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2. 


3. $2 \vec{v}=(\vec{v}+\vec{w})+(\vec{v}-\vec{w})=\left[\begin{array}{l}5 \\ 1\end{array}\right]+\left[\begin{array}{l}1 \\ 5\end{array}\right]=\left[\begin{array}{l}6 \\ 6\end{array}\right]$, so $\vec{v}=\left[\begin{array}{l}3 \\ 3\end{array}\right]$
$2 \vec{w}=(\vec{v}+\vec{w})-(\vec{v}-\vec{w})=\left[\begin{array}{l}5 \\ 1\end{array}\right]-\left[\begin{array}{l}1 \\ 5\end{array}\right]=\left[\begin{array}{c}4 \\ -4\end{array}\right]$, so $\vec{w}=\left[\begin{array}{c}2 \\ -2\end{array}\right]$

5. $\vec{u}+\vec{v}+\vec{w}=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$
$2 \vec{u}+2 \vec{v}+w=\left[\begin{array}{c}-2 \\ 3 \\ 1\end{array}\right]$
The three vectors lie in a plane because $-\vec{u}-\vec{v}=\vec{w}$.
6. Every combination of $\vec{v}=(1,-2,1)$ and $\vec{w}=(0,1,-1)$ has components that add to 0 .

$$
3 \vec{v}+9 \vec{w}=(3,3,-6)
$$

$(3,3,6)$ is impossible because its components don't add up to 0 .
7. Let $\vec{v}=\left[\begin{array}{l}2 \\ 1\end{array}\right]$ and $\vec{w}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$. Note that $0 \vec{v}+0 \vec{w}$ is the zero vector, so can't be seen as an arrow on the diagram. The other eight combinations are seen below:

8. Let the other diagonal be $\vec{x}$ (see diagram below).


Then $\vec{w}+\vec{x}=\vec{v}$. So $\vec{x}=\vec{v}-\vec{w}=\left[\begin{array}{l}5 \\ 0\end{array}\right]$.
9. - Shifting $(1,1)$ to the origin means subtracting it from all other points. So the points become $(0,0),(3,1)$, and $(0,2)$. Let $\vec{v}=\left[\begin{array}{l}3 \\ 1\end{array}\right]$ and $\vec{w}=\left[\begin{array}{l}0 \\ 2\end{array}\right]$. Then $\vec{v}+\vec{w}=\left[\begin{array}{l}3 \\ 3\end{array}\right]$. So (adding $(1,1)$ back) the fourth point is $(4,4)$

- Similarly with $(4,2)$. Let $\vec{v}=\left[\begin{array}{l}-3 \\ -1\end{array}\right]$ and $\vec{w}=\left[\begin{array}{c}-3 \\ 1\end{array}\right]$. Then $\vec{v}+\vec{w}=\left[\begin{array}{c}-6 \\ 0\end{array}\right]$. So (adding $(4,2)$ back), the fourth point is $(-2,2)$.
- Lastly with (1,3). Let $\vec{v}=\left[\begin{array}{c}0 \\ -2\end{array}\right]$ and $\vec{w}=\left[\begin{array}{c}3 \\ -1\end{array}\right]$. Then $\vec{v}+\vec{w}=\left[\begin{array}{c}3 \\ -3\end{array}\right]$. So (adding $(1,3)$ back), the fourth point is $(4,0)$.

(The question only asks for two of the sketches above.)

12. It is the $x-y$ plane.
13. (a) $V=\overrightarrow{0}$.
(b) $\overrightarrow{0}-\vec{v}=-\vec{v}$. If $\vec{v}$ is $2: 00$, then $-\vec{v}$ is $8: 00$.
(c) $\theta=30^{\circ}$, so $\vec{v}=\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.
14. $\vec{v}=[3,5,7]^{T}, \vec{w}=[1,0,-1]^{T} v_{1}+w_{1}=4 ; v_{2}+w 2=5 ; v_{3}+w_{3}=6 ; v_{1}-w_{1}=2 ; v_{2}-w_{2}=5$, $v_{3}-w_{3}=8$. This is a question with $\underline{6}$ unknown numbers...
15. $2 c-d=1,-c+2 d-e=0,-d+2 e=0$. By manual elimination, we see that $c=\frac{3}{4}, d=\frac{1}{2}$, $e=\frac{1}{4}$ is a solution.

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4. These are all -1 .
5. Unit vector in the direction of $\vec{u}_{1}$ is $[1 / \sqrt{10}, 3 / \sqrt{10}]$. Unit vector in the direction of $\vec{w}_{1}$ is $[2 / 3,1 / 3,2 / 3]$. Unit vector perpendicular to $\vec{u}_{1}$ is $[1 / \sqrt{10},-3 / \sqrt{10}]$ (or the negative of this). Unit vector perpendicular to $\vec{w}_{1}$ is $[1 / \sqrt{2}, 0,-1 / \sqrt{2}]$ (or the negative of this). Also $[1 / \sqrt{5},-2 / \sqrt{5}, 0]$, the negative of this, or any linear combination of these normalized to length 1.
6. The length of this vector is 3 . A unit vector in the same direction is $[1 / 3, \ldots, 1 / 3]$. A unit vector perpendicular to it is $[0,1 / \sqrt{8},-1 / \sqrt{8}, 1 / \sqrt{8},-1 / \sqrt{8}, 1 / \sqrt{8},-1 / \sqrt{8}, 1 / \sqrt{8},-1 / \sqrt{8}]$. There are many others.
7. $|\vec{v}|^{2}=4^{2}+2^{2}=20,|\vec{w}|^{2}=(-1)^{2}+2^{2}=5$. $|\vec{v}+\vec{w}|=|(-3,4)|=(-3)^{2}+4^{2}=25$. So $|\vec{v}|^{2}+|\vec{w}|^{2}=|\vec{v}+\vec{w}|^{2}$.
8. Rule 2 says $\vec{u} \cdot(\vec{v}+\vec{w})=\vec{u} \cdot \vec{w}+\vec{u} \cdot \vec{w}$ (this is the distributive property for dot products).

Let $\vec{u}=\vec{v}+\vec{w}$. Then subbing into the above:

$$
\begin{aligned}
(\vec{v}+\vec{w}) \cdot(\vec{v}+\vec{w}) & =(\vec{v}+\vec{w}) \cdot \vec{v}+(\vec{v}+\vec{w}) \cdot \vec{w} \\
& =\vec{v} \cdot \vec{v}+\vec{w} \cdot \vec{v}+\vec{v} \cdot \vec{w}+\vec{w} \cdot \vec{w} \\
& =\|\vec{v}\|^{2}+2 \vec{v} \cdot \vec{w}+\|\vec{w}\|^{2}
\end{aligned}
$$

The left hand of the first equation is $\|\vec{v}+\vec{w}\|^{2}$, so we're done.
22. a. $v_{1}^{2} w_{1}^{2}+2 v_{1} w_{1} v_{2} w_{2}+v_{2}^{2} w_{2}^{2} \leq v_{1}^{2} w_{1}^{2}+v_{1}^{2} w_{2}^{2}+v_{2}^{2} w_{1}^{2}+v_{2}^{2} w_{2}^{2}$
b. The first and last term on each side cancel out, so we get

$$
2 v_{1} w_{1} v_{2} w_{2} \leq v_{1}^{2} w_{2}^{2}+v_{2}^{2} w_{1}^{2}
$$

Subtracting the left hand side over gives us

$$
0 \leq v_{1}^{2} w_{2}^{2}+v_{2}^{2} w_{1}^{2}-2 v_{1} w_{1} v_{2} w_{2}=\left(v_{1} w_{2}-v_{2} w_{1}\right)^{2} .
$$

Note that working all this backwards gives us Schwartz.
29. $\|\vec{v}-\vec{w}\|^{2}=\|\vec{v}\|^{2}-2\|\vec{v}\|\|\vec{w}\| \cos \theta+\|\vec{w}\|^{2}$.

So max value of $\|\vec{v}-\vec{w}\|$ is when $\cos \theta=-1$ (vectors are anti-parallel), giving

$$
\|\vec{v}-\vec{w}\|^{2}=5^{2}+2 \cdot 5 \cdot 3+3^{2}=64, \text { so }\|\vec{v}-\vec{w}\|=8 \text {. }
$$

Min length is when $\cos \theta=1$ (vectors are parallel), giving

$$
\|\vec{v}-\vec{w}\|^{2}=5^{2}-2 \cdot 5 \cdot 3+3^{2}=4, \text { so }\|\vec{v}-\vec{w}\|=2 \text {. }
$$

Min value of $\vec{v} \cdot \vec{w}$ is -15 (when the two vectors are anti-parallel). Max value is 15 (when the two vectors are parallel).
31. If $x+y+z=0$, then $z=-x-y$, so

$$
\vec{v}=\left[\begin{array}{c}
x \\
y \\
-x-y
\end{array}\right] \text {, and } \vec{w}=\left[\begin{array}{c}
-x-y \\
x \\
y
\end{array}\right] .
$$

So

$$
\vec{v} \cdot \vec{w}=-x^{2}-x y+x y-x y-y^{2}=-\left(x^{2}+x y+y^{2}\right),
$$

and

$$
\|\vec{v}\|=\|\vec{w}\|=\sqrt{x^{2}+y^{2}+(-x-y)^{2}}=\sqrt{2\left(x^{2}+y^{2}+x y\right)} .
$$

Therefore

$$
\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}=\frac{-\left(x^{2}+x y+y^{2}\right)}{2\left(x^{2}+y^{2}+x y\right)}=-\frac{1}{2} .
$$

So the angle between any two such vectors is always $\cos ^{-1}\left(-\frac{1}{2}\right)=120^{\circ}$.

## Additional Problem

$$
\begin{gathered}
\vec{v}^{T} \vec{w}=21 . \\
\vec{v} \vec{w}^{T}=\left[\begin{array}{ccc}
3 & -5 & 7 \\
6 & -10 & 14 \\
12 & -20 & 28
\end{array}\right] .
\end{gathered}
$$

$\vec{w} \vec{v}^{T}=\left(\vec{v} \vec{w}^{T}\right)^{T}$, so the outer product is not commutative. Multiplying the other way gives the transpose.

