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1. (a) $\left(\begin{array}{ccc}1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$
(b) $\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1\end{array}\right)$
(c) $\left(\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0\end{array}\right)$
2. $E_{21}=\left(\begin{array}{rrr}1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1\end{array}\right), E_{31}=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1\end{array}\right), E_{32}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1\end{array}\right), M=\left(\begin{array}{ccc}1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1\end{array}\right)$.
3. Let $A=\left(\begin{array}{lll}1 & 2 & 4 \\ 1 & 2 & 4 \\ 1 & 2 & 4\end{array}\right)$, and let $\vec{x}=[1,2,3]^{T}$. Then $A \vec{x}=[16,16,16]^{T}$. Elimination produces two rows of zeros, so there is only one pivot.
4. (a) To invert that step you should add 7 times row 1 to row 3 .
(b) $E^{-1}=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1\end{array}\right)$
(c) $E E^{-1}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ -7 & 0 & 1\end{array}\right)\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 7 & 0 & 1\end{array}\right)=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$
5. $E_{21}=\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right), E_{32}=\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{2}{3} & 1 & 0 \\ 0 & 0 & 0 & 1\end{array}\right), E_{43}=\left(\begin{array}{cccc}1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{3}{4} & 1\end{array}\right)$
6. The equation is $\left(\begin{array}{lll}1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9\end{array}\right)\left(\begin{array}{l}a \\ b \\ c\end{array}\right)=\left(\begin{array}{c}4 \\ 8 \\ 14\end{array}\right)$. Augmenting and row reducing, we get $\left(\begin{array}{lll|l}1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2\end{array}\right)$. By back sub, this gives use $c=1, b=1, a=2$.
7. $[A \mid \vec{u}]=\left(\begin{array}{cc|c}2 & 3 & 1 \\ 4 & 1 & 17\end{array}\right)$ becomes $[U \mid \vec{c}]=\left(\begin{array}{cc|c}2 & 3 & 1 \\ 0 & -5 & 15\end{array}\right)$. So $\vec{x}=[5,-3]^{T}$.
8. $[A \mid \vec{b}]=\left(\begin{array}{lll|l}1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 5 & 7 & 6\end{array}\right)$ reduces to $\left(\begin{array}{rrr|r}1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 3\end{array}\right)$. This gives $0=3$ as the last equation, so no solution. Changing the 6 to a 3 gives a solution (actually, an infinite number of them see later).

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32. $A X=I . x_{1}, x_{2}$, and $x_{3}$ are exactly what you get on the right by augmenting the identity to $A$ and doing Gauss-Jordan.

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7. (a) eqn $1+$ eqn 2 must have right hand side 0 , but since row $1+$ row $2=$ row 3 , we get two equations with the same coefficients, but different outputs. Hence no solution.
(b) There is a solution exactly if $b_{1}+b_{2}=b_{3}$.
(c) It becomes $0=b_{3}$.
8. $(A \mid I)=\left(\begin{array}{ll|ll}1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1\end{array}\right) \xrightarrow{r_{2}-2 r_{1} \rightarrow r_{2}}\left(\begin{array}{rr|rr}1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1\end{array}\right) \xrightarrow{r_{1}-3 r_{2} \rightarrow r_{1}}\left(\begin{array}{rr|rr}1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1\end{array}\right)=\left(I \mid A^{-1}\right)$
9. $\left(\begin{array}{ll}1 & 2 \\ 2 & 6\end{array}\right) \xrightarrow{r_{2}-2 r_{1} \rightarrow r_{2}}\left(\begin{array}{ll}1 & 2 \\ 0 & 2\end{array}\right) \xrightarrow{\frac{r_{2}}{2} \rightarrow r_{2}}\left(\begin{array}{ll}1 & 2 \\ 0 & 1\end{array}\right) \xrightarrow{r_{1}-2 r_{2} \rightarrow r_{1}}\left(\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right) . \quad$ So $E_{21}=$ $\left(\begin{array}{cc}1 & 0 \\ -2 & 1\end{array}\right), D^{-1}=\left(\begin{array}{cc}1 & 0 \\ 0 & \frac{1}{2}\end{array}\right)$, and $E_{12}=\left(\begin{array}{cc}1 & -2 \\ 0 & 2\end{array}\right)$. So $A^{-1}=\left(\begin{array}{cc}1 & -2 \\ 0 & 2\end{array}\right)\left(\begin{array}{cc}1 & 0 \\ 0 & \frac{1}{2}\end{array}\right)\left(\begin{array}{rr}1 & 0 \\ -2 & 1\end{array}\right)=\left(\begin{array}{rr}3 & -1 \\ -1 & \frac{1}{2}\end{array}\right)$.
10. $\left(\begin{array}{lll|lll}1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \xrightarrow{r_{2}-2 r_{1} \rightarrow r_{2}}\left(\begin{array}{lll|rrr}1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right) \xrightarrow{r_{2}-3 r_{3} \rightarrow r_{2}}\left(\begin{array}{rrr|rrr}1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1\end{array}\right)$

$$
\begin{aligned}
& \left(\begin{array}{lll|lll}
1 & 1 & 1 & 1 & 0 & 0 \\
1 & 2 & 2 & 0 & 1 & 0 \\
1 & 2 & 3 & 0 & 0 & 1
\end{array}\right) \xrightarrow[r_{3}-r_{1} \rightarrow r_{3}]{r_{2}-r_{1} \rightarrow r_{2}}\left(\begin{array}{lll|lll}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & -1 & 1 & 0 \\
0 & 1 & 2 & -1 & 0 & 1
\end{array}\right) \xrightarrow{r_{3}-r_{2} \rightarrow r_{3}}\left(\begin{array}{lll|rrr}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & -1 & 1 & 0 \\
0 & 0 & 1 & 0 & -1 & 1
\end{array}\right) \\
& \xrightarrow[r_{1}-r_{3} \rightarrow r_{2}]{r_{2}-r_{3} \rightarrow r_{2}}\left(\begin{array}{lll|rrr}
1 & 1 & 0 & 1 & 1 & -1 \\
0 & 1 & 0 & -1 & 2 & -1 \\
0 & 0 & 1 & 0 & -1 & 1
\end{array}\right) \xrightarrow{r_{1}-r_{2} \rightarrow r_{1}}\left(\begin{array}{rrr|rrr}
1 & 0 & 0 & 2 & -1 & 0 \\
0 & 1 & 0 & -1 & 2 & -1 \\
0 & 0 & 1 & 0 & -1 & 1
\end{array}\right)
\end{aligned}
$$

30. $A=\left(\begin{array}{ccc}a & b & b \\ a & a & b \\ a & a & a\end{array}\right) \xrightarrow[r_{3}-r_{1} \rightarrow r_{3}]{r_{2}-r_{1} \rightarrow r_{2}}\left(\begin{array}{ccc}a & b & b \\ 0 & a-b & 0 \\ 0 & a-b & a-b\end{array}\right) \xrightarrow{r_{3}-r_{2} \rightarrow r_{3}}\left(\begin{array}{ccc}a & b & b \\ 0 & a-b & 0 \\ 0 & 0 & a-b\end{array}\right)$. So if $a \neq 0$ and $a \neq b$, we have non-zero pivots $a, a-b$, and $a-b$. So the matrix is invertible.

If $c=0$, we have a row of all zeros, so $C$ is not invertible. If $c=2$, the first two rows are the same, so $C$ is not invertible. If $c=7$, the second and third columns are the same, so $C$ is also not invertible (at this stage, students might have to argue that if $c=7$, the transpose of $A$ isn't invertible, so neither is $A$ ).

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5. $A=\left(\begin{array}{lll}2 & 1 & 0 \\ 0 & 4 & 2 \\ 6 & 3 & 5\end{array}\right) \xrightarrow{r_{3}-3 r_{1} \rightarrow r_{3}}\left(\begin{array}{lll}2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5\end{array}\right)=U . E=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1\end{array}\right)$. So $L=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1\end{array}\right)$.
6. $A=\left(\begin{array}{lll}1 & 1 & 1 \\ 2 & 4 & 5 \\ 0 & 4 & 0\end{array}\right) \xrightarrow{r_{2}-2 r_{1} \rightarrow r_{2}}\left(\begin{array}{lll}1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 4 & 0\end{array}\right) \xrightarrow{r_{3}-2 r_{2} \rightarrow r_{3}}\left(\begin{array}{ccc}1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -6\end{array}\right)=U . E_{21}=\left(\begin{array}{rrr}1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$, and $E_{32}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1\end{array}\right)$. So $L=\left(\begin{array}{lll}1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1\end{array}\right)$.
7. $A=\left(\begin{array}{lll}1 & 0 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 5\end{array}\right) \xrightarrow[r 3-3 r_{1} \rightarrow r_{3}]{r 2-2 r_{1} \rightarrow r_{2}}\left(\begin{array}{lll}1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 4 & 2\end{array}\right) \xrightarrow{r_{3}-2 r_{2} \rightarrow r_{3}}\left(\begin{array}{lll}1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right)=U$. So $E_{21}=$ $\left(\begin{array}{ccc}1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1\end{array}\right), E_{31}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1\end{array}\right), E_{32}=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1\end{array}\right)$. So $L=\left(\begin{array}{ccc}1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1\end{array}\right)$
8. (a) $E_{32} E_{31} E_{21}=\left(\begin{array}{rrr}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1\end{array}\right)\left(\begin{array}{rrr}1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1\end{array}\right)\left(\begin{array}{rrr}1 & 0 & 0 \\ -a & 1 & 0 \\ 0 & 0 & 1\end{array}\right)=\left(\begin{array}{rrr}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1\end{array}\right)\left(\begin{array}{ccc}1 & 0 & 0 \\ -a & 1 & 0 \\ -b & 0 & 1\end{array}\right)=\left(\begin{array}{ccc}1 & 0 & 0 \\ -a & 1 & 0 \\ a c-b & -c & 1\end{array}\right)=$ E
(b) $E_{21}^{-1} E_{31}^{-1} E_{32}^{-1}=\left(\begin{array}{lll}1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1\end{array}\right)\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1\end{array}\right)=\left(\begin{array}{lll}1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ b & c & 1\end{array}\right)=\left(\begin{array}{lll}1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1\end{array}\right)=$ $L=A$.
9. $\left(\begin{array}{ll}1 & 0 \\ 4 & 1\end{array}\right)\binom{c_{1}}{c_{2}}=\binom{2}{11}$. We get $c_{1}=2$ and $4 c_{1}+c_{2}=11$, so $8+c_{2}=11$, or $c_{2}=3$. Then $\left(\begin{array}{ll}2 & 4 \\ 0 & 1\end{array}\right)\binom{x_{1}}{x_{2}}=\binom{2}{3}$. We get $x_{2}=3$ and $2 x_{1}+4 x_{2}=2$, or $2 x_{1}+12=2$, so $x_{1}=-5$. $\left(\begin{array}{ll}1 & 0 \\ 4 & 1\end{array}\right)\left(\begin{array}{ll}2 & 4 \\ 0 & 1\end{array}\right)=\left(\begin{array}{cc}2 & 4 \\ 8 & 17\end{array}\right)$.

So consider the augmented matrix $\left(\begin{array}{cc|c}2 & 4 & 2 \\ 8 & 17 & 11\end{array}\right) \xrightarrow{r_{2}-4 r_{1} \rightarrow r_{2}}\left(\begin{array}{ll|l}2 & 4 & 2 \\ 0 & 1 & 3\end{array}\right)$. This gives $x_{2}=3$ and $2 x_{1}+4 x_{2}=2$, or $2 x_{1}+12=2$, so $x_{1}=-5$, as before.
16. $\left(\begin{array}{lll}1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1\end{array}\right)\left(\begin{array}{l}c_{1} \\ c_{2} \\ c_{3}\end{array}\right)=\left(\begin{array}{l}4 \\ 5 \\ 6\end{array}\right)$. So $c_{1}=4$. We get $c_{1}+c_{2}=5$, or $4+c_{2}=5$, so $c_{2}=1$; and $c_{1}+c_{2}+c_{3}=6$, so $4+1+c_{3}=6$, or $c_{3}=1$.

$$
\begin{aligned}
& \left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=\left(\begin{array}{l}
4 \\
1 \\
1
\end{array}\right) . \text { So } x_{3}=1 \text {. We get } x_{2}+x_{3}=1 \text {, or } x_{2}+1=1 \text {, so } x_{2}=0 \text {; and } \\
& x_{1}+x_{2}+x_{3}=4 \text {, or } x_{1}+0+1=4 \text {, so } x_{1}=3 .
\end{aligned}
$$

$$
A=\left(\begin{array}{lll}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 1 & 1
\end{array}\right)\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{array}\right)=\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 2 & 2 \\
1 & 2 & 3
\end{array}\right)
$$

