30. (a) \[
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
\]
(b) \[
\begin{pmatrix}
1 & -1 \\
0 & 1
\end{pmatrix}
\]
(c) \[EM = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}, FEM = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, EFEM = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, EEFEM = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, FEEFEM = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\]
(d) \[M^{-1} = FEEFE, \text{ so } M = E^{-1}F^{-1}E^{-1}E^{-1}F^{-1} = ABAAB.\]

10. \[A^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1/3 & 0 \\ 0 & 1/4 & 0 & 0 \\ 1/5 & 0 & 0 & 0 \end{pmatrix}, B^{-1} = \begin{pmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{pmatrix}\]

11. (a) \[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
(b) \[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

8. You have \(n\) choices for where the 1 in the first row goes. That leaves \((n - 1)\) choices for where the 1 in the second row goes, then \((n - 2)\) choices for where the 1 goes in the third row, and so on. Altogether, there are \(n!\) possibilities.

13. (a) \[
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}
\]
(b) \[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\]

22. First matrix: \[P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & -2/3 \\ 0 & 0 & 2/3 \end{pmatrix}, U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -3/2 & -1 \\ 0 & 0 & 1/3 \end{pmatrix}.\]

Second matrix: \[P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 0 & -2/3 \\ 0 & 0 & 2/3 \end{pmatrix}, U = \begin{pmatrix} 2 & 4 & 1 \\ 0 & -1 & 1/2 \\ 0 & 0 & -1/2 \end{pmatrix}.\]
39. (a) All diagonal entries are 1, so for each \( i \), \( q_i^T q_i = 1 \), but \( ||q_i||^2 = q_i^T q_i \).
(b) All non-diagonal entries are 0, so for each \( i \neq j \), \( q_i^T q_j = 0 \).
(c) \( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \). (There are three other possibilities)

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9. (a) \( \begin{pmatrix} x \\ y \end{pmatrix} \) with \( x \) and \( y \) integers. (Note: this is a lattice, an important structure in other areas of math.)
(b) Two lines intersecting at the origin.

10. (a) Yes;
(b) No (doesn’t contain the zero vector);
(c) No (not closed under addition);
(d) Yes;
(e) Yes;
(f) No (Not closed under scalar multiplication, e.g. by a negative scalar).

12. Many possibilities, like \( v_1 = [0, 0, -2]^T \), and \( v_2 = [4, 0, 0]^T \).

13. \( P_0 \) is given by \( x + y - 2z = 0 \). Many answers. E.g. \( v_1 = [1, 1, 1] \) and \( v_2 = [2, 0, 1] \). Then \( v_1 + v_2 = [3, 1, 2] \), and \( 3 + 1 - 2 \times 2 = 0 \), as required.

16. Suppose \( P \) is a plane through \((0, 0, 0)\) and \( L \) is a line through \((0, 0, 0)\). The smallest vector space containing both \( P \) and \( L \) is either a point or a line (that is, it’s either the zero vector space, or \( L \) itself).

20. (a) Only for multiples of \([1, 2, -1]^T\).
(b) Any vector with \( b_1 + b_3 = 0 \).

22. First system: all vectors in \( \mathbb{R}^3 \); Second system: all vectors for which \( b_3 = 0 \); Third system: all vectors for which \( b_2 = b_3 \).

23. If we add an extra column \( \vec{b} \) to a matrix \( A \), then the column space gets larger unless \( \vec{b} \in C(A) \).

For example, if we add the column \([1, 1, 0]^T\) to the matrix \( \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \), the column space doesn’t get larger, but if we add the column \([0, 0, 1]^T\), it does. \( A\vec{x} = \vec{c} \) is solvable exactly if the column space doesn’t get larger because in that case, \( \vec{c} \in C(A) \), which is exactly the condition necessary for the equation to have a solution.
24. For two square matrices, any non-singular matrix $A$ and singular matrix $B$ will do. Specifically, if $B$ is the zero matrix, we’re done.

27. (a) False. This set doesn’t contain the zero vector, so can’t be a subspace.
(b) True.
(c) True.
(d) False. For example, if $A = I$, then $C(A) = \mathbb{R}^n$, but $C(A - I) = \{0\}$.

28. Many examples. Easiest for the first part: \[
\begin{bmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}.
\]
For the second part, any rank 1 matrix will do. For example, a matrix all of whose columns are the same and are not all zeros.

32. $C(AB) \subseteq C(A)$, so by adding the columns of $AB$ to the matrix $A$ (to get $[AAB]$), we don’t expand the column space. If (e.g.) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $A^2 = 0$, so $C(A^2)$ is smaller than $C(A)$. An $n$ by $n$ matrix has $C(A) = \mathbb{R}^n$ exactly when $A$ is an invertible matrix.

**Extra Problems**

1. A quadratic is $y = ax^2 + bx + c$. The system of equations is therefore:

\[
\begin{align*}
-7 &= a + b + c \\
-16 &= 4a + 2b + c \\
-33 &= 9a + 3b + c
\end{align*}
\]

Or

\[
\begin{bmatrix}
1 & 1 & 1 \\
4 & 2 & 1 \\
9 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c
\end{bmatrix}
= \begin{bmatrix}
-7 \\
-16 \\
-33
\end{bmatrix}.
\]

The $PA = LU$ decomposition (with MPP) is:

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
4 & 2 & 1 \\
9 & 3 & 1
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{3} & 1 & 0 \\
\frac{1}{3} & 1 & 1 \\
0 & 0 & \frac{1}{3}
\end{bmatrix}
\begin{bmatrix}
9 & 3 & 1 \\
0 & \frac{3}{5} & \frac{1}{5} \\
0 & 0 & \frac{1}{5}
\end{bmatrix}
\]

So by fwd-sub and back-sub, we get $c = -6$, $b = 3$, $a = -4$. This can be checked by plugging in the $x$ values.

2. 

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 & -1 \\
-2 & -7 & 4 \\
5 & 1 & -3
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
\frac{2}{5} & 1 & 0 \\
\frac{3}{5} & \frac{3}{5} & 1
\end{bmatrix}
\begin{bmatrix}
5 & 1 & -3 \\
0 & \frac{33}{5} & \frac{14}{5} \\
0 & 0 & \frac{1}{5}
\end{bmatrix}
\]