot 4}{\sqrt{1^2 + 1^2 + 2^2 \sqrt{2^2 + 1^2 + 4^2}}} = \frac{11}{\sqrt{126}}
\]
Question 2 (5 points) Show that if $A^T A = 0$, then $A = 0$. You may not assume that $A$ is any particular size. Rather, you need to argue in general.

Solution: If $A = [\vec{c}_1 \ \vec{c}_2 \ \ldots \ \vec{c}_n]$, where $\vec{c}_i \in \mathbb{R}^m$ are the columns of $A$, then $A^T = \begin{pmatrix} \vec{c}_1^T \\ \vec{c}_2^T \\ \vdots \\ \vec{c}_n^T \end{pmatrix}$. The $(i, j)$ entry in $A^T A$ is the dot product of the $i^{th}$ row of $A^T$ and the $j^{th}$ column of $A$. But the $i^{th}$ row of $A^T$ is the $i^{th}$ column of $A$, so the $(i, j)$ entry in $A^T A$ is the dot product of the $i^{th}$ column of $A$ and the $j^{th}$ column of $A$:

$$A^T A = \begin{pmatrix} \vec{c}_1 \cdot \vec{c}_1 & \vec{c}_1 \cdot \vec{c}_2 & \cdots & \vec{c}_1 \cdot \vec{c}_n \\ \vec{c}_2 \cdot \vec{c}_1 & \vec{c}_2 \cdot \vec{c}_2 & \cdots & \vec{c}_2 \cdot \vec{c}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{c}_n \cdot \vec{c}_1 & \vec{c}_n \cdot \vec{c}_2 & \cdots & \vec{c}_n \cdot \vec{c}_n \end{pmatrix}$$

Specifically, every entry on the diagonal of $A^T A$ is the dot product of a column of $A$ with itself:

$$(A^T A)_{i, i} = \vec{c}_i \cdot \vec{c}_i = ||\vec{c}_i||^2.$$ If all the diagonal entries are 0, this means the magnitude of every column is 0. Since the only vector with magnitude 0 is the zero vector, all the columns of $A$ are $\vec{0}$.

Question 3 (6 points) One of the following is a vector subspace of $\mathbb{R}^3$. The other is not. Decide which is which. Justify your choices using the definition of a vector space. Answers with no justification will receive no credit.

(a) All vectors $[b_1, b_2, b_3]^T$ such that $b_1 + b_2 - 2b_3 = 0$.

Solution: Let $V$ be the set of all such vectors. We need $c\vec{v} + d\vec{w} \in V$ for every $\vec{v}, \vec{w} \in V$ and $c, d \in \mathbb{R}$. Let $\vec{v} = [b_1, b_2, b_3]^T$ and $\vec{w} = [c_1, c_2, c_3]^T$. Then

$$c\vec{v} + d\vec{w} = [cb_1, cb_2, cb_3]^T + [dc_1, dc_2, dc_3]^T = [cb_1 + dc_1, cb_2 + dc_2, cb_3 + dc_3]^T.$$ We check that this sum is in $V$:

$$(cb_1 + dc_1) + (cb_2 + dc_2) - 2(cb_3 + dc_3) = c(b_1 + b_2 - 2b_3) + d(c_1 + c_2 - 2b_3) = c \cdot 0 + d \cdot 0 = 0,$$
so $c\vec{v} + d\vec{w} \in V$ as well. So $V$ is a vector space.

(b) All vectors $[b_1, b_2, b_3]^T$ such that $b_1 \leq b_2 \leq b_3$.

Solution: Suppose that $\vec{v} = [b_1, b_2, b_3]^T$ satisfies $b_1 \leq b_2 \leq b_3$. We show that the set of such vectors isn’t closed under scalar multiplication. Specifically

$$-1 \cdot \vec{v} = -1 \cdot [b_1, b_2, b_3]^T = [-b_1, -b_2, -b_3]^T.$$
This vector is not in the given set. So the set is not a vector space.
Question 4 (24 points) Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 2 & 10 \end{bmatrix}$

(a) Find the $PA = LU$ decomposition of $A$ using maximal partial pivoting. You should get $U = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Solution:

$$U = \begin{bmatrix} 2 & 4 & 6 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & -1/2 & 1 & 0 \\ 1/2 & 0 & 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) What is the rank of the matrix $A$?

**Solution:** Rank of $A$ is 2.

(c) Write down a basis for $C(A)$, the column space of $A$.

**Solution:** $[1, 1, 2, 2]^T, [3, 2, 4, 2]^T$

Question continues on the next page...
(d) Find a basis for $N(A)$, the nullspace of $A$. You may find $RREF(A)$ along the way to help, but you are not required to do so.

Solution: Continuing from $U$ above:

\[
\begin{bmatrix}
2 & 4 & 6 \\
0 & -2 & 4 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\overset{r_2/2\rightarrow r_2}{\Rightarrow}
\begin{bmatrix}
1 & 2 & 3 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\overset{r_1\rightarrow r_1}{\Rightarrow}
\begin{bmatrix}
1 & 0 & 7 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The only free variable is $x_3$, so letting $x_3 = 1$, we get the only special solution to be

\[
\begin{bmatrix}
-7 \\
2 \\
1
\end{bmatrix}
\]

This is the basis for $N(A)$.

(e) Let $\vec{c} = [1, 3, 6, 5]^T$. By finding a particular solution $\vec{x}_p$ and using your answer to the previous part, find the complete solution of $A\vec{x} = \vec{c}$.

Solution: Note that there was a typo in the question. As written, $\vec{c}$ is not in the column space, so this equation has no solution.

If we let $\vec{c} = [4, 3, 6, 4]^T$ (as was the original intent), we can either augment this to $A$ and row reduce, or notice that it is the sum of the first two columns of $A$. So $\vec{x}_p = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

The complete solution is therefore

\[
\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + d \begin{bmatrix} 7 \\ 2 \\ 1 \end{bmatrix} \text{ for any } d \in \mathbb{R}.
\]