**Positive Matrices and Perron's Theorem**

**Def.** (For this section - careful!)
If $P$ is an $n \times n$ matrix over $\mathbb{R}$,
$P > 0$ mean all entries $> 0$

"Positive"

A vector $v > 0$ also mean all entries $> 0$.
$\sum v_i > 0$ also and $A > B$ mean $A - B > 0$
e etc.

**Theorem (Perron):**
Every $n \times n$ matrix $P$ has a "dominant" eigenvalue
$\lambda(P) \in \mathbb{R}$ which satisfies:

I) $\lambda(P) > 0$ and its eigenvector $h$ is also $> 0$.
\[Ph = \lambda(P)h.\]

II) $\lambda(P)$ is multiplicity one and has no
    associated generalized eigenvectors
    i.e. $\lambda(P)$ is simple?

III) For any other eigenvalue, $\lambda$,
\[|\lambda| < \lambda(P).\]

IV) There is no other eigenvector $> 0$.

Proof: Given $P$, let $p(P) \in \mathbb{R}_+$ be the $x \geq 0$ number $\lambda$
such that $\exists x$, with $x \neq 0, x \geq 0$ and
\[Px \geq \lambda x.\]
Lemma. If \( P > 0 \) then

1) \( P(x) \) contains a positive number
2) \( P(x) \) is bounded
3) \( P(x) \) is closed (includes all limits)

**Proof.** Given any \( x > 0 \), clearly \( Px > 0 \) so there must exist a \( \lambda > 0 \) such that \( Px > \lambda x \).

So 1) to write

2) Let \( \zeta \) be the row vector \((1,1,\ldots,1)\)

and consider resealed \( x \) such that

\[
\frac{\zeta x}{\zeta} = \sum x_i = 1
\]

Then \( Px \geq \lambda x \) gives

\[
\frac{\zeta Px}{\zeta} \geq \lambda
\]

Let \( \zeta P \) have largest component \( b \) so \( b \zeta \geq \zeta P \)

thus \( \lambda \leq \frac{\zeta Px}{\zeta} \leq b \zeta x = b \)

thus bounding \( \lambda \).

3) Read - uses analytical,

Thus, by lemma, \( P(x) \) has a max value, \( x_{\text{max}} > 0 \).

This is the later eigenval - used to prove properties
and that it's an eigenval!

**Eigenv.** We know \( P(x) \geq x_{\text{max}} \) for \( t > 0, t \to 0^+ \)

so we need to show \( = \) holds.

Suppose it didn't:

\[
\sum p_{ij} x_j \geq x_{\text{max}} x_i \quad \text{all } i
\]

but \( \sum p_{ij} x_j > x_{\text{max}} x_i \quad \text{some } k \).
3. \text{Let } x = h + (0, 0, \ldots, 0, \ldots) \text{ for } k^{th} \text{ spot}

So \( P x > P h \text{ for all components} \)

\text{Suppose } \sum p_{kj} x_j = \lambda_{\max} x_k + B
\text{ for some } B > 0.

then, \( \sum_{k} p_{kj} x_j = \lambda_{\max} x_k + B + e \left( p_{kk} - \lambda_{\max} \right) \text{ for some } e \in \mathbb{R} \)

So if \( e < \frac{B}{p_{kk} - \lambda_{\max}} \)

then \( P x > \lambda_{\max} x \).
\text{ ... contradicts } \lambda_{\max} \text{ being max!}

So \( \lambda_{\max} \) is eigenvalue

- \( h > 0 \): Since \( \lambda_{\max} > 0 \) and \( h > 0 \)
  we have \( P h > 0 \) \implies h > 0

- Simple: Suppose \( y \) is another eigenvector \( \lambda_{\min} \)
  - not multiple of \( h \).
  Then \( h + cy \) is eigenvector of \( \lambda_{\max} \)
  but then varying \( c \) we can make \( h + cy \leq 0 \).
4. Next suppose we have a generalized eigenvector

\[(P - \lambda \text{max})^2 y = 0\]

i.e. \[(P - \lambda \text{max}) y = c \cdot h\]

so \[Py = \lambda_{\text{max}} y + ch\].

WLOG \(c > 0\) (by \(y \geq y - y\))

\[y > 0\] (by \(y = y + bh\))

so then \[Py > \lambda_{\text{max}} y\]

which violates \(\lambda\) being max!

iii) Let \(x\) be another eigenvector \(\neq \lambda_{\text{max}}\) in \(e\).

\[\sum_j \begin{pmatrix} \rho_j \\ \psi_j \end{pmatrix} y_j = x y_i\]

so \[\sum_j |\rho_j| |y_j| > |\sum_j \begin{pmatrix} \rho_j \\ \psi_j \end{pmatrix} y_j| \quad \text{by } \Delta\text{-ineq.}\]

\[= 1x1jy_j| = |xjy_j| \quad \text{(x)}\]

so \(1x1\) belongs to \(p(P)\) with "\(h\)" vecs.

Let \((1y_1, 1:1)\)

If \(1x1 = \lambda_{\text{max}}\) then the above vec must be \(h\).

\[|y_i| = c \cdot h_i\] for some \(c\).

and \((x)\) is an equality.

equality holds in \(\Delta\text{-ineq}\) for vecs aligned

i.e. \[y_i = ce^{i\theta} h_i\] for all \(i\)

\[\Rightarrow y = ce^{i\theta} h \Rightarrow 1x1 = \lambda_{\text{max}}\]
5. IV) If \( \lambda \) is an eigenvalue of \( P \) and \( y \) is an eigenvector of \( P' \) with the same eigenvalue \( \lambda = \lambda' \), then \( y(\lambda) = 0 \).

Let \( P > 0 \) be diagonalizable at \( P \).

Since \( P > 0 \) it has diagonalizable eigenvectors \( \xi > 0 \).

So \( \xi(y) = 0 \) for any eigenvector \( \xi \) of \( P \) not \( \xi \).

\( \Rightarrow \) \( y \) must have non-negative entries.

A "stochastic matrix" is a square matrix with entries \( \geq 0 \) and row sum to 1.

E.g. \( \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \).

If \( S \) has entries \( s_{ij} \) one can think of \( s_{ij} \) as transition probabilities defining a "Markov Chain".

A "Markov Process" is a process that when the future depends only on current state and not the past.

E.g. A system has \( \Xi \) - a collection of "populations" each person in some state at time \( t \).

\( \text{Populations at time } t+1 = S \times \text{Pop at time } t \)

If \( \xi = (1,1,\ldots) \) row vector then \( \xi S = \xi \) by col sum to 1.

So total population unchanged in this model.
6. Theorem. If $S$ is a positive stochastic matrix.

I) Dominant eigenvalue $= 1$.

II) If $x \neq 0$ then

\[ \lim_{N \to \infty} S^N x = c1 \]

so $x$ is constant $c \neq 0$.

Proof. I) If $S > 0$ then $x$ is $S^T$.

$S^T$ has eigenvalue 1 go $x^T (\cdot | 1) = $ element

so $S$ has eigenvalue 1 too as $S$ is

II) Let $x = \sum \frac{e_n}{h}$

Let $S$ has Jordan Normal Form $J$

One block is 1

other blocks are $B = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ with $|\lambda| < 1$.

So $\lim_{N \to \infty} B^N = 0$.

So $J^\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = $ projection on to $h$.

Note $\left( S^N x, \xi \right) = \left( x, (S^T)^N \xi \right) = \left( x, \xi \right)$

$= c(h, \xi) \implies c \neq 0$ if $x \neq 0$.

So any Markov chain leads to a stab in a true
matrix yields a unique stable population.