Page 190

4. When they exist, these are not usually unique.

   (a) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

   (b) Impossible. This must be a 3×3 matrix, which means $dim(C(A)) + dim(N(A)) = 3$.

   (c) $\begin{pmatrix} 1 & 0 \end{pmatrix}$ (I think this really must be the simplest possible example).

   (d) $\begin{pmatrix} 9 & 3 \\ 3 & \text{-}1 \end{pmatrix}$

   (e) Impossible. Since $C(A)^\perp = N(A^T)$ and $C(A^T)^\perp = N(A)$, then if $C(A) = C(A^T)$, then $N(A) = N(A^T)$.

18. The combination $r_3 - 2r_2 + r_1 = \vec{0}$. So multiples of $[1, -2, 1]$ are in $N(A^T)$. Similarly, since $c_3 - 2c_2 + c_1 = \vec{0}$, multiples of $[1, -2, 1]$ are also in $N(A)$.

25. (a) True. These are both the rank of the matrix.

    (b) False. Left nullspace of A^T is the nullspace of A, which lives in a different dimension than the left nullspace of A for any non-square matrix.

    (c) False. Row space and column space are the same for any invertible matrix.

    (d) True. This is trivially true if $A^T = A$, and multiplying the right side by -1 doesn't change the space spanned by the columns.

Page 203

9. If $A^T A\vec{x} = \vec{0}$, then $A\vec{x} = \vec{0}$. Reason: $A\vec{x}$ is in the nullspace of A^T and also in the <u>column space</u> of A and those spaces are <u>orthogonal complements</u>.

14. The RREF of $[A|B]$ is $\begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. So if $A = [c_1|c_2]$ and $B = [d_1|d_2]$, then $d_1 = 3c_1 + c_2$.

    So $d_1 = \begin{pmatrix} 5 \\ 6 \\ 5 \end{pmatrix}$ lies in both column spaces.

21. If S is spanned by $(1, 2, 2, 3)$ and $(1, 3, 3, 2)$, then $S^\perp$ is spanned by $(0, -1, 1, 0)$ and $(-5, 1, 0, 1)$. This is the same as solving $A\vec{x} = \vec{0}$ for $A = \begin{pmatrix} 1 & 2 & 2 & 3 \\ 1 & 3 & 3 & 2 \end{pmatrix}$ (i.e. finding the nullspace of this matrix.)

28. (a) Two planes in $\mathbb{R}^3$ cannot be orthogonal. In this case, the vector $[1, -1, 0]^T$ and its multiples form the intersection of the two planes.

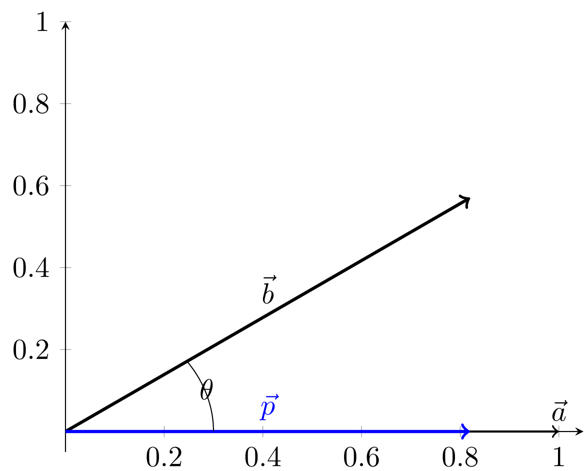
    (b) The orthogonal complement of a 2-D subspace of $\mathbb{R}^5$ is 3-D, but there are only two basis vectors given.

(c) The subspaces spanned by $[1,0]^T$ and $[1,1]^T$ only intersect at the origin, but aren't orthogonal.

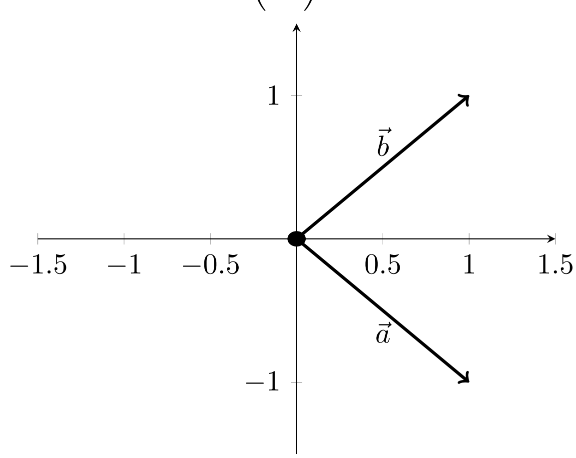
30. Since $C(B) \subseteq N(A)$, we get that dim $C(B) \leq$ dim $N(A)$. A has four columns, so dim $N(A)$+dim $C(A) = 4$, so dim $N(A) = 4-$dim $C(A)$. Putting these two together, we get dim $C(B) \leq 4-$ dim $C(A)$, so dim $C(A)+$ dim $C(B) \leq 4$.

Page 215

2. (a) $\hat{x} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \dfrac{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}}{\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix}} = \cos\theta$. So $\vec{p} = \hat{x}\vec{a} = \cos\theta\begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}\cos\theta \\ 0\end{pmatrix}$.



(b) $\hat{x} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \dfrac{\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 \\ 1\end{pmatrix}}{\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1 \\ -1\end{pmatrix}} = 0$. So $\vec{p} = 0\vec{a} = \begin{pmatrix}0 \\ 0\end{pmatrix}$.



5. $P_1 = \frac{1}{9}\begin{pmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{pmatrix}$, and $P_2 = \frac{1}{9}\begin{pmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{pmatrix}$. Note that $P_1 P_2 = 0$. This makes sense, because the vectors are orthogonal, so projecting onto one, then the other always gives the zero vector.

6. $\vec{p_1} = P_1\vec{b} = \frac{1}{9}\begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$; $\vec{p_2} = P_2\vec{b} = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -2 \end{pmatrix}$; $\vec{p_3} = P_3\vec{b} = \frac{1}{9}\begin{pmatrix} 4 \\ -2 \\ 4 \end{pmatrix}$. $p_1 + p_2 + p_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

7. $P_3 = \frac{1}{9}\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}$. The three do indeed add up to I.

11. (a) $\vec{p} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$; $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$.

    (b) $\vec{p} = \begin{pmatrix} 4 \\ 4 \\ 6 \end{pmatrix}$; $\vec{e} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

12. $P_1 = \frac{1}{9}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $P_2 = \frac{1}{9}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $P_1\vec{b} = \frac{1}{9}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$. P_2^2 is indeed the identity.

17. $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$. When P projects on the column space of A, $I - P$ projects onto its left nullspace.

19. Two possible vectors are $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$. Then $P = \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$.

20. The vector $\vec{e} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ (the vector of coefficients) is perpendicular to the plane. So $Q = \frac{\vec{e}\vec{e}T}{\vec{e}^T\vec{e}} = \frac{1}{6}\begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{pmatrix}$. So $P = I - Q = \frac{1}{6}\begin{pmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{pmatrix}$, as before.

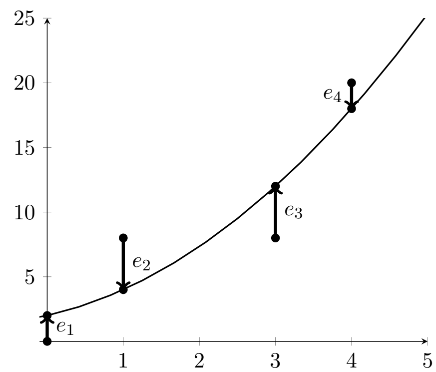
27. $(I-P)^2 = I^2 - IP - PI + P^2 = I - 2P + P = I - P$. When P projects onto the column space of A, $I - P$ projects onto the left nullspace of A.

31. You would take the dot product of $\vec{b} - \vec{p}$ with each of the $\vec{a}_i$'s. If all the dot products are zero, then $\vec{p}$ is the orthogonal projection of $\vec{b}$ onto the subspace spanned by the $\vec{a}_i$'s.

Page 229

1. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$. $\vec{b} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$. So $A^T A\hat{x} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$ and $A^T\vec{b} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$. $\hat{x} = (A^T A)^{-1}A^T\vec{b} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. So $\vec{p} = A\hat{x} = \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix}$, and $E = |\vec{e}|^2 = (\vec{b} - \vec{p})^2 = 44$.

9. $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{pmatrix}$. $\hat{x} = (A^T A)^{-1}A\vec{b} = \begin{pmatrix} 2 \\ \frac{4}{3} \\ \frac{2}{3} \end{pmatrix}$. Best fit parabola is $2 + \frac{4}{3}x + \frac{2}{3}x^2$, and $\vec{p} = \begin{pmatrix} 2 \\ 4 \\ 12 \\ 18 \end{pmatrix}$.



25. Consider the matrix $\left(\begin{array}{cc|c} 1 & t_1 & b_1 \\ 1 & t_2 & b_2 \\ 1 & t_3 & b_3 \end{array}\right)$. The row echelon form of this is

$$\left(\begin{array}{cc|c} 1 & t_1 & b_1 \\ 0 & 1 & \frac{b_2 - b_1}{t_2 - t_1} \\ 0 & 0 & (b_3 - b_1) - \frac{b_2 - b_1}{t_2 - t_1}(t_3 - t_1) \end{array}\right)$$

So we need $(b_3 - b_1) - \frac{t_3 - t_1}{t_2 - t_1}(b_2 - b_1) = 0$, or

$$\frac{b_3 - b_1}{t_3 - t_1} = \frac{b_2 - b_1}{t_2 - t_1}.$$

That is, the slope between (t_1, b_1) and (t_2, b_2) must be the same as the slope between (t_1, b_1) and (t_3, b_3)....which makes sense!

28. If the columns of A are not independent, find the row echelon form of A to figure out the pivot columns. Remove all free columns from A to create a matrix B with the same column space, but with full row rank. Then $P = B(B^T B)^{-1}B^T$ is the projection onto $C(B) = C(A)$.

Extra Problem

1. Create a matrix B that is A augmented by $\vec{c}$. Its RREF is $\begin{pmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$. Since the last row is not all zeros, $\vec{c}$ cannot be written as a linear combination of the columns of A, so it's not in $C(A)$.

2. $A^TA = \begin{pmatrix} 14 & 5 & 13 & 1 \\ 5 & 5 & -5 & 10 \\ 13 & -5 & 41 & -28 \\ 1 & 10 & -28 & 29 \end{pmatrix}$, and $A^T\vec{c} = [22, 12, 8, 14]^T$.

3. We saw in part 1 of the question that A has rank 2, so $N(A)$ is two-dimensional. Since we know that $N(A^TA) = N(A)$, $N(A^TA) \neq \{\vec{0}\}$, so $A^TA\hat{x} = A^T\vec{c}$ does not have a unique solution.

4. Form the augmented matrix $[A^TA|A^T\vec{c}]$. Its RREF is $\begin{pmatrix} 1 & 0 & 2 & -1 & 10/9 \\ 0 & 1 & -3 & 3 & 58/45 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. So $\hat{x}_p = [10/9, 58/45, 0, 0]^T = \frac{1}{45}[50, 58, 0, 0]^T$. The special solutions of are $\hat{s}_1 = [-2, 3, 1, 0]^T$ and $\hat{s}_2 = [1, -3, 0, 1]^T$, so the general solution of the normal equation is

$$\hat{x} = \frac{1}{45}\begin{pmatrix} 50 \\ 58 \\ 0 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} -2 \\ 3 \\ 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -3 \\ 0 \\ 1 \end{pmatrix}.$$

5. $\vec{p} = A\hat{x} = \frac{1}{45}[-8, 100, 266]^T$, and $\vec{e} = \vec{c} - \vec{p} = \frac{1}{45}[8, -10, 4]^T$.

6. Since A doesn't have full column rank, we can't just plug it into the formula for projection matrices. However, we can if we drop the two free columns (the 3$^{\text{rd}}$ and 4$^{\text{th}}$). If C is the matrix consisting of the two remaining pivot columns of A, then $C(A) = C(C)$, so $P = C(C^TC)^{-1}C^T = \frac{1}{45}\begin{pmatrix} 29 & 20 & -8 \\ 20 & 20 & 10 \\ -8 & 10 & 41 \end{pmatrix}$.

7. The projection matrix onto $C(A^T)$ is the matrix $E(E^TE)^{-1}E^T$, where E is the matrix formed by the pivot columns of A^T. By taking the RREF of A^T, we see that the first two columns are pivot columns. So $E = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 5 & 4 \\ -4 & -2 \end{pmatrix}$, giving $P = \begin{pmatrix} 19 & 9 & 11 & 8 \\ 9 & 6 & 0 & 9 \\ 11 & 0 & 22 & -11 \\ 8 & 9 & -11 & 19 \end{pmatrix}$. This gives

$$\hat{x}_{\text{row}} = P\hat{x}_p = \frac{1}{1485}\begin{pmatrix} 1472 \\ 798 \\ 550 \\ 922 \end{pmatrix}.$$

So we get

$$\hat{x} = \frac{1}{1485}\begin{pmatrix}1472\\798\\550\\922\end{pmatrix} + c_1\begin{pmatrix}-2\\3\\1\\0\end{pmatrix} + c_2\begin{pmatrix}1\\-3\\0\\1\end{pmatrix}.$$