Page 242

4. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Any non-square matrix with orthonormal columns will do.

   (b) Any vector and the the zero vector.

   (c) $\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. There are many other answers.

10. (a) If $c_1\vec{q_1} + c_2\vec{q_2} + c_3\vec{q_3} = \vec{0}$, and the q_i's are orthonormal, then by taking dot product with q_1, we get $c_1\vec{q_1} \cdot \vec{q_1} + c_2\vec{q_2} \cdot \vec{q_1} + c_3\vec{q_3} \cdot \vec{q_1} = \vec{0} \cdot \vec{q_1} \Rightarrow c_1 = 0$. Similarly, by taking dot products with q_2 and q_3, we get that $c_2 = c_3 = 0$.

    (b) If $Q\vec{x} = \vec{0}$, then $Q^TQ\vec{x} = Q^T\vec{0} = \vec{0}$. Since Q is orthonormal, $Q^TQ = I$, so we get $\vec{x} = \vec{0}$.

11. (a) First normalize $\vec{a}$ to get $q_1 = \frac{1}{10}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix}$. Then $v_2 = \vec{b} - \vec{q_1}^T\vec{b}\vec{q_1} = \begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}$. Normalize to get

    $q_2 = \frac{1}{10}\begin{pmatrix} -7 \\ 3 \\ 4 \\ -5 \\ 1 \end{pmatrix}$.

    (b) We want to project $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ onto the plane. Let $Q = [q_1|q_2]$. Then $P = QQ^T =$

    $\frac{1}{50}\begin{pmatrix} 25 & -9 & -12 & 20 & 0 \\ -9 & 9 & 12 & 0 & 12 \\ -12 & 12 & 16 & 0 & 16 \\ 20 & 0 & 0 & 25 & 15 \\ 0 & 12 & 16 & 15 & 25 \end{pmatrix}$. So $P\vec{v} = \frac{1}{50}\begin{pmatrix} 25 \\ -9 \\ -12 \\ 20 \\ 0 \end{pmatrix}$.

15. (a) By GS on the matrix $[A|e_1]$ where e_1 is $[1, 0, 0]^T$, we get $q_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$, $q_2 = \frac{1}{\sqrt{66}}\begin{pmatrix} 7 \\ 1 \\ 4 \end{pmatrix}$,

    $q_3 = \frac{1}{\sqrt{11}}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$.

    (b) $q_3 \in N(A^T)$.

(c) If $Q = [q_1|q_2]$, the least squares solution is $Q^T \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -11\sqrt{6} \\ \frac{37}{\sqrt{66}} \end{pmatrix}$.

16. Project $\vec{b}$ onto $\vec{a}$: $\dfrac{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix}}{\begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix}} \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix} = \frac{2}{7} \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix}$. $\vec{q}_1 = \frac{1}{7} \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix}$, $\vec{e}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{7} \begin{pmatrix} 4 \\ 5 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} -1 \\ 4 \\ -4 \\ -4 \end{pmatrix}$.

This has length 1, so it's also $\vec{q}_2$.

20. (a) True, since if Q is orthogonal, $Q^T Q = I$, so $Q^{-1} = Q^T$, which satisfies the same.

(b) True, since for any $\vec{x} \in \mathbb{R}^2$, $|Q\vec{x}|^2 = (Q\vec{x})^T (Q\vec{x}) = \vec{x}^T Q^T Q \vec{x} = \vec{x}^T \vec{x} = |x|^2$.

23. $q_1 = a_1$, $q_2 = \frac{1}{3} a_2 - \frac{2}{3} a_1$, $q_3 = \frac{1}{5} v_3 - \frac{2}{5} v_2$. $Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $R = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

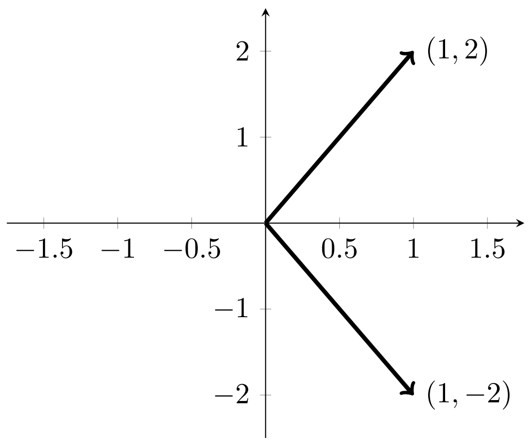
24. (a) A simple basis is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$.

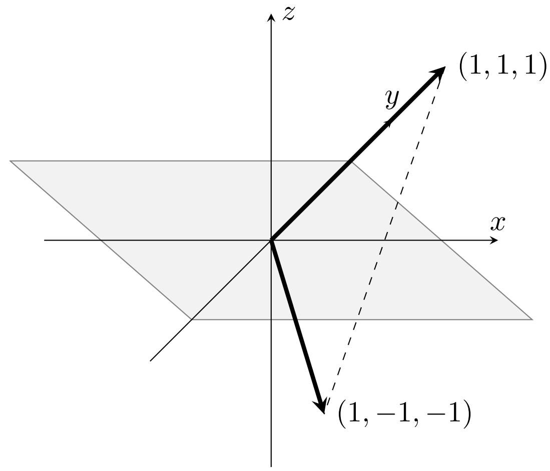
(b) Coefficients of the hyperplane form a basis: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$. (Or: append $[1, 0, 0, 0]$ to the matrix formed by the basis above and do GS to get the same result.)

(c) $\vec{b_1} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 3 \end{pmatrix}$, and $\vec{b_2} = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

32. $Q_1 = I - 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. $Q_2 = I - 2 \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$.

Note that Q_1 reflects in the x axis, and Q_2 reflects in the plane $y = -z$.





33. The only such matrix is the identity.

34. (a) $Q\vec{u} = I\vec{u} - 2\vec{u}\vec{u}^T\vec{u} = \vec{u} - 2\vec{u} = -\vec{u}$, where the second to last equality is true because $\vec{u}^T\vec{u} = 1$.

    (b) If $\vec{v} \perp \vec{u}$, then $Q\vec{v} = \vec{v} - 2\vec{u}\vec{u}^T\vec{v} = \vec{v}$, since $\vec{u}^T\vec{v} = 0$.