2. \( \det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -4 \\ -2 & \lambda - 3 \end{vmatrix} = (\lambda - 1)(\lambda - 3) - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) \). So the eigenvalues are \( \lambda = 5, -1 \). These have eigenvectors corresponding the nullspaces of \( \begin{pmatrix} 4 & -4 \\ -2 & 2 \end{pmatrix} \) and \( \begin{pmatrix} 2 & -4 \\ -2 & -4 \end{pmatrix} \) resp. These have RREF’s \( \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \) resp. So \( E_5 \) has basis \([1, 1]^T\), and \( E_{-1} \) has basis \([2, -1]^T\). Similarly, \( A + I \) has eigenvalues 6 and 0 corresponding to the same eigenvectors. \( A + I \) has the same eigenvectors as \( A \). Its eigenvalues are increased by 1.

3. \( A \) has eigenvalues 2 and -1 with eigenvectors \([1, 1]^T\) and \([2, -1]^T\) resp. \( A^{-1} \) has eigenvalues \( \frac{1}{2} \) and -1 with eigenvectors \([1, 1]^T\) and \([2, -1]^T\). \( A^{-1} \) has the same eigenvectors as \( A \). When \( A \) has eigenvalues \( \lambda_1 \) and \( \lambda_2 \), its inverse has eigenvalues \( \frac{1}{\lambda_1} \) and \( \frac{1}{\lambda_2} \).

4. \( A: \lambda = 2, -3 \) with eigenvectors \([1, 1]^T\) and \([3, -2]^T\). \( A^2: \lambda = 4, 9 \) with the same eigenvector. \( A^2 \) has the same eigenvectors as \( A \). When \( A \) has eigenvalues \( \lambda_1 \) and \( \lambda_2 \), \( A^2 \) has eigenvalues \( \lambda_1^2 \) and \( \lambda_2^2 \). \( \lambda_1^2 + \lambda_2^2 = 13 = tr(A^2) \).

5. The eigenvalues of both \( A \) and \( B \) are 1 and 3. The eigenvalues of \( A + B \) are 5 and 3. Eigenvalues of \( A + B \) are not equal to eigenvalues of \( A \) plus eigenvalues of \( B \).

6. The only of \( A \) is 1. Same for \( B \). The eigenvalues of \( AB \) are \( 2 - \sqrt{3} \) and \( 2 + \sqrt{3} \). Same for \( BA \).

(a) The eigenvalues of \( AB \) are not equal to the eigenvalues of \( A \) times the eigenvalues of \( B \).

(b) The eigenvalues of \( AB \) are equal to the eigenvalues of \( BA \).

12. \( E_0 \) has basis vector \([2, -1, 0]^T\), and \( E_1 \) has basis vectors \([1, 2, 0]^T\) and \([0, 0, 1]^T\). \([1, 2, 1]\) is an eigenvector of \( P \) with no zero components.

13. \[ P = \tilde{u} \tilde{u}^T = \frac{1}{36} \begin{pmatrix} 1 & 1 & 3 & 5 \\ 1 & 1 & 3 & 5 \\ 3 & 3 & 9 & 15 \\ 5 & 5 & 15 & 25 \end{pmatrix} \]

(a) \( P \tilde{u} = \tilde{u} \) comes from \( (\tilde{u} \tilde{u}^T) \tilde{u} = \tilde{u}(\tilde{u}^T \tilde{u}) \). Then \( \tilde{u} \) is an eigenvector with eigenvalue 1.

(b) If \( \tilde{v} \perp \tilde{u} \), then \( P \tilde{v} = (\tilde{u} \tilde{u}^T) \tilde{v} = \tilde{u}(\tilde{u}^T \tilde{v}) = \tilde{0} \). Then \( \lambda = 0 \).

(c) Note that \( C(P) = \tilde{u} \), so \( E_0 \) is exactly \( C(P)^\perp = N(P^T) \). So we want a basis for the left nullspace of \( P \). Computing this, we find eigenvectors \([5, 0, 0, -1]^T\), \([0, 5, 0, -1]^T\) and \([0, 0, 5, -3]^T\).
15. The first matrix has complex eigenvalues $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ and $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, as well as 1. The second has eigenvalue $-1$ as well as 1.

17. The quadratic formula gives the eigenvalues $\lambda = \left( a + d + \sqrt{(a-d)^2 + 4bc} \right)/2$ and $\lambda = \left( a + d - \sqrt{(a-d)^2 + 4bc} \right)/2$. Their sum is $a + d$ (Note: this is the trace of $A$!). If $A$ has $\lambda_1 = 3$ and $\lambda_2 = 4$, then $\det(A - \lambda I) = (3 - \lambda)(4 - \lambda)$.

19. (a) Yes. Rank is 2.
   (b) Yes. $|B^T| = |B| = 0 \cdot 1 \cdot 2 = 0$. So $|B^T B| = 0$.
   (c) No.
   (d) Yes. These are $\frac{1}{\lambda + 1}$ for each eigenvalue $\lambda$ of $B$, so 1, $\frac{1}{2}$, and $\frac{1}{5}$.

21. The eigenvalue of $A$ equal the eigenvalues of $A^T$. This is because $\det(A - \lambda I) = \det(A^T - \lambda I)$. That is true because $(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I$. So the determinants are the same.

Almost no matrices have the same eigenvectors as their transposes. For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has eigenvector $[1, 0]^T$, but $A^T$ has eigenvector $[0, 1]^T$.

27. Rank($A$) = 1. It’s eigenvalues are 0 (with an AM of 3), and 4 (AM = 1). Rank($C$) = 2. Its eigenvalues are 0 (AM = 2) and 2 (AM = 2).

32. (a) $\vec{u}$ is a basis for the nullspace, and the vectors $\vec{v}$ and $\vec{w}$ are a basis for the column space.
   (b) A particular solution for $A\vec{x} = \vec{v} + \vec{w}$ is $\vec{x} = \frac{1}{3}\vec{v} + \frac{1}{5}\vec{w}$.
   (c) $A\vec{x} = \vec{u}$ has no solution. If it did then $\vec{u}$ would be in the column space.

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2. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 0 & 5 \end{pmatrix}$

4. (a) False.
   (b) True.
   (c) True.
   (d) False.

11. (a) True.
   (b) False.
   (c) False.

15. $A^k = XA^kX^{-1}$ approaches the zero matrix as $k \to \infty$ if and only if every $\lambda$ has absolute values less than 1. $A_1$ has eigenvalues 1 and $-0.3$, so does not satisfy $\lim_{k \to \infty} A_1^k = 0$. On the other hand $A_2$ has eigenvalues 0.9 and 0.3, so $\lim_{k \to \infty} A_2^k = 0$. 
16. \( \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -0.3 \end{pmatrix} \), \( X = \begin{pmatrix} 9 & 1 \\ 4 & -1 \end{pmatrix} \). \( \lim_{k \to \infty} \Lambda^k = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \). \( \lim_{k \to \infty} X\Lambda^kX^{-1} = \frac{1}{13} \begin{pmatrix} 9 & 9 \\ 4 & 4 \end{pmatrix} \). In the columns of this matrix, you see the normalized eigenvector corresponding to \( \lambda = 1 \).

18. \( \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \). So \( A^k = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 3^k & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 + 3^k & 1 - 3^k \\ -3^k & 1 + 3^k \end{pmatrix} \).

31. \( A - \lambda_1I = X\Lambda X^{-1} - \lambda_1I = X(\Lambda - \lambda_1I)X^{-1} \). The matrix in the parentheses has a zero in the \((1, 1)\) position. Likewise, for each \( \lambda_i \), the corresponding matrix will have a zero in the \((i, i)\) position. We get
\[
(A - \lambda_1I)(A - \lambda_2I) \cdots (A - \lambda_nI) = X(\Lambda - \lambda_1I)(\Lambda - \lambda_2I) \cdots (\Lambda - \lambda_nI)X^{-1}.
\]
The product of the matrices in parentheses is zero.

**Extra Question**

1. \[
\chi_A(\lambda) = \begin{vmatrix} \lambda - 9 & 1 & 2 & -4 \\ 4 & \lambda - 2 & -1 & 2 \\ -8 & 0 & \lambda & -4 \\ 10 & -2 & -3 & \lambda + 4 \end{vmatrix} = \begin{vmatrix} \lambda - 9 & 1 & 2 & -4 \\ 4 & \lambda - 2 & -1 & 2 \\ -8 & 0 & \lambda & -4 \\ 2\lambda - 8 & 0 & 1 & \lambda - 4 \end{vmatrix} = (\lambda - 1)(\lambda - 2)(\lambda^2 - 4\lambda + 4) = (\lambda - 1)(\lambda - 2)^2 = (\lambda - 1)(\lambda - 2)^3
\]
So \( A \) has two eigenvalues: 1 and 2.

2. \[
N(I - A) = N\left(\begin{pmatrix} -8 & 1 & 2 & -4 \\ 4 & -1 & -1 & 2 \\ -8 & 0 & 1 & -4 \\ 10 & -2 & -3 & 5 \end{pmatrix}\right) = N\left(\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right)
\]
So \( \vec{v}_1 = [-\frac{1}{2}, 0, 0, 1]^T \), with eigenvalue 1.
\[ N(2I - A) = N \begin{pmatrix} -7 & 1 & 2 & -4 \\ 4 & 0 & -1 & 2 \\ -8 & 0 & 2 & -4 \\ 10 & -2 & -3 & 6 \end{pmatrix} = N \begin{pmatrix} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

So \( v_2 = [\frac{1}{4}, -\frac{1}{4}, 1, 0]^T \), and \( v_3 = [-\frac{1}{2}, \frac{1}{2}, 0, 1] \), with eigenvalue 2.

3. \( \lambda = 1 \) has \( AM = GM = 1 \). \( \lambda = 2 \) has \( AM = 3 \), but \( GM = 2 \).

4. \( A \) is not diagonalizable, as there are not enough linearly independent eigenvectors to create a square matrix of eigenvectors.