End of Term Assignment 1

• This is an assignment with exam-style questions.

• There are 60 points available in total.

• You should use a spreadsheet on questions where its use is indicated.

• When you have completed the assignment, you should scan it in and upload it to Gradescope.

• All work done should be your own, completed individually with no assistance from anyone else.

• Sign the honor code below to indicate that you agree to this.

Name: ________________________________

Honor Code:

I have adhered to the Duke Community Standard in completing this exam.

Signature: ________________________________
1. (15 pts) (Spreadsheet question) Suppose that $\frac{dy}{dt} = \frac{y}{y+t}$, and $y(0) = 1$. Use Euler’s method with $\Delta t = 0.5$ to estimate $y(5)$. Is your estimate an overestimate or underestimate of the true answer? Explain.

To show your answer this question, write down the values of $y(1)$, $y(2)$, $y(3)$, $y(4)$, and $y(5)$ from your spreadsheet, as well as the values of your slopes at all these points:

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</thead>
<tbody>
<tr>
<td>$y(t)$</td>
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<td>$y'(t)$</td>
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</table>
2. (16 pts) Suppose that \( \frac{dy}{dx} = \frac{y}{x} \).

(a) Find the equilibrium solutions of this differential equation, if any.

(b) By sketching the slopefield for the differential equation, determine whether any equilibria you found in part (a) are stable or unstable.

(c) Show that \( \frac{d^2y}{dx^2} = 0 \). (Hint: use the quotient rule!)

(d) Suppose that \( y(1) = 2 \). If you use Euler’s method to estimate \( y(5) \), will your answer be an overestimate, underestimate, or neither? Explain your answer using part (c).

(e) Using separation of variables or otherwise, solve the above differential equation with initial condition \( y(1) = 2 \).
3. (12 pts) For each of the following, state whether it is true or false. If it is false, provide a counterexample or a reason it is false.

(a) The differential equation \( \frac{dy}{dt} = e^{-y^2}e^t \) is separable.

(b) The differential equation \( \frac{dy}{dt} = e^{-y^2}e^t \) can be solved using separation of variables.

(c) Suppose that a differential equation has an unstable equilibrium at \( y = 1 \) and no other equilibria. Then if \( y(x) \) is a solution with \( y(0) = 0 \), then \( \lim_{x \to \infty} y(x) \) must be \( -\infty \).

(d) All solutions to the differential equation \( \frac{dy}{dt} = y^6(y - 1)^2(t - 1) \) have critical points at \( t = 1 \).
4. (11 pts) Consider the differential equation \( \frac{dy}{dt} = t^2 - y. \)

(a) Show that \( y(t) = ce^{-t} + t^2 - 2t + 2 \) is a solution to this equation for any value of \( c. \)

(b) Show that \( \frac{d^2y}{dt^2} = 2t - t^2 + y. \)

(c) Consider a solution of the equation that passes through the point \((2, 2)\). At this point, is the solution increasing or decreasing? Is it concave up or concave down?

5. (6 pts) Find a value of \( n \) for which the function \( y = \frac{x-1}{x+1} \) is a solution of the differential equation \( \frac{1}{2} \frac{dy}{dx} = \left(\frac{y}{x-1}\right)^n. \) (Hint: plug in to both sides of the equation!)