22. First matrix: \( P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 3 & 4 \\ 0 & -\frac{3}{2} & -1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}. \)

Second matrix: \( P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} \frac{1}{2} & 1 & 0 \\ \frac{2}{3} & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad U = \begin{pmatrix} 2 & 4 & 1 \\ 0 & -1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{pmatrix}. \)

39. (a) All diagonal entries are 1, so for each \( i \), \( q_i^T q_i = 1 \), but \( ||q_i||^2 = q_i^T q_i \).

(b) All non-diagonal entries are 0, so for each \( i \neq j \), \( q_i^T q_j = 0 \).

(c) \( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \). (There are three other possibilities)

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9. (a) \( \begin{pmatrix} x \\ y \end{pmatrix} \) with \( x \) and \( y \) integers. (Note: this is a lattice, an important structure in other areas of math.)

(b) Two lines intersecting at the origin.

10. (a) Yes;

(b) No (doesn’t contain the zero vector);

(c) No (not closed under addition);

(d) Yes;

(e) Yes;

(f) No (Not closed under scalar multiplication, e.g. by a negative scalar).

12. Many possibilities, like \( v_1 = [0, 0, -2]^T \), and \( v_2 = [4, 0, 0]^T \).

13. \( P_0 \) is given by \( x + y - 2z = 0 \). Many answers. E.g. \( v_1 = [1, 1, 1] \) and \( v_2 = [2, 0, 1] \). Then \( v_1 + v_2 = [3, 1, 2] \), and \( 3 + 1 - 2 \times 2 = 0 \), as required.

16. Suppose \( P \) is a plane through (0, 0, 0) and \( L \) is a line through (0, 0, 0). The smallest vector space containing both \( P \) and \( L \) is either a point or a line (that is, it’s either the zero vector space, or \( L \) itself).

20. (a) Only for multiples of \( [1, 2, -1]^T \).

(b) Any vector with \( b_1 + b_3 = 0 \).
22. First system: all vectors in \( \mathbb{R}^3 \); Second system: all vectors for which \( b_3 = 0 \); Third system: all vectors for which \( b_2 = b_3 \).

23. If we add an extra column \( \vec{b} \) to a matrix \( A \), then the column space gets larger unless \( \vec{b} \in \text{C}(A) \).

For example, if we add the column \([1, 1, 0]^T\) to the matrix \(
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\), the column space doesn’t get larger, but if we add the column \([0, 0, 1]^T\), it does. \( A\vec{x} = \vec{c} \) is solvable exactly if the column space doesn’t get larger because in that case, \( \vec{c} \in \text{C}(A) \), which is exactly the condition necessary for the equation to have a solution.

24. For two square matrices, any non-singular matrix \( A \) and singular matrix \( B \) will do. Specifically, if \( B \) is the zero matrix, we’re done.

27. (a) False. This set doesn’t contain the zero vector, so can’t be a subspace.

(b) True.

(c) True.

(d) False. For example, if \( A = I \), then \( \text{C}(A) = \mathbb{R}^n \), but \( \text{C}(A - I) = \{0\} \).

28. Many examples. Easiest for the first part: \(
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\). For the second part, any rank 1 matrix will do. For example, a matrix all of whose columns are the same and are not all zeros.

32. \( \text{C}(AB) \subseteq \text{C}(A) \), so by adding the columns of \( AB \) to the matrix \( A \) (to get \([AAB] \)), we don’t expand the column space. If (e.g.) \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \), then \( A^2 = 0 \), so \( \text{C}(A^2) \) is smaller than \( \text{C}(A) \). An \( n \) by \( n \) matrix has \( \text{C}(A) = \mathbb{R}^n \) exactly when \( A \) is an invertible matrix.
Extra Problem

A quadratic is $y = ax^2 + bx + c$. The system of equations is therefore:

$$
-7 = a + b + c \\
-16 = 4a + 2b + c \\
-33 = 9a + 3b + c
$$

Or

$$
\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -7 \\ -16 \\ -33 \end{pmatrix}.
$$

Gaussian elimination takes the augmented matrix

$$
\begin{pmatrix} 1 & 1 & 1 & -7 \\ 4 & 2 & 1 & -16 \\ 9 & 3 & 1 & -33 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & -7 \\ 0 & -2 & -3 & 12 \\ 0 & 0 & 1 & -6 \end{pmatrix}
$$

so by back-sub, we get $c = -6$, $b = 3$, $a = -4$. This can be checked by plugging in the $x$ values.