1. (25 pts) In each case state whether $V$ is a subspace or not. If it is not, say why this is.

a) $V \subset \mathbb{R}^2$, where $V = \{(x_1, x_2) : x_1 + x_2 = 1\}$.

No: It does not contain $(0,0)$.

b) $V \subset \mathbb{R}^2$, where $V = \{(x_1, x_2) : x_1 \geq 0\}$.

No: Not closed under scalar multiplication by $-1$.

c) $V \subset \mathbb{R}^n$, where $V = \{x \in \mathbb{R}^n : Ax = 3x\}$ and $A$ is some $n \times n$ matrix.

Yes.

d) $V \subset \mathcal{P}_2$ (i.e., polynomial of degree at most 2) where $V$ is given by all such polynomials, $f(t)$, such that $\int_0^1 f(t) \, dt = 0$.

Yes.
e) $V$ is a subset of all functions from $[0, 1]$ to $\mathbb{R}$, given by those that satisfy $f(0) = 1$.

No: In this case the zero vector is the function that maps $[0, 1]$ to 0. This is not in the subset.
2. (20 pts) For each of the following cases find $C(A)$, $R(A)$, $N(A)$ and $N(A^T)$ (i.e., give a basis or say the subspace is trivial).

a) $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

<table>
<thead>
<tr>
<th>Space</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(A)$</td>
<td>${(1,0)}$</td>
</tr>
<tr>
<td>$R(A)$</td>
<td>${(1)}$</td>
</tr>
<tr>
<td>$N(A)$</td>
<td>${}$ (i.e., subspace is trivial)</td>
</tr>
<tr>
<td>$N(A^T)$</td>
<td>${(0,1)}$</td>
</tr>
</tbody>
</table>

b) $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

<table>
<thead>
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</tr>
</thead>
<tbody>
<tr>
<td>$C(A)$</td>
<td>${(1)}$</td>
</tr>
<tr>
<td>$R(A)$</td>
<td>${(1,1)}$</td>
</tr>
<tr>
<td>$N(A)$</td>
<td>${(1, -1)}$</td>
</tr>
<tr>
<td>$N(A^T)$</td>
<td>${}$</td>
</tr>
</tbody>
</table>

c) $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$.

<table>
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<tr>
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</tr>
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<tbody>
<tr>
<td>$C(A)$</td>
<td>${(1,2)}$</td>
</tr>
<tr>
<td>$R(A)$</td>
<td>${(1,1)}$</td>
</tr>
<tr>
<td>$N(A)$</td>
<td>${(1, -1)}$</td>
</tr>
<tr>
<td>$N(A^T)$</td>
<td>${(-2,1)}$</td>
</tr>
</tbody>
</table>
3. (20 pts) Let

\[ U = \text{Span} \left( \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right), \quad V = \text{Span} \left( \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right), \]

be subspaces of \( \mathbb{R}^3 \).

a) Compute \( \dim(U \cap V) \).

Finding the intersection is just like the previous test!

It is spanned by the single vector \( (1, 1, -2) \). So the dimension is 1.

b) Find a basis for \( (U \cap V)^\perp \).

Since \( U \cap V \) is a one-dimensional space in \( \mathbb{R}^3 \), the subspace \( (U \cap V)^\perp \) must have dimension \( 3 - 1 = 2 \). So we need two linearly independent vectors perpendicular to \( (1, 1, -2) \) to form a basis.

Two such vectors are, for example, \( (-1, 1, 0) \) and \( (2, 0, 1) \).
4. (15 pts) Let $A$ be given by the product

$$
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
1 & 2 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 & 0 & 1 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

Solve $Ax = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Call this decomposition $A = LU$. Let $y = Ux$. So we first solve

$$Ly = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} y = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$ 

This is easily solved by forward-substitution to give

$$y = \begin{bmatrix} 2 \\ 3 \\ -8 \end{bmatrix}.$$ 

Now solve $y = Ux$ by back-substitution to give

$$x = \begin{bmatrix} 5 \\ -3 \\ -8 \end{bmatrix}.$$ 

Naturally you could ignore completely that you were given an $LU$-decomposition and just solve for $x$ directly, but that’s more work!
5. (20 pts) If $U$ and $V$ are subspaces of $\mathbb{R}^n$, prove

a) $(U + V)\perp = U\perp \cap V\perp$.

b) Hence, $(U \cap V)\perp = U\perp + V\perp$.

Let $x \in (U + V)\perp$. That is, $x$ is perpendicular to any vector of the form $u + v$, $u \in U$, $v \in V$. If we let $v = 0$ then we see that $x$ is perpendicular to any vector in $U$. If we let $u = 0$ then we see that $x$ is perpendicular to any vector in $V$. Thus $x \in U\perp \cap V\perp$. This shows $(U + V)\perp \subset U\perp \cap V\perp$.

Now let $x \in U\perp \cap V\perp$. This means that $x$ satisfies $x.u = 0$ for any vector $u \in U$ and it satisfies $x.v = 0$ for any vector $v \in V$. So $x$ satisfies $x.(u + v) = 0$ for any vector $u \in U$ and any vector $v \in V$. That is $x \in (U + V)\perp$. This shows $(U + V)\perp \supset U\perp \cap V\perp$.

So $(U + V)\perp = U\perp \cap V\perp$.

Apply the operation $\perp$ to both sides of the above equality. This gives

$$(U + V)^\perp = U + V = (U\perp \cap V\perp)^\perp.$$ But this is true for any subspaces $U$ and $V$. So we can substitute $U\perp$ for $U$, and $V\perp$ for $V$ giving

$$(U \cap V)^\perp = U\perp + V\perp.$$