4.7. COHOMOLOGY

4.7.2 Lie Algebra Cohomology

We presented De Rahm cohomology first as it’s probably the most common theory of cohomology one is first exposed to, especially in mathematical physics. That is not to say however, that it is the simplest. In this section we will consider a theory of cohomology that makes no mention of differentiable functions and where the $d$ operator acts on the generators of the exterior algebra themselves.

We will write down the simplest nontrivial system. All we will assume is that the complex is based on exterior powers and that the differential is linear and satisfies the Leibniz rule. First we’ll base it on the complex of exterior powers of a vector space as in (4.91). It is most natural here to use the dual $V^\ast$. To be precise, if $\omega \in \Lambda^p V^\ast$, then

$$\omega(v_1, v_2, \ldots, v_p) \in \mathbb{R},$$

where all the $v_i$’s are vectors in $V$. Furthermore, exchanging any of the arguments in (4.109) swaps the sign. That is

$$\omega(v_1, v_2, \ldots, v_p) = -\omega(v_2, v_1, \ldots, v_p),$$

etc.

We can naturally define a wedge product on $\Lambda^\bullet V^\ast$ if we keep everything antisymmetrized. That is, if $\alpha, \beta, \gamma, \ldots$ are in $V^\ast$ then

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

$$(\alpha \wedge \beta \wedge \gamma)(v_1, v_2, v_3) = \alpha(v_1)\beta(v_2)\gamma(v_3) + \alpha(v_2)\beta(v_3)\gamma(v_1) + \alpha(v_3)\beta(v_1)\gamma(v_2) - \alpha(v_2)\beta(v_1)\gamma(v_3) - \alpha(v_3)\beta(v_2)\gamma(v_1) - \alpha(v_1)\beta(v_3)\gamma(v_2),$$

etc.

**Exercise 17:** If $\omega \in \Lambda^2 V^\ast$ and $\gamma \in V^\ast$ then use (4.111) to prove

$$(\omega \wedge \gamma)(v_1, v_2, v_2) = \omega(v_1, v_2)\gamma(v_3) + \omega(v_2, v_3)\gamma(v_1) + \omega(v_3, v_1)\gamma(v_2).$$

So we’ll define a complex

$$0 \longrightarrow \Lambda^0 V^\ast \overset{d}{\longrightarrow} \Lambda^1 V^\ast \overset{d}{\longrightarrow} \Lambda^2 V^\ast \overset{d}{\longrightarrow} \cdots$$

We start with $\Lambda^0 V^\ast = \mathbb{R}$. Let’s define $d$ on this to be zero.

**Exercise 18:** Show this is actually forced on us if we want $d$ to be linear and satisfy the Leibniz rule.

Next consider $\omega \in \Lambda^1 V^\ast = V^\ast$. Here, $d$ maps to $\Lambda^2 V^\ast$. We’ll allow this $d$ to be arbitrary but we can use it to define a bracket $[,]$ on $V$ by saying, for any $\omega$

$$(d\omega)(v_1, v_2) = -\omega([v_1, v_2]).$$
The negative sign is just a convention.) The antisymmetry tells us that swapping \( v_1 \) and \( v_2 \) must change the sign. That is, the bracket must obey
\[
[v, w] = -[w, v]. \tag{4.115}
\]

The next thing we need to consider is the Leibniz rule for this complex. Here something interesting happens. Suppose
\[
\omega \in \Lambda^p V^* \\
\eta \in \Lambda^q V^*. \tag{4.116}
\]

Since it’s a complex, we will assume \( d^2 \omega = d^2 \eta = 0 \). (We will find how to impose this shortly.) Now consider \( d \) acting on \( \omega \wedge \eta \). A first guess for the Leibniz rule would be
\[
d(\omega \wedge \eta) = (d\omega) \wedge \eta + \omega \wedge (d\eta). \tag{4.117}
\]
But it follows from this that
\[
d^2 (\omega \wedge \eta) = 2(d\omega) \wedge (d\eta) \quad \text{and there’s no real reason why this should always be zero. Indeed, if we imposed that this were always zero then we would force} \\
d = 0, \text{ which is rather uninteresting. Things work out much more nicely if we define the Leibniz rule to be} \\
d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge (d\eta). \tag{4.118}
\]
Now it is easy to show that \( d^2 \omega = d^2 \eta = 0 \) implies \( d^2 (\omega \wedge \eta) = 0 \) automatically. This sign play is the first example of all kinds of sign rules we will impose later when we study superalgebras in section 4.8.

To compute \( d \) further, suppose \( \omega \in \Lambda^2 V^* \). So we consider \( \omega = \alpha \wedge \beta \) for some \( \alpha, \beta \in V^* \) and then we extend by linearity to all of \( \Lambda^2 V^* \). Using the Leibniz rule we get
\[
(d\omega)(v_1, v_2, v_3) = (d(\alpha \wedge \beta))(v_1, v_2, v_3) \\
= ((d\alpha) \wedge \beta)(v_1, v_2, v_3) - (\alpha \wedge (d\beta))(v_1, v_2, v_3) \\
= -\alpha([v_1, v_2])\beta(v_3) - \alpha([v_2, v_3])\beta(v_1) - \alpha([v_3, v_1])\beta(v_2) + \ldots \\
= -\omega([v_1, v_2], v_3) - \omega([v_2, v_3], v_1) - \omega([v_2, v_3], v_1), \tag{4.119}
\]
where we’ve also used (4.111), (4.112) and (4.114).

We can apply to same kind of argument to any \( \Lambda^p V^* \) and, in general, we find by a hefty combinatorial computation that if \( \omega \in \Lambda^p V^* \) then
\[
(d\omega)(v_1, v_2, \ldots, v_{p+1}) = \sum_{i<j} (-1)^{i+j} \omega([v_i, v_j], v_1, v_2, \ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots, v_{p+1}), \tag{4.120}
\]
where the hat denotes omission.

Now, if \( \omega \) is any element of \( V^* \) then
\[
d^2 \omega(v_1, v_2, v_3) = \omega([[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2]). \tag{4.121}
\]
So $d^2 \omega = 0$ for all $\omega \in V^*$ implies

$$[[v_1, v_2], v_3] + [[v_2, v_3], v_1] + [[v_3, v_1], v_2] = 0.$$  \hspace{1cm} (4.122)

By the Leibniz rule, this condition is also sufficient to guarantee that $d^2 \omega = 0$ for any $\omega \in \Lambda^p V^*$. But this is exactly the Jacobi identity! What we have shown therefore is that we can define this complex on $V$ precisely when $V$ is a Lie algebra. Not surprisingly, the cohomology coming from this complex is called \textit{Lie algebra cohomology}.

So given any Lie algebra $\mathfrak{g}$ we may compute the Lie algebra cohomology groups $H^k(\mathfrak{g})$ as the cohomology groups of the complex we have just constructed. We won’t use the following result but there an interesting connection to the De Rham cohomology we computed earlier. Certain Lie algebras $\mathfrak{g}$ can be exponentiated to give Lie groups $G$ which are compact (that is, have finite volume). If this is the case, then the Lie algebra of $\mathfrak{g}$ coincides with the De Rham cohomology of $G$. For a proof of this see [27].

\textit{Exercise 19:} Prove that the Lie algebra cohomology of $\mathfrak{su}(2)$ satisfies $H^k = 0$ for $k \neq 0$ or 3; and that $H^0$ and $H^3$ are dimension one. This is the De Rham cohomology of the 3-sphere.

The expression (4.120) is a bit ugly and we can write the action of $d$ in a nicer way which will prove very useful later on. Suppose a basis of $V = \mathfrak{g}$ is given by $X_1, X_2, \ldots$. Then introduce a basis $c^1, c^2, \ldots$ of $\mathfrak{g}^*$ such that

$$c^j(X_i) = \delta^j_i.$$ \hspace{1cm} (4.123)

Now we’d like to make these $c^j$’s a basis for $\Lambda^* \mathfrak{g}$ and so we will let them \textit{anticommute}:

$$c^i c^j = -c^j c^i.$$ \hspace{1cm} (4.124)

In other words, we will implicitly write the wedge product as usual multiplication on the $c^j$’s. Then $c_i c_j$ is an element of $\Lambda^2 \mathfrak{g}^*$ and we have rules like

$$c^i c^j(X_k, X_l) = \delta^i_k \delta^j_l - \delta^j_k \delta^i_l,$$ \hspace{1cm} (4.125)

and so on as we had above. Now $[X_i, X_j]$ is an element of $\mathfrak{g}$ as so can be expanded in terms of the basis:

$$[X_i, X_j] = C^k_{ij} X_k,$$ \hspace{1cm} (4.126)

to define the \textit{structure constants} $C^k_{ij}$. Clearly

$$C^k_{ji} = -C^k_{ij}.$$ \hspace{1cm} (4.127)

Now

$$(dc^i)(X_j, X_k) = -c^i([X_j, X_k])$$
$$= -c^i C^l_{jk} X_l$$
$$= -C^i_{jk},$$ \hspace{1cm} (4.128)
which is the same thing, using the above formulae, as saying
\[ dc^i = -\frac{1}{2} C_{jk}^i c^j c^k. \] (4.129)

Now the nice thing about (4.129) is that it is all we need to know to compute \( d \) for any of the higher \( d \)'s in the Lie algebra complex (4.113). We simply write any element of \( \Lambda^p g^* \) in terms of the basis generated by the \( c^j \)'s and then use (4.129) and the signed Leibniz rule (4.118).

### 4.7.3 Lie algebra extensions

Let’s return to the issue of central extensions and projective representations. We will address the question locally near the identity. That is, we will study the issue at the Lie algebra level.

Note first that an abelian Lie group has a Lie algebra where the bracket is always zero. The real line \( \mathbb{R} \) with zero bracket is therefore the Lie algebra associated to a one-dimensional abelian Lie group. Using the exponential map \( \exp(ix) \) we can view it as the Lie algebra of \( U(1) \).

Let us suppose that \( G \) is a Lie group with Lie algebra \( g \) and let us suppose we want to centrally extend \( g \) by \( \mathbb{R} \) to get \( h \). This means that we want to consider a new Lie algebra which, as vector space, looks like \( g \oplus \mathbb{R} \) but which, as a Lie algebra might “mix up” the \( g \) and \( \mathbb{R} \) factors. Saying that \( \mathbb{R} \) is central amounts to the statement that \( [X,Y] = 0 \) for all \( X \in \mathbb{R} \) and \( Y \in h \).

To be more precise, we need a new Lie algebra \( h \) which contains a central \( \mathbb{R} \) and which is isomorphic to \( g \) is we divide out by \( \mathbb{R} \). This is the same as saying the following complex is an exact sequence:
\[ 0 \rightarrow \mathbb{R} \xrightarrow{i} h \xrightarrow{q} g \rightarrow 0. \] (4.130)

Note that the maps in (4.130) are Lie algebra maps. That is \( q([X,Y]) = [q(X),q(Y)] \), etc. Given that \( h = g \oplus \mathbb{R} \) as vector spaces, we can construct a vector space map
\[ \sigma : g \rightarrow h, \] (4.131)
such that \( q \circ \sigma \) is the identity map on \( g \). But the map \( \sigma \) need not respect the bracket structure since \( h \) need not be isomorphic as Lie algebra to \( g \oplus \mathbb{R} \). We can measure exactly how much the bracket structure is messed up by saying
\[ [\sigma(X),\sigma(Y)] = \sigma([X,Y]) + f(X,Y), \] (4.132)
for some function \( f \). If we apply \( q \) to the above equation and use the fact that \( q \) is a Lie algebra map, we get that \( q \circ f(X,Y) = 0 \). This, in turn implies that \( f(X,Y) \) must have image in \( \mathbb{R} \). Furthermore, it is clear from (4.132) that \( f \) is antisymmetric in \( X \) and \( Y \). This all means that \( f \) is a map
\[ f : \Lambda^2 g \rightarrow \mathbb{R}, \] (4.133)
or \( f \in \Lambda^2 g^* \). The next step we leave as an exercise.
Exercise 20: Given the fact that $\mathbb{R}$ is central, use the Jacobi identity to show that $df = 0$, where $d$ is the differential in Lie algebra cohomology from the previous section.

Given this, it should come as no surprise that central extensions of $\mathfrak{g}$ are given by the Lie algebra cohomology group $H^2(\mathfrak{g})$. To state this properly we need to be precise by what exactly we mean by two extensions being “the same”. Suppose we replace our map $\sigma$ by a new map $\sigma' : \mathfrak{g} \rightarrow \mathfrak{h}$ so that $\sigma = \sigma' + k$, where $k$ is some linear map $k : \mathfrak{g} \rightarrow \mathbb{R}$. Substituting this change into (4.132) we get

$$f(X,Y) = f'(X,Y) + k([X,Y]). \quad (4.134)$$

In Lie cohomology language this is $f' = f + dk$. This can be written in terms of a map from $\mathfrak{g}$ to itself as taking $(X, c) \in \mathfrak{g} \oplus \mathbb{R}$ to $(X, k(X))$. This map is actually an isomorphism of Lie algebras. So the change (4.134) is uninteresting. So, all told, the group $H^2(\mathfrak{g})$ exactly classifies these central extensions of $\mathfrak{g}$.

Exercise 21: The following is the simplest nontrivial case a Lie algebra central extension. Let $\mathfrak{g} = \mathbb{R}^2$ with zero bracket. That is, if $X$ and $Y$ form a basis for $\mathfrak{g}$ then $[X,Y] = 0$. Show that $H^2(\mathfrak{g}) = \mathbb{R}$ and describe the Lie algebra structure on the extensions. Have we already seen this algebra?

Being able to use the universal cover $\tilde{G}$ to avoid the issues of projective representations turns out to be given by whether the central extensions can be seen locally by $H^2(\mathfrak{g})$. That is, we have the following due to Bargmann [28]

**Theorem 7** If $H^2(\mathfrak{g}) = 0$ then every projective representation of $G$ can be lifted to a linear representation of $\tilde{G}$.

Fortunately the group $H^2(\mathfrak{g})$ if often 0 for cases of interest to physics. In particular, if $\mathfrak{g}$ is “semi-simple” then Whitehead’s second Lemma asserts that $H^2(\mathfrak{g}) = 0$. (See [29] for example.) This includes any Lie algebra associated to a non-abelian Lie group which is compact, such as SU($N$) or SO($N$).

However, we now turn to the symmetry group of string theory and we are not so lucky. The issue of central extensions becomes important.

### 4.7.4 The Virasoro Algebra

The symmetry group that we are interested in is that of a structureless string. We analyzed the string earlier as if it were made up of particles but the point of string theory is that the string should be fundamental.

We have parametrized where we are on the string by $\sigma$. Specifying a value of $\sigma$ locates a constituent particle in the string. To render such particles meaningless we need to declare that all parametrizations of the string are equivalent. So the desired symmetry group is the group of reparametrizations of the string.
Let us focus on the closed string, which is a circle $S^1$ and let $\sigma$ vary from 0 to $2\pi$ to parametrize this circle. Let $\sigma'$ be a function of $\sigma$ which

1. Is continuous and can be differentiated an arbitrary number of times and remain continuous.

2. Has a well-defined inverse which is also continuous and can be differentiated an arbitrary number of times.

3. Retains the orientation of the circle. That is, it parametrizes the circle in the same direction.

We can clearly compose such functions, and the identity function $\sigma'(\sigma) = \sigma$ is also such a function. As such, we get a group, which is called Diff($S^1$), standing for diffeomorphisms of the circle.

Now, Diff($S^1$) is a Lie group, but it is an infinite-dimensional Lie group. We want to study its algebra $\mathfrak{diff}(S^1)$, which will also be infinite dimensional. As explained earlier, to do this we need to study elements of Diff($S^1$) which are very close to the identity.

So we want to consider a transformation

$$\sigma \to \sigma + \epsilon f(\sigma), \quad (4.135)$$

where $\epsilon$ is very small in some sense. Now $f(\sigma)$ must obviously be a periodic function in $\sigma$, so we can Fourier decompose it. It’s easier to work over the complex numbers for now, so we’ll consider basic transformations of the form$^{10}$

$$\sigma \to \sigma + i\epsilon e^{in\sigma}, \quad (4.136)$$

for any integer $n$. (We can make everything real again at the end if we want by considering linear combinations, e.g., $\sigma \to \sigma + c(e^{in\sigma} + e^{-in\sigma})$ for real $c$.)

Earlier we used matrices to represent a Lie algebra but that’s a bit awkward here as the matrices will be infinite dimensional. We need some other way to “represent” the algebra. The best way turns out to be to consider functions on $S^1$. After all, functions on $S^1$ form a vector space so we can think of linear operators taking functions on $S^1$ to functions on $S^1$ as matrices, albeit infinite-dimensional.

Let us suppose we have some (complex valued) function $g: S^1 \to \mathbb{C}$. Under the change (4.136) we get

$$g(\sigma) \to g(\sigma + i\epsilon e^{in\sigma})$$

$$\to g(\sigma) + i\epsilon e^{in\sigma} \frac{d}{d\sigma} g(\sigma) + \ldots. \quad (4.137)$$

Let $L_n$ denote the element of the Lie algebra associated to (4.136), that is, the transformation is given, to first order in $\epsilon$ by $(1 + \epsilon L_n)$. This implies, in terms of our functions representation

$$L_n = ie^{in\sigma} \frac{d}{d\sigma}. \quad (4.138)$$

$^{10}$We use $i\epsilon$ rather than $\epsilon$ to make equation (4.140) nicer.
Note that this is indeed a linear operator as desired. By the Leibniz rule,

\[ L_m L_n g(\sigma) = -e^{im\sigma} \frac{d}{d\sigma} \left( e^{im\sigma} \frac{dg}{d\sigma} \right) = -ie^{i(m+n)\sigma} \frac{d}{d\sigma} g(\sigma) - e^{i(m+n)\sigma} \frac{d^2 g}{d\sigma^2} \]

(4.139)

It immediately follows that

\[ [L_m, L_n] = (m - n)L_{m+n}. \]  (4.140)

So this is the Lie algebra that must be a symmetry of fundamental string theory. In order to analyze the quantum mechanics of this we need to consider projective representations, which in turn means we need to consider central extensions of this algebra.

**Theorem 8** The Lie algebra cohomology group \( H^2(\mathfrak{diff}(S^1)) \) is one-dimensional.

We prove this by explicitly constructing a representative of a cohomology class. Let us use \( g \) to denote \( \mathfrak{diff}(S^1) \) and let \( f \in \Lambda^2 g^* \) be our representative. The statement that \( df(L_0, L_m, L_n) = 0 \) can be written

\[ f([L_0, L_m], L_n) + f(L_m, [L_0, L_n]) = f(L_0, [L_m, L_n]). \]  (4.141)

Now, for a cohomology class we are free to replace \( f \) by \( f + dk \), where \( k \in \Lambda^1 g^* = g^* \). Let us try to do this to make the right hand side of the above equation zero.

So using (4.114), and the known bracket relation,

\[ (f + dk)(L_0, [L_m, L_n]) = f(L_0, [L_m, L_n]) - k([L_0, [L_m, L_n]]) \]

(4.142)

To make this zero we just define \( k \) by (replacing \( m + n \) by \( n \))

\[ k(L_n) = \frac{1}{n} f(L_0, L_n), \]  (4.143)

for \( n \neq 0 \). Note that \( k(L_0) \) can be anything for now. We’ll come back to this later. So, replacing \( f \) by \( f + dk \) we can assume

\[ f([L_0, L_m], L_n) + f(L_m, [L_0, L_n]) = 0. \]  (4.144)

Applying the commutator relation to this gives

\[ -(m + n)f(L_m, L_n) = 0. \]  (4.145)
Thus \( f(L_m, L_n) \) vanishes unless \( m + n = 0 \). So, we can get all the information about \( f \) by defining the numbers

\[
f_m = f(L_m, L_{-m}),
\]

where \( f_m = -f_{-m} \). Next consider \( df(L_1, L_m, L_{-1-m}) \). The vanishing of this gives

\[
-(m - 1)f_{m+1} + (m + 2)f_m - (2m + 1)f_1 = 0.
\]

This recursion relation specifies all the \( f_m \) given \( f_1 \) and \( f_2 \). So we have at most a 2 dimensional set of possible \( f \)'s. Suppose we set \( f_1 = 0 \). Then the relation

\[
f_{m+1} = \frac{m + 2}{m - 1} f_m,
\]

has an obvious solution

\[
f_m = (m - 1)m(m + 1) = m^3 - m,
\]

Any multiple of this is also a solution and so we have a one-dimensional set of solutions

\[
f_m = \frac{c}{12}(m^3 - m),
\]

by varying a constant \( c \). The factor of 12 is a standard convention and will indeed prove to be convenient. We might get more solutions by varying \( f_1 \) but we claim this varies \( f \) by something exact. To see this recall that we used up our shift \( f \to f + dk \) to get (4.144) except that we never specified \( k(L_0) \). We need to use up this last degree of freedom. If \( k(L_0) = B \), then \( f \to f + dk \) replaces \( f_m \) by

\[
(f + dk)(L_m, L_{-m}) = f_m - k([L_m, L_{-m}])
\]

\[
= f_m - 2Bm.
\]

That is, we shift \( f_m \) by a term linear in \( m \). This also shifts \( f_1 \) by a constant and thus, as claimed, shifts of \( f_1 \) are boring for cohomology classes.

We have therefore proved out result. The Lie algebra cohomology group \( H^2(\mathfrak{g}) \) is one-dimensional and is given by (4.150). Note that we could use the fact that \( f_m \) is defined only up to linear shifts in \( m \) to rewrite \( f_m \) as \( cm^3/12 \). This is sometimes done, but it is perhaps more common to leave it in the form (4.150).

Since \( H^2(\mathfrak{g}) \) is one-dimensional, all central extensions of \( \text{diff}(S^1) \) are given by (4.150) are are parametrized by a single real number \( c \). Let \( \mathfrak{Vir} \) denote this new algebra. That is, we have an exact sequence

\[
0 \rightarrow \mathbb{R} \xrightarrow{i} \mathfrak{Vir} \xrightarrow{q} \text{diff}(S^1) \rightarrow 0.
\]

To write down the commutation relations we use (4.132). Let \( \sigma(L_n) \) also be denoted \( L_n \), and let \( i(1) \) be denoted \( e \) in the above sequence. Thus, a basis for \( \mathfrak{Vir} \) is given by the \( L_n \)'s and \( e \). The commutation relations for \( \mathfrak{Vir} \) are then

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n} e,
\]

\[
[L_m, e] = 0,
\]

\[
[L_m, L_{-n}] = (m + n)L_{m+n}.
\]
where $\delta_{m+n}$ is an abbreviation for $\delta_{m+n,0}$, which will be commonly used below. This algebra is known as the Virasoro algebra after work by Virasoro [30], although it was analyzed by mathematicians in earlier work [31,32]. The number $c \in \mathbb{R}$ is known as the central charge. When we see some representations of this algebra, the generator $e$ will always be mapped to 1, so you often see the Virasoro algebra (4.153) written without the $e$.

Note that ambiguity in the definition of $k(L_0)$ means that we are free to shift $L_0 \rightarrow L_0 + Be$, for any $B$. One can see directly from (4.153) that this recovers once again that this shifts the last term by something linear in $m$. 
