Page 142

20. There are many. Anything whose RREF is $R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ will do the trick.

21. Impossible, since rank + nullity $= 3$, so the column space and row space cannot have the same dimension.

31. (a) Rank is 1. RREF has first row all 1's, and all other entries 0.

   (b) Rank is 2. RREF is $\begin{pmatrix} \boxed{1} & 0 & \text{-1} & \text{-2} \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

   (c) Rank is 1. RREF is $\begin{pmatrix} \boxed{1} & \text{-1} & 1 & \text{-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

33. (c) is the only correct definition of rank.

Page 158

4. The RREF of the matrix is $\begin{pmatrix} \boxed{1} & 3 & 0 & 0 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. So columns 1 and 3 form a basis for $C(A)$.

Note that $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}$, so the particular solution is $x_p = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$. By backsub, we

find that the basis for $N(A)$ is $\left(\begin{pmatrix} \text{-3} \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \text{-2} \\ 1 \end{pmatrix} \right)$. Therefore the complete solution is

$$\vec{x} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} \text{-3} \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ \text{-2} \\ 1 \end{pmatrix}.$$

5. The RREF of $\begin{pmatrix} 1 & 2 & \text{-2} & | & b_1 \\ 2 & 5 & \text{-4} & | & b_2 \\ 4 & 9 & \text{-8} & | & b_3 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & \text{-2} & | & 5b_1 - 2b_2 \\ 0 & 1 & 0 & | & b_2 - 2b_1 \\ 0 & 0 & 0 & | & b_3 - b_2 - 2b_1 \end{pmatrix}$. So the system has a solution if

and only if $b_3 - b_2 - 2b_1 = 0$. If that's the case, then $\vec{x}_p = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix}$, and the special

solution is $[2, 0, 1]^T$. So the general solution is $\vec{x} = \begin{pmatrix} 5b_1 - 2b_2 \\ b_2 - 2b_1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

10. Easiest solution is $A = \begin{pmatrix} 1 & 0 & \text{-}1 \\ 0 & 1 & \text{-}1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, but there are many.

16. The largest possible rank of a 3 by 5 matrix is $\underline{3}$. Then there is a pivot in every <u>row</u> of U and R. The solution to $A\vec{x} = \vec{b}$ *always exists*. The column space of A is $\mathbb{R}^3$. An example is

$$A = \begin{pmatrix} ① & 0 & 0 & 1 & 0 \\ 0 & ① & 0 & 0 & 1 \\ 0 & 0 & ① & 1 & 1 \end{pmatrix}.$$

22. If $A\vec{x} = \vec{b}$ has infinitely many solutions, then $N(A)$ has non-zero dimension. So if one solution to $A\vec{x} = \vec{B}$ exists (i.e. $\vec{B} \in C(A)$), you can add any vector in the nullspace to get another solution. If $\vec{B} \notin C(A)$, then there is no solution.

24. (a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, or any matrix with full column rank, but not full row rank.

(b) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, or any matrix with full row rank, but not full column rank.

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, or any matrix with neither full row rank, nor free column rank.

(d) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, or any invertible matrix

25. (a) $r < m$

(b) $r = m < n$

(c) $r = n < m$

(d) $r = m = n$

33. $A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$.

34. (a) $rank(A) = 3$. The complete solution to $A\vec{x} = \vec{0}$ is $\vec{x} = c \begin{pmatrix} 2 \\ 3 \\ 1 \\ 0 \end{pmatrix}$.

(b) $rref(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) The rank is 3, so $C(A)$ is a 3-D subspace of $\mathbb{R}^3$, i.e. $\mathbb{R}^3$ itself. So all vectors are in the column space, which is the same as saying the equation has a solution for any vector.

Page 175

7. The v_i's are linearly dep. if there is a set of non-zero c_1, c_2, c_3 that solve $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$. By plugging in the expressions for each v_i in terms of the w_i's, we get

$$(c_2 + c_3)w_1 + (c_1 - c_3)w_2 - (c_1 + c_2)w_3 = 0.$$

One set of solutions is $c_1 = c_3 = 1$, $c_2 = -1$. $[v_1 v_2 v_3] = [w_1 w_2 w_3]A$, where A is the singular matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \text{-}1 \\ \text{-}1 & \text{-}1 & 0 \end{pmatrix}.$$

8. We want to show that the v_i's are independent. So write

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \Leftrightarrow \vec{0} = c_1\vec{w}_2 + c_1\vec{w}_3 + c_2\vec{w}_1 + c_2\vec{w}_3 + c_3\vec{w}_1 + c_3\vec{w}_2$$
$$\Leftrightarrow \vec{0} = (c_2 + c_3)\vec{w}_1 + (c_1 + c_3)\vec{w}_2 + (c_1 + c_2)\vec{w}_3$$

Since the $\vec{w}_i$'s are independent, this means that

$$c_2 + c_3 = 0$$
$$c_1 + c_3 = 0$$
$$c_1 + c_2 = 0$$

or $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. By row-reducing, we see that the matrix has rank 3, so its nullspace is just the zero vector. So the only solution is $c_1 = c_2 = c_3 = 0$. Therefore the v_i's are independent.

12. The vector $\vec{b}$ is in the subspace spanned by the columns of A when $\underline{A\vec{x} = \vec{b}}$ has a solution. The vector $\vec{c}$ is in the row space of A when $\underline{A^T\vec{y} = \vec{c}}$ has a solution. It is *false* that if the zero vector is in the row space then the rows are dependent. In fact, the zero vector is always in the row space by definition of a space.

16. (a) $[1, 1, 1, 1]^T$ (or any multiple of this.)

(b) $[1, -1, 0, 0]^T$, $[0, 1, -1, 0]^T$, $[0, 0, 1, -1]^T$ (there are many others.)

(c) $[1, -1, -1, 0]^T$, $[0, 0, 1, -1]^T$ (there are many others.)

3

(d) $C(I)$ has basis $[1,0,0,0]^T$, $[0,1,0,0]^T$, $[0,0,1,0]^T$, $[0,0,0,1]^T$ (or any basis of $\mathbb{R}^4$). The basis of $N(I)$ is the empty set.

21. (a) The equation $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$ because <u>A has full column rank, since its columns are independent, so $N(A) = \{\vec{0}\}$.</u>

(b) If $\vec{b}$ is in $\mathbb{R}^5$, then $A\vec{x} = \vec{b}$ is solvable because the basis vectors <u>span</u> $\mathbb{R}^5$.

22. (a) True.

(b) False. Consider the standard basis for $\mathbb{R}^6$. Remove, say, $[0,0,0,0,0,1]^T$. Then if S contains any vector with a non-zero last coordinate, the remaining set cannot be a basis for S.

45. Let $v_1, \ldots, v_\alpha$ be a basis for V, and let $w_1, \ldots, w_\beta$ be a basis for W. Then we know $\alpha + \beta > n$. Since any basis of $\mathbb{R}^n$ has n vectors in it, the set $\{v_1, \ldots, v_\alpha, w_1, \ldots, w_\beta\}$ must be linearly dependent. Therefore there exists a set of scalars $c_1, \ldots, c_\alpha, d_1, \ldots, d_\beta$, not all zeros such that

$$c_1 v_1 + \ldots + c_\alpha v_\alpha + d_1 w_1 + \ldots d_\beta w_\beta = \vec{0}.$$

Therefore $c_1 v_1 + \ldots + c_\alpha v_\alpha = -(d_1 w_1 + \ldots d_\beta w_\beta)$ is a non-zero vector in $V \cap W$.

Extra Problem

1. $Rank(A) = 6$.

2. • Basis for $C(A)$: (column space):
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -11 \\ 0 & 0 & 0 & 1 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 1 & -4 \end{pmatrix}$$

• Basis for $N(A^T)$ (left nullspace):
$$\begin{pmatrix} 5 & 0 & -11 & -6 & -4 & -4 & -1 \end{pmatrix}$$

• Basis for $C(A^T)$ (row space):
$$\begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & -2 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 \end{pmatrix}$$

• Basis for $N(A)$ (nullspace):
$$\begin{pmatrix} 5 & -3 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & -1 & 2 & 1 & 0 & 0 \\ 3 & -3 & -1 & 1 & 0 & 2 & -2 & 0 & 1 & 0 \\ 2 & 3 & 2 & 2 & 0 & -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

3. $RREF([A|c]) = \begin{pmatrix} 1 & 0 & 0 & 0 & -5 & 0 & 0 & 2 & -3 & -2 & 9 \\ 0 & 1 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & -3 & 1 \\ 0 & 0 & 1 & 0 & -2 & 0 & 0 & 2 & 1 & -2 & 5 \\ 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 & -1 & -2 & 6 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 2 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

By setting all the free variables to zero, we get $x_p = [9, 1, 5, 6, 0, 0, -3, 0, 0, 0]$.

4. x_p is a linear combination of basis vectors in $C(A^T)$ (the row space) and $N(A)$ (the nullspace).

5. In order to express x_p as a linear combination of basis vectors in $C(A^T)$ (the row space) and $N(A)$ (the nullspace), create a 10×11 matrix whose first six columns are the basis vectors for $C(A^T)$, the next four columns the basis vectors for $N(A)$, and the last column x_p. Then the last column of this matrix's RREF is the coefficients of x_p as a linear combination of these basis vectors. (Bonus points: The coefficient vector turns out to be $\frac{1}{193} \begin{pmatrix} 26620 & 13768 & -35461 & -11647 & -13815 & 2005 & -14067 & 35507 & 30730 & 12138 \end{pmatrix}$.)