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15. (a) The intersection of two planes through $(0, 0, 0)$ is probably a line in \mathbb{R}^3 , but it could be a plane. It can't be Z!
- (b) The intersection of a plane through $t(0, 0, 0)$ with a line through $(0, 0, 0)$ is probably a point, but it could be a line.
- (c) Let $\vec{x}, \vec{y} \in S \cap T$. Then $\vec{x}, \vec{y} \in S$ and $\vec{x}, \vec{y} \in T$ (since S and T are vector spaces). So $\vec{x} + \vec{y} \in S$ and also $\vec{x} + \vec{y} \in T$. Therefore $\vec{x} + \vec{y} \in S \cap T$. Similarly, if $c \in \mathbb{R}$, then $c\vec{x} \in S$ and $c\vec{x} \in T$ (again, since they are vector spaces). So $c\vec{x} \in S \cap T$. Therefore $S \cap T$ is a vector space.
19. From left to right: $C(A)$ is all multiples of $[1, 0, 0]^T$, in other words the x -axis in \mathbb{R}^3 ; $C(B)$ is the x - y plane in \mathbb{R}^3 ; $C(C)$ is all multiples of the vector $[1, 2, 0]$ – a line.
26. If A is any 5 by 5 invertible matrix, then its column space is \mathbb{R}^5 , since the equation $A\vec{x} = \vec{c}$ has solution $\vec{x} = A^{-1}\vec{c}$ for any vector \vec{c} .
29. If the 9 by 12 system $A\vec{x} = \vec{b}$ is solvable for every \vec{b} , then $C(A) = \underline{\mathbb{R}^9}$.

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1. (a) $\begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ (also accept the row echelon form, $\begin{pmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$). Free variables are x_2, x_4 , and x_5 .
- (b) $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (also accept the row echelon form, $\begin{pmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 0 \end{pmatrix}$). Only x_3 is free.
2. (a) $[-2, 1, 0, 0, 0]^T$, $[0, 0 - 2, 1, 0]^T$, and $[0, 0, -3, 0, 1]^T$.
- (b) $[1, -1, 1]^T$.
4. $RREF(A) = \begin{pmatrix} 1 & -3 & -5 \\ 0 & 0 & 0 \end{pmatrix}$, so the special solutions are $[3, 1, 0]^T$ and $[5, 0, 1]^T$. $RREF(B) = \begin{pmatrix} 1 & -3 & -5 \\ 0 & 0 & 1 \end{pmatrix}$, so the special solution is $[0, 1, 0]^T$. For an n by m matrix the number of pivot variables plus the number of free variables is n .
5. (a) False. The square zero matrix, for example, has all variables free.
- (b) True. If A is invertible, then the unique solution to $A\vec{x} = \vec{c}$ is $\vec{x} = A^{-1}\vec{c}$. Equivalently, the RREF of an invertible matrix is the identity. Equivalently, all columns (or rows) of an invertible matrix are lin. ind.
- (c) True. The number of columns is the number of variables. So certainly, there are no more free variables than columns.
- (d) True. There can be no more than one pivot variable per row.

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8. Suppose column 4 of a 3 by 5 matrix is all zero. Then x_4 is certainly a free variable. The special solution for this variable is the vector $\vec{x} = [0, 0, 0, 1, 0]$
9. x_5 is a free variable. The corresponding special solution is $[-1, 0, 0, 0, 1]$.
10. Suppose an m by n matrix has r pivots. The number of special solutions is $\underline{n - r}$. The nullspace contains only $\vec{x} = \vec{0}$ when $r = \underline{n}$. The column space is all of \mathbb{R}^m when $r = \underline{m}$.
11. The nullspace of a 5 by 5 matrix contains only $\vec{x} = \vec{0}$ when the matrix has 5 pivots. The column space is \mathbb{R}^5 when there are 5 pivots. This is true because having full column rank in a square matrix makes it invertible.
14. Column 5 has no pivot. $N(A)$ is one-dimensional, with basis $[1, 0, 1, 0, 1]^T$, since $1 \cdot c_1 + 0 \cdot c_2 + 1 \cdot c_3 + 0 \cdot c_4 + 1 \cdot c_5 = \vec{0}$. Such a matrix might be
- $$\begin{pmatrix} \textcircled{1} & 0 & 0 & 0 & -1 \\ 0 & \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 & -1 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{pmatrix}.$$
17. $\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{pmatrix}$. Note that $c_1 + c_2 + 2c_3 = \vec{0}$, so the required vector is in the nullspace, and that the pivot columns (1 and 2) are a basis for the column space.
22. If $AB = 0$, then the column space of B is contained in the nullspace of A . If $\vec{c} \in C(B)$, then $\vec{c} = B\vec{x}$ for some vector \vec{x} . Since $AB\vec{x} = 0\vec{x} = \vec{0}$, and by associativity, we get $\vec{0} = A(B\vec{x}) = A\vec{c}$. So $\vec{c} \in N(A)$.

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2. Taking each vector to be the column of a matrix, then finding its RREF gives the first three columns as pivot columns. Hence a maximal linearly independent set is the first three vectors.