A Variational Principle for KPP Front Speeds in Temporally Random Shear Flows

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Abstract

We establish the variational principle of Kolmogorov-Petrovsky-Piskunov (KPP) front speeds in temporally random shear flows with sufficiently decaying correlations. A key quantity in the variational principle is the almost sure Lyapunov exponent of a heat operator with random potential. To prove the variational principle, we use the comparison principle of solutions, the path integral representation of solutions, and large deviation estimates of the associated stochastic flows. The variational principle then allows us to analytically bound the front speeds. The speed bounds imply the linear growth law in the regime of large root mean square shear amplitude at any fixed temporal correlation length; and the sublinear growth law if the temporal decorrelation is also large enough, the so called bending phenomenon.

1 Introduction

Reaction-diffusion front propagation in strongly time dependent random media arises in premixed flame propagation problems ([9, 26, 35, 36, 42, 43] and references), interacting particle systems ([28, 10] and references) and population biology ([37] and references). A fundamental issue is to characterize, bound and compute the large time front speed, an upscaled quantity that depends on statistics of the random media in a highly nontrivial manner. In

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combustion literature, ad hoc and formal procedures abound for approximation, such as closures and renormalization group methods [35, 43]. In this paper, we establish a variational principle for the propagation speeds of KPP reaction-diffusion fronts through temporally random shear flows. The variational characterization then allows us to estimate and compute the statistical properties of front speeds with both accuracy and ease. Related computational results will be presented separately [34].

The model equation is:

$$u_t = \frac{1}{2}\Delta_z u + B \cdot \nabla_z u + f(u), \tag{1}$$

where $u = u(z,t)$, $z = (x,y) \in \mathbb{R} \times \mathbb{R}^{n-1}$, $n \geq 2$; $B = (b(y,t), 0, \ldots, 0)$, $b(y,t)$ is a stationary Gaussian process in t, with a deterministic profile in y, to be made more precise later. The nonlinear function $f(u) \in C^1([0,1])$ is a KPP nonlinearity: $f(u) > 0$ for $u \in (0,1)$, $f(0) = f(1) = 0$, $f'(0) = \sup_{u\in(0,1)} f(u)/u$. An example is $f(u) = u(1-u)$.

For compactly supported initial data bounded between 0 and 1, solutions of (1) develop into propagating fronts separating the domain into a region where $u \approx 1$ and the rest where $u \approx 0$, which correspond to burned (hot) and unburned (cold) states in combustion. In case B is periodic in z and t, KPP type front dynamics and speeds have been recently studied for both shear and more general incompressible flows [22, 25, 26, 30, 32, 29]. Exact traveling front solutions exist [30, 32, 29], extending those in spatially periodic media, [5, 7, 40, 39], see also [4] and [41] for reviews.

For temporally random shear flows, it is more efficient to study front solutions asymptotically without constructing exact traveling fronts. This line of work goes back to Gärtner and Freidlin [18], [16] where variational principles of KPP front speeds in spatially periodic media are obtained by combining large deviation techniques and Feynman-Kac representation formulas of KPP solutions. See also [16] and references therein for related results on KPP fronts through one dimensional spatial random media. We shall further develop this approach to treat the temporally random shear flows which generate more complexities in path integrals and unbounded variations in time.

Let us make precise our assumptions on the shear field. The function $b(y,t) = b(y,t,\hat{\omega})$ is a mean zero Gaussian random field over (y,t), periodic in y with period L for each fixed t, and stationary in t for each fixed y. The field b is defined over probability space $(\hat{\Omega}, \hat{\mathcal{F}}, Q)$ and has covariance function

$\Gamma(y_1, y_2, t_1, t_2) = E_Q[b(y_1, t_1)b(y_2, t_2)]$. The following assumptions hold on $b(y, t)$:

A1: (Periodicity in y) Let $C_P^{0,1}(D)$ denote the space of Lipschitz continuous functions that are periodic on the period cell $D = [0, L]^{n-1}$. For each $\hat{\omega} \in \hat{\Omega}$, there is a continuous map $J(\cdot, \hat{\omega}) : [0, +\infty) \to C_P^{0,1}(D)$ such that $b(\cdot, t, \hat{\omega}) = J(t, \hat{\omega})$.

A2: (Stationarity in t) For each $s \in \mathbb{R}^+$ there is a measure preserving transformation $\tau_s : \hat{\Omega} \to \hat{\Omega}$ such that $b(y, \cdot + s, \hat{\omega}) = b(y, \cdot, \tau_s \hat{\omega})$. Hence, Γ depends only on y_1, y_2 and $|t - s|$.

A3: (Ergodicity) The transformation τ_s is ergodic: if a set $A \in \hat{\mathcal{F}}$ is invariant under the transformation τ_s, then either $Q(A) = 0$ or $Q(A) = 1$.

A4: The field b is mean zero, almost surely continuous in (y, t), and has uniformly bounded variance:

$$E[b(y, t)] = 0 \quad E[b(y, t)^2] \le \sigma^2 \quad \text{for all } \; y \in D, t \ge 0. \tag{2}$$

A5: (Decay of Temporal Correlations) The function $\hat{\Gamma}(r) = \sup_{y_1, y_2} \Gamma(y_1, y_2, 0, r)$ is integrable over $[0, \infty)$:

$$\int_0^\infty \hat{\Gamma}(r)\, dr = p_1 < \infty \tag{3}$$

for some finite constant $p_1 > 0$. This constant will appear later in estimates of the front speed.

A6: There is a finite constant $p_2 > 0$ such that

$$|\Gamma(y_1, y_2, s, t) - \Gamma(y_1, y_3, s, t)| \le p_2 |y_3 - y_2| \hat{\Gamma}(|s - t|).$$

For example, a field satisfying Assumptions A1-A6 might have the form $b(y, t, \hat{\omega}) = \sum_{j=1}^N b_1^j(y) b_2^j(t, \hat{\omega})$, where the functions $b_1^j(y)$ are deterministic, Lipschitz continuous and periodic over D, and the functions $b_2^j(t, \hat{\omega})$ are mean zero, stationary Gaussian fields in t.

Before stating the main results, let us define the family of Markov processes associated with the linear part of the operator in (1). For a fixed $\hat{\omega} \in \hat{\Omega}$ and for each $z \in \mathbb{R}^n$, $t \geq 0$, let $Z^{z,t}(s) = (X^{z,t}(s), Y^{z,t}(s)) \in \mathbb{R}^n$ solve the Itô equation:

$$dZ^{z,t}(s) = B(Z^{z,t}(s), t-s)\, ds + dW(s), \quad s \in [0,t] \tag{4}$$

with initial condition $Z^{z,t}(0) = z = (x,y) \in \mathbb{R}^n$, where $W(s) = (W_1(s), W_2(s)) \in \mathbb{R}^n$ is the n-dimensional Wiener process with $W(0) = 0$. Because of the shear structure of B, we therefore have

$$\begin{aligned} X^{z,t}(s) &= x + \int_0^s b(y + W_2(\tau), t-\tau)\, d\tau + W_1(s) \\ Y^{z,t}(s) &= y + W_2(s). \end{aligned} \tag{5}$$

Let $P^{z,t}$ denote the corresponding family of measures on $C([0,t];\mathbb{R}^n)$. As we will see, the KPP front speed depends on large deviations of the random variable

$$\eta_z^t(\kappa t) = \frac{z - Z^{z,t}(\kappa t)}{\kappa t}, \tag{6}$$

which is the average velocity of a trajectory over time interval $[0, \kappa t]$. The need for the parameter κ results from the time dependence of the field $b(y,t)$ and will become more apparent later.

Now we state the main results. First, the following proposition allows us to characterize the speed of propagation:

Proposition 1.1 *Assume that A1-A6 hold for the process $b(y,t)$. There is a set $\hat{\Omega}_0 \subset \hat{\Omega}$ such that $Q(\hat{\Omega}_0) = 1$ and for any $\hat{\omega} \in \hat{\Omega}_0$ and any $\lambda \in \mathbb{R}^n$, the limit*

$$\mu(\lambda, z) = \mu(\lambda) = \lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\lambda\cdot(Z^{z,t}(t) - z)}\right] \tag{7}$$

exists uniformly over $z \in \mathbb{R}^n$ and locally uniformly over $\lambda \in \mathbb{R}^n$. The limit $\mu(\lambda)$ is a finite constant for all $\hat{\omega} \in \hat{\Omega}$ and independent of $z \in \mathbb{R}^n$. Moreover, $\mu(\lambda) \geq 0$, and $\mu(\lambda)$ is both convex and super-linear: $\mu(\lambda)/|\lambda| \to +\infty$ as $|\lambda| \to \infty$.

If we let $S(c)$ be the Legendre transform of $\mu(\lambda)$

$$S(c) = \sup_{\lambda\in\mathbb{R}^n} [c \cdot \lambda - \mu(\lambda)], \tag{8}$$

then we find that the speed of propagation can be bounded above in terms of S.

Theorem 1.1 (Upper bound on front speed) *Let $b(y, t, \hat{\omega})$ satisfy assumptions A1 - A6. Let $u(z, t, \hat{\omega})$ solve (1) with initial condition $u(z, 0, \hat{\omega}) = u_0(z)$, where $u_0(z) \in [0, 1]$ has compact support and is independent of $\hat{\omega}$. Then, for any closed set $F \subset \{c \in \mathbb{R}^n | \ S(c) - f'(0) > 0\}$,*

$$\lim_{t\to\infty} \sup_{c\in F} u(ct, t, \hat{\omega}) = 0$$

uniformly in $c \in F$, for almost every $\hat{\omega} \in \hat{\Omega}$.

Furthermore, the speed of propagation can be bounded below in terms of the function S:

Theorem 1.2 (Lower bound on front speed) *Let $b(y, t, \hat{\omega})$ satisfy assumptions A1 - A6. Let $u(z, t, \hat{\omega})$ solve (1) with initial condition $u(z, 0, \hat{\omega}) = u_0(z)$, where $u_0(z) \in [0, 1]$ has compact support and is independent of $\hat{\omega}$. Then, for any compact set $K \subset \{c \in \mathbb{R}^n | \ S(c) - f'(0) < 0\}$,*

$$\lim_{t\to\infty} \inf_{c\in K} u(ct, t, \hat{\omega}) = 1 \tag{9}$$

uniformly in $c \in K$, for almost every $\hat{\omega} \in \hat{\Omega}$.

Therefore, if for each unit vector $e \in \mathbb{R}^n$ we define the constant $c^* = c^*(e) > 0$ by the **variational formula:**

$$c^*(e) = \inf_{\lambda\cdot e>0} \frac{\mu(\lambda) + f'(0)}{\lambda \cdot e}, \tag{10}$$

we see from the definition of S that the front spreads asymptotically with speed equal to $c^*(e)$ in the direction of the vector e. Although the solution u depends on $\hat{\omega} \in \hat{\Omega}$ since B is a random variable over $\hat{\Omega}$, the function $S(c)$ and the speeds $c^*(e)$ are independent of $\hat{\omega}$. They are almost surely constant with respect to $\hat{Q}$, a consequence of the ergodicity assumption A3. Hence, we will refer to the constant $c^*(e)$ as the front speed in the direction e. We will frequently suppress the dependence of u on $\hat{\omega}$ for clarity of notation. If the initial data is front-like and aligned with the shear ($u_0(z, t) = \chi_{\{x<0\}}(x, y)$), then this variational formula reduces to the one derived by Berestycki and Nirenberg [7] for waves traveling in a cylinder with a time-independent, spatially periodic flow.

Theorems 1.1 and 1.2 extend our recent results on KPP front speeds in temporally periodic incompressible flows [32, 29] and classical results of Gärtner and Freidlin (see [18] and Theorem 7.3.1, p. 494 of [16]) where they treated the case of spatially periodic advecting flows. Our proofs are built on those, with additional ingredients to handle both the time-dependence and the stochastic nature of the field B. For example, in the periodic case, $\mu(\lambda)$ is the principal eigenvalue of a periodic-parabolic operator [32, 29], and perturbation theory [21] implies that $\mu(\lambda)$ is differentiable in λ. It then follows from Theorem 7.1.1 and Theorem 7.1.2 of [16] that the random variables $\eta_z^t(t)$ satisfy a large deviation principle with convex rate function $S(c)$ given by (8). However, if $\mu(\lambda)$ is not known to be differentiable, the large deviation property needs a new proof. In the present case, $\mu(\lambda)$ is not an eigenvalue of a linear operator, so we cannot readily apply the perturbation theory [21] to get differentiability. Instead, we will show that a rate function exists and is convex, and that it continues to satisfy (8). In fact, $\mu(\lambda)$ is related to the almost sure principal Lyapunov exponent of a heat operator with random potential [42], known as the parabolic Anderson problem ([8, 12] and references). Dynamical aspects of principal Lyapunov exponents as an extension of principal eigenvalues are recently studied in [27]. Regularity of $\mu(\lambda)$ is an interesting problem in itself.

The paper is organized as follows. In Section 2, we prove some important technical bounds on the process $Z^{z,t}(s)$. In Section 3, we prove Proposition 1.1 which defines the principal Lyapunov exponent $\mu(\lambda)$. In section 4, we prove Theorem 1.1, the upper bound on the front speed. In Section 5, we adapt the method of [16] to prove Theorem 1.2, the lower bound on the front speed. A technical estimate (Lemma 5.1) and a large deviations estimate on the process $Z^{z,t}(s)$ are needed here; these are proven in Sections 6 and 7. We will make frequent use of the subadditive ergodic theorem and the Borell inequality for Gaussian fields [1, 2, 23]. In Section 8, we use the variational formula (10) to derive analytical estimates on the front speed c^* based on properties of the Lyapunov exponent μ. In particular, we demonstrate that if the shear field is $\delta \sum_{j=1}^N b_1^j(y) b_2^j(t, \hat{\omega})$, the front speed c^* grows linearly with large δ. On the other hand, if the shear field is $\delta \sum_{j=1}^N b_1^j(y) b_2^j(\delta t, \hat{\omega})$ with the temporal correlation length decreasing accordingly, the speed enhancement is no faster than $O(\sqrt{\delta})$. This is analogous to the decrease of front speeds with increasing frequency of temporally oscillating periodic shear flows [22, 30, 32]. The reduction of speed enhancement due to rapid temporal decorrelation is known as the bending phenomenon in combustion literature [3, 13, 22], here we obtain a rigorous proof in the stochastic setting. We note that linear (large δ) and quadratic (small δ) speed growth laws are known for deterministic flow patterns with channel structures ([4, 5, 6, 11, 19, 24, 38] and references), also for spatially random shears inside infinite cylinders [33, 31] or white in time Gaussian shears in the entire space [42]. Our variational bounds show consistent results for temporally random shears with sufficiently decaying correlations. The speed growth laws known to date are not sensitive to the form of the nonlinearity as long as fronts propagate out of the initial data. In this sense, KPP plays the role of a solvable model and KPP front speeds carry universal properties, similar to Burgers equation for conservation laws [14].

Though the arguments in our proofs rely on the periodicity of $b(y,t)$ in y to provide compactness in the y dimensions, they can be easily modified to solve the same problem in an infinite cylinder with the zero Neumann boundary condition on the sides of the cylinder, a case considered originally by Berestycki and Nirenberg [7] for time-independent, spatially-periodic shear flow. In the infinite cylinder, the compactness property remains, and the process $Z^{z,t}(s)$ just needs to be reflected when it hits the boundary $\mathbb{R} \times \partial\Omega$. It is also not necessary for the process $b(y,t)$ to be Gaussian. Our proofs of the bounds in Sections 1, 6, and 7 shall rely on the powerful Borell inequality for Gaussian process, yet it is easy to see that if the estimates of Section 1, 6, and 7 hold for a given process, the main results extend.

2 Estimates on $Z^{z,t}(s)$

In this section we derive some technical estimates on the process $Z^{z,t}(s)$ which follow from our structural assumptions on the field B and the Borell inequality for Gaussian fields.

Let use first note that by changing variables $r = s - t$, $v = s + t$, it is easy to see from our assumptions on the field B that

$$\begin{aligned} \int_0^T \int_0^T \sup_{y_1,y_2} \Gamma(y_1,y_2,s,t)\,ds\,dt &\leq 2\int_0^{\sqrt{2}T} \int_0^{T/\sqrt{2}} \hat{\Gamma}(r)\,dr\,dv \\ &\leq 2\sqrt{2}p_1 T \end{aligned} \tag{11}$$

and for $H \in [0, T]$,

$$\int_0^T \int_H^T \sup_{y_1, y_2} \Gamma(y_1, y_2, s, t)\, ds\, dt \leq \sqrt{2}|T - H| \int_0^{T/\sqrt{2}} \hat{\Gamma}(r)\, dr \leq \sqrt{2}|T - H| p_1. \quad (12)$$

Let $\rho(s) \in C([0, +\infty), \mathbb{R}^{n-1})$ with $\rho(0) = 0$ be fixed. For $y \in D$, define $\rho_y(s) = y + \rho(s)$. For fixed $t > 0$, the integral

$$f(y, s) = \int_0^s b(\rho_y(\tau), t - \tau)\, d\tau$$

is a Gaussian random field over $M = D \times [0, t]$, with respect to the measure Q. The Borell inequality for Gaussian fields states that if $\|f\| = \sup_{(y,s)\in M} f(y, s)$ is almost surely finite, then $E_Q[\|f\|] < \infty$ and for any $u > 0$,

$$Q(\|f\| - E[\|f\|] > u) \leq e^{-\frac{u^2}{2\sigma_t^2}} \quad (13)$$

where $\sigma_t^2 = \sup_{(y,s)\in M} E_Q[f^2]$ (see [1]). By (11), $\sigma_t^2 \leq 2\sqrt{2}p_1 t$. So, using inequality (13), we can control deviations of $\|f\|$, if we bound the growth of $E[\|f\|]$.

Lemma 2.1 *There is a finite constant $C > 0$ such that*

$$E_Q[\|f\|] \leq C t^{1/2}. \quad (14)$$

Proof of Lemma 2.1: The expectation $E[\|f\|]$ can be bounded by the metric entropy relation

$$E[\|f\|] \leq C \int_0^\delta (\log N(\epsilon))^{1/2}\, d\epsilon$$

where $\delta = diam(M)/2$ in the metric

$$d((x, s), (y, z)) = E\left[(f(x, s) - f(y, z))^2\right]^{1/2}$$

and $N(\epsilon)$ is the minimum number of ϵ-balls required to cover M (see [1]). Using (11) and (12), a straightforward computation shows that

$$E\left[(f(x, s) - f(y, s))^2\right] \leq C|x - y|t$$

and

$$E\left[(f(y,s)-f(y,z))^2\right] \le C|s-z|$$

for some finite constant C, independent of ρ. Therefore,

$$d((x,s),(y,z)) \le C_1\left(|s-z|\right)^{1/2} + C_2(|x-y|t)^{1/2},$$

and there is a constant C_3 independent of t and ϵ such that $d((x,s),(y,z)) \le \epsilon$ whenever $|s-z| \le C_3\epsilon^2$ and $|x-y| \le \frac{C_4\epsilon^2}{t}$. For $\epsilon \in (0, diam(M)/2]$, we have the bound

$$N(\epsilon) \le \max(C_5\frac{t^2}{\epsilon^4}, 1)$$

and

$$\begin{aligned} E[\|f\|] &\le C\int_0^{C_5^{1/4}t^{1/2}} (\log(C_5\frac{t^2}{\epsilon^4}))^{1/2}\, d\epsilon \\ &= C_6 t^{1/2}\int_0^1 (\log(\frac{1}{\epsilon^4}))^{1/2}, d\epsilon \le C_7 t^{1/2}. \end{aligned} \tag{15}$$

□

Note that the constants depend on the assumed properties of the process b and the size of the domain D, but not on the particular function $\rho(s)$. If $u \ge 2C_7t^{1/2}$, then by (13),

$$Q\left(\|f\| > u\right) \le e^{-(u-E\|f\|)^2/2\sigma_t^2} \le e^{-u^2/8\sigma_t^2} \le e^{-u^2/32p_1t}.$$

It now follows that

Lemma 2.2 *For any $\eta > 0$ and for $t \ge t_0 = t_0(\eta) = (2C_7/\eta)^2$,*

$$Q\left(\sup_{y\in D,\, s\in[0,t]} \int_0^s b(y+\rho(\tau), t-\tau)\, d\tau > \eta t\right) \le e^{-\eta^2 t/32p_1}$$

for any $\rho \in C([0,\infty), \mathbb{R}^{n-1})$, $\rho(0) = 0$.

In applying this lemma, the continuous function ρ will be a realization of the Wiener process $W_2(s) \in \mathbb{R}^{n-1}$.

Lemma 2.3 *For $\eta > 0$, $z \in \mathbb{R}^n$, define the Markov time*

$$\tau_{\eta,z}(t) = \min\{s \geq 0 |\ |X^{z,t}(s) - x| \geq \eta t\}.$$

with $\tau_{\eta,z}(t) = +\infty$ if the set on the right is empty. Then there are constants K_1, K_2 such that

$$Q\left(P\left(\inf_{z\in\mathbb{R}^n} \tau_{\eta,z}(t) \leq t\right) > e^{-K_2\eta^2 t/2}\right) \leq K_1 e^{-K_2\eta^2 t/2}$$

for all $t > 0$.

Proof of Lemma 2.3: Note that for the Wiener process $W_1(s)$ with $W_1(0) = 0$

$$\begin{aligned} P\left(\sup_{s\in[0,t]} |W_1(s)| \geq \eta t\right) &\leq 2\sqrt{\frac{2}{\pi}} \int_{\eta\sqrt{t}}^{\infty} e^{-x^2/2}\, dx \\ &\leq K_1 e^{-\eta^2 t/2}. \end{aligned} \tag{16}$$

The point of the lemma is that at large times and almost surely with respect to Q, the process $X^{z,t}(s)$ also behaves like a Wiener process in the sense that a bound like (16) holds. By definition of $\tau_{\eta,z}(t)$,

$$P\left(\inf_{z\in\mathbb{R}^n} \tau_{\eta,z}(t) \leq t\right) = P\left(\sup_{s\in[0,t],z\in\mathbb{R}^n} |f_t(y,s) + W_1(s)| \geq \eta t\right).$$

Using Tchebyshev's inequality, (16), and Lemma 2.2 we see that for any $\eta > 0$, $\alpha > 0$:

$$\begin{aligned} Q\left(P\left(\inf_{z\in\mathbb{R}^n} \tau_{\eta,z}(t) \leq t\right) > \alpha\right) &\leq \alpha^{-1} E_Q P\left(\sup_{s\in[0,t],z\in\mathbb{R}^n} |f_t(y,s) + W_1(s)| \geq \eta t\right) \\ &= \alpha^{-1} E_P Q\left(\sup_{s\in[0,t],z\in\mathbb{R}^n} |f_t(y,s) + W_1(s)| \geq \eta t\right) \\ &\leq \alpha^{-1} E_P Q\left(\sup_{s\in[0,t],z\in\mathbb{R}^n} |f_t(y,s)| \geq \eta t/2\right) \\ &\quad + \alpha^{-1} P\left(\sup_{s\in[0,t],z\in\mathbb{R}^n} |W_1(s)| \geq \eta t/2\right) \\ &\leq \alpha^{-1}(2e^{-\eta^2 t/32p_1} + K_1 e^{-\eta^2 t/8}) \leq \alpha^{-1} K_1 e^{-K_2\eta^2 t} \end{aligned}$$

for t sufficiently large, for some constants $K_1, K_2 > 0$. The result now follows from a choice of $\alpha = e^{-K_2\eta^2 t/2}$. $\square$

Lemma 2.4 *There are constants $K_1, K_2 > 0$ such that, except on a set of Q-measure zero,*

$$\sup_{z\in\mathbb{R}^n} P\left(\tau_{\eta,z}(t) \le t\right) \le K_1 e^{-K_2\eta^2 t} \tag{17}$$

for t sufficiently large depending on $\hat{\omega}$ and η.

Proof of Lemma 2.4: Lemma 2.3 and the Borel-Cantelli lemma imply that outside a set of Q-measure zero

$$P\left(\inf_{z\in\mathbb{R}^n} \tau_{\eta,z}(k) \le k\right) \le e^{-K_2\eta^2 k/2} \tag{18}$$

if $k \in \mathbb{Z}$ is sufficiently large. Now we want to extend this to all real t sufficiently large. Let $t \in [k, k+1]$, $t = k + \tau$, $\tau \in [0, 1]$.

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,t]} |X^{z,t}(s) - x_0| \ge t\eta\right) \\
\le\ &\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\tau]} |X^{z,t}(s) - x_0| \ge t\eta/2\right) + \\
&\quad + \sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[\tau,t]} |X^{z,t}(s) - X^{z,t}(\tau)| \ge t\eta/2\right)
\end{aligned}$$

By the Markov property, this is bounded by

$$\begin{aligned}
\le\ &\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\tau]} |X^{z,t}(s) - x_0| \ge t\eta/2\right) + \\
&\quad + \sup_{\bar{z}\in\mathbb{R}^n} P\left(\sup_{s\in[0,k]} |X^{\bar{z},k}(s) - \bar{x}_0| \ge t\eta/2\right) \le \\
\le\ & P\left(\sup_{z\in\mathbb{R}^n, t\in[k,k+1], s\in[0,1]} |X^{z,t}(s) - x_0| \ge k\eta/2\right) + \\
&\quad + P\left(\sup_{\bar{z}\in\mathbb{R}^n, s\in[0,k]} |X^{\bar{z},k}(s) - \bar{x}_0| \ge k\eta/2\right)
\end{aligned} \tag{19}$$

By (18), the second term on the right side of (19) is bounded (Q-a.s.) by

$$P\left(\sup_{\bar{z}\in\mathbb{R}^n, s\in[0,k]} |X^{\bar{z},k}(s) - \bar{x}_0| \geq k\eta/2\right) \leq e^{-K_3\eta^2 k} \tag{20}$$

for $k \in \mathbb{Z}$ sufficiently large. To bound the other term in (19), it suffices to show that

$$P\left(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2\right) \leq e^{-K_4\eta^2 k} \tag{21}$$

for $k \in \mathbb{Z}$ sufficiently large, where

$$f_k(y,s,r) = \int_0^s b(W_2^y(\tau), r + k - \tau)\, d\tau.$$

Note that $f_k(y,s,r)$ is a centered Gaussian field (with respect to Q) over $D \times [0,1] \times [0,1]$, and its distribution is invariant with respect to $k > 0$, due to the stationarity of $b(y,t)$. For any fixed path $W_2^y(\omega)$, the Borell inequality implies that for k sufficiently large

$$Q\left(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2\right) \leq K_5 e^{-K_6\eta^2 k^2}$$

for some constants $K_5, K_6 > 0$, independent of k and the realization $W_2^y(\omega)$. Therefore, proceeding as in the proof of Lemma 2.2, we see that

$$Q\left(P(\sup_{y\in D, r\in[0,1], s\in[0,1]} |f_k(y,s,r)| \geq k\eta/2) \geq e^{-K_6\eta^2 k^2/2}\right) \leq K_7 e^{-K_6\eta^2 k^2/2}.$$

Now (21) follows from the Borel-Cantelli lemma. We complete the proof by combining (20) and (21). □

The next estimate gives a coarse bound on very large excursions of the random process $X^{z,t}$, the first component of the process $Z^{z,t}$:

Lemma 2.5 *There are constants $K_1, K_2 > 0$ independent of $\kappa \in (0,1]$ such that, except on a set of Q-measure zero,*

$$\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\kappa t]} |X^{z,t}(s) - x| \geq \eta t \;\mid W_2^0 \in \Lambda\right) \leq K_1 e^{-K_2\eta^2 t/\kappa} \tag{22}$$

for any open set $\Lambda \subset \mathbb{R}^n$, for any $\kappa \in (0,1]$, $\eta > 0$, and for t sufficiently large depending on $\hat{\omega}$, κ, and η.

In particular, the lemma holds with $\Lambda = \mathbb{R}^n$. Using the fact that the $Y^{z,t}$ component of the process $Z^{z,t}$ is a Wiener process, the lemma implies the following corollary:

Corollary 2.1 *There are constants $K_1, K_2 > 0$ independent of $\kappa \in (0,1]$ such that, except on a set of Q-measure zero,*

$$\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - z| \geq \eta t\right) \leq K_1 e^{-K_2\eta^2 t/\kappa} \tag{23}$$

for any $\kappa \in (0,1]$, $\eta > 0$, and for t sufficiently large depending on $\hat{\omega}$, κ, and η.

Proof of Lemma 2.5: Lemma 2.4 encompasses the special case that $\kappa = 1$ and $\Lambda = \mathbb{R}^n$. For $\kappa < 1$, modify the preceding bounds for the field

$$f(y,s) = \int_0^s b(\rho_y(\tau), t-\tau)\, d\tau$$

considered over $M_\kappa = D \times [0, \kappa t]$. Now we have $\sigma_t^2 = \sup_{(x,s)\in M} E_Q[f^2] \leq p_1 \kappa t$, so we find that $E[\|f\|] \leq C\sqrt{\kappa t}$ for some constant $C > 0$. Then, just as in Lemma 2.3, we have

$$Q\left(P\left(\sup_{s\in[0,\kappa t], z\in\mathbb{R}^n} |X^{z,t}(s) - x| \geq \eta t\right) > e^{-K_2\eta^2 t/2\kappa}\right) \leq K_1 e^{-K_2\eta^2 t/2\kappa},$$

and for $\Lambda = \mathbb{R}^n$, the rest follows as in the proof of Lemma 2.3. To bound the more general conditional probability in Lemma 2.5 (with $\Lambda \neq \mathbb{R}^n$), observe that whenever $P(W_2^0 \in \Lambda) > 0$

$$\begin{aligned}
&Q\left(P\left(\sup_{s\in[0,\kappa t], z\in\mathbb{R}^n} |X^{z,t}(s) - x| \geq \eta t \,|W_2^0 \in \Lambda\right) > e^{-K_2\eta^2 t/2\kappa}\right) = \\
&\quad = Q\left(P\left(\sup_{s\in[0,\kappa t], z\in\mathbb{R}^n} |X^{z,t}(s) - x| \geq \eta t, W_2^0 \in \Lambda\right) > P\left(W_2^0 \in \Lambda\right) e^{-K_2\eta^2 t/2\kappa}\right) \\
&\quad \leq \frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_Q P\left(\sup_{s\in[0,\kappa t], z\in\mathbb{R}^n} |X^{z,t}(s) - x| \geq \eta t, W_2^0 \in \Lambda\right) \\
&\quad \leq \frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0 \in \Lambda\right)} E_P\left(\chi_{W_2^0\in\Lambda} Q(\sup_{s\in[0,\kappa t], z\in\mathbb{R}^n} |X^{z,t}(s) - x| \geq \eta t)\right)
\end{aligned} \tag{24}$$

By Lemma 2.2, the probability $Q(\sup_{s\in[0,\kappa t],z\in\mathbb{R}^n}|X^{z,t}(s)-x|\geq\eta t)$ is bounded independently of the realization of W_2^0, so the right hand side of (24) is bounded by

$$\frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0\in\Lambda\right)}E_P\left(\chi_{W_2^0\in\Lambda}Q(\sup_{s\in[0,\kappa t],z\in\mathbb{R}^n}|X^{z,t}(s)-x|\geq\eta t)\right)$$

$$\leq\frac{e^{K_2\eta^2 t/2\kappa}}{P\left(W_2^0\in\Lambda\right)}E_P\left(\chi_{W_2^0\in\Lambda}K_1e^{-K_2\eta^2 t/\kappa}\right)=K_1e^{-K_2\eta^2 t/2\kappa}$$

Then the rest follows as in Lemma 2.3. □

3 The Lyapunov Exponent $\mu(\lambda)$

In this section we prove Proposition 1.1. We study the limit

$$\mu(\lambda,z)\quad=\quad\lim_{t\to\infty}\frac{1}{t}\log E\left[e^{-\lambda\cdot(Z^{z,t}(t)-z)}\right].\tag{25}$$

Notice that this is equivalent to the limit

$$\mu(\lambda,z)\quad=\quad\frac{\lambda^2}{2}+\lim_{t\to\infty}\frac{1}{t}\log\phi(z,t)\tag{26}$$

where $\phi(z,t)>0$ solves that auxiliary initial value problem

$$\phi_t=\frac{1}{2}\Delta_z\phi+(B-\lambda)\cdot\nabla\phi-\lambda\cdot B(z,t)\phi.\tag{27}$$

$$\phi(z,0)\equiv 1.$$

To see this, use the Feynman-Kac representation to express ϕ as

$$\phi=\tilde{E}_z\left[e^{-\int_0^t\lambda\cdot B(\tilde{Z}^{z,t}(s),t-s)ds}\right],$$

where $\tilde{Z}$ solves

$$d\tilde{Z}^{z,t}(s)\quad=\quad\left(B(\tilde{Z}^{z,t}(s),t-s)-\lambda\right)\,ds+dW(s).$$

This induces a measure $\tilde{P}^{z,t}$ that is absolutely continuous with respect to $P^{z,t}$. The Girsanov theorem [20] implies that:

$$\frac{d\tilde{P}}{dP}=e^{-\int_0^t\lambda\cdot dW(s)-\frac{1}{2}|\lambda|^2 t}.$$

Hence

$$\phi = \tilde{E}_z\left[e^{-\int_0^t \lambda\cdot B(\tilde{Z}^{z,t}(s),t-s)ds}\right] = e^{-\frac{|\lambda|^2}{2}t}E_z\left[e^{-\lambda\cdot(Z^{z,t}(t)-z)}\right],$$

establishing (26). Without the drift term, the equation (27) is called the parabolic Anderson problem (see [8] and [12]). We will denote by $\rho(\lambda)$ the limit $\lim_{t\to\infty}\frac{1}{t}\log\phi(z,t)$. Therefore, $\mu(\lambda)$ exists independent of z if and only if $\rho(\lambda)$ exists independent of z.

The proof that $\mu(\lambda)$ exists almost surely with respect to Q, independent of z, relies on the sub-additive ergodic theorem and a Harnack-type estimate based on techniques in [12], provided we assume the necessary decay of temporal correlation of the process $B(y,t)$. Following [12], we define for any continuous path $W \in C([0,t],\mathbb{R}^n)$ the exponential

$$\xi(t,W) = e^{-\int_0^t \lambda_1 b(W_2(s)+z,t-s)\,ds - \lambda_1 W_1(t) - \lambda_2 W_2(t)}.$$

which is the exponential term $e^{-\lambda\cdot(Z^{z,t}(t)-z)}$ for a fixed realization of the Wiener process. For any fixed path W, $\xi(t,W)$ is lognormal with mean

$$E_Q[\xi(t,W)] = e^{\frac{\lambda^2\hat{\sigma}^2}{2}}e^{-\lambda_1 W_1(t)-\lambda_2 W_2(t)} \tag{28}$$

where

$$\hat{\sigma}^2 = \int_0^t\int_0^t \Gamma(X_s,X_r,s,r)\,ds\,dr \le 2\sqrt{2}p_1 t, \tag{29}$$

by (11). Note that $\hat{\sigma}^2$ is bounded independently of the particular path W.

For $0 \le s < t$, define the random variables

$$\begin{aligned} q^z(\lambda,s,t) &= E_{z,t}[e^{-\lambda\cdot(Z(t-s)-z)}] \\ q_I(\lambda,s,t) &= \inf_{z\in D} q^z(\lambda,s,t) \\ q_S(\lambda,s,t) &= \sup_{z\in D} q^z(\lambda,s,t). \end{aligned}$$

Using the sub-additive ergodic theorem, we will show that the limits

$$\lim_{t\to\infty}\frac{1}{t}\log q_I(\lambda,0,t) = \mu_I(\lambda) \tag{30}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\log q_S(\lambda,0,t) = \mu_S(\lambda) \tag{31}$$

exist and are finite, almost surely with respect to Q. Then we will show $\mu_I(\lambda) = \mu_S(\lambda)$, and therefore $\mu(\lambda) = \mu_I(\lambda) = \mu_S(\lambda)$ is well-defined, independently of z. By the Markov property of the Wiener process we have for any $s < r < t$:

$$\begin{aligned} q_I(\lambda, s, t) &= \inf_z E_{z,t}\left[e^{-\lambda\cdot(Z^{z,t}(t-r)-z)} E[e^{-\lambda\cdot(Z^{a,r}(r-s)-a)}|Z^{z,t}(t-r)=a]\right] \\ &\geq \inf_z E_{z,t}\left[e^{-\lambda\cdot(Z^{z,t}(t-r)-z)} \inf_a E_{a,r}[e^{-\lambda\cdot(Z^{a,r}(r-s)-a)}]\right] \\ &= q_I(\lambda, s, r) q_I(\lambda, r, t). \end{aligned}$$

Therefore, $\log(q_I(\lambda, s, t))$ is super-additive:

$$\log(q_I(\lambda, s, t)) \geq \log(q_I(\lambda, s, r)) + \log(q_I(\lambda, r, t))$$

for any $0 \leq s < r < t$. Similarly, the function $\log(q_S(\lambda, s, t))$ is sub-additive. By the stationarity of B, $\tau_r \log(q_I(\lambda, s, t)) = \log(q_I(\lambda, s+r, t+r))$ for any $r \geq 0$.

In order to apply the ergodic theorem, we must show that $\log q_I$ and $\log q_S$ are integrable. First,

$$\begin{aligned} E_Q[\log(q_I(\lambda, s, t))] &\leq \log E_Q[q_I(\lambda, s, t)] \\ &\leq \log \inf_z E_P E_Q[e^{-\lambda\cdot(Z^{z,t}(t-s)-z)}], \qquad (32) \\ &= \log \inf_z E_P[e^{\frac{\lambda^2\hat{\sigma}^2}{2}} e^{-\lambda_1 W_1(t-s)-\lambda_2 W_2(t-s)}] \\ &\leq \frac{|\lambda|^2}{2}|t-s| + \log e^{\frac{\lambda^2\sigma^2}{2}} \\ &\leq \frac{|\lambda|^2}{2}|t-s| + \sqrt{2}\lambda^2 p_1 |t-s|. \end{aligned}$$

By Jensen's Inequality

$$\begin{aligned} E_Q\left[\log(q_I(\lambda, s, t))\right] &\geq E_Q\left[\log(E_P e^{\inf_z -\lambda\cdot Z^{z,t}(t-s)})\right] \\ &\geq E_P E_Q\left[\inf_z -\lambda\cdot(Z^{z,t}(t-s)-z)\right] \\ &= \frac{|\lambda|^2}{2}|t-s| + E_P E_Q\left[\inf_z -\int_0^t \lambda_1 b(W_2(s)+z, t-s)\, ds\right]. \end{aligned}$$

This last term is finite, by the Borell inequality. Note also that if $M(\hat{\omega}) =$

$\sup_{t\in[0,1],z\in D}|B(z,t)|$, then

$$\begin{aligned}\sup_{s,t\in[0,1]}|\log q_I(\lambda,s,t)| &\leq |\lambda_1|M(\hat\omega)+\sup_{s,t\in[0,1]}|\log E[e^{-\lambda_1W_1(t-s)-\lambda_2W_2(t-s)}]| \\ &\leq |\lambda_1|M(\hat\omega)+\frac{|\lambda|^2}{2},\end{aligned}$$

and the latter is integrable with respect to Q. It now follows from the subadditive ergodic theorem (Theorem 2.5 of [2]) and the continuity of $q(\lambda,0,t)$ with respect to t that the limit (30) exists almost surely and is finite:

$$\lim_{t\to\infty}\frac{\log(q_I(\lambda,0,t))}{t}=\sup_t\frac{E_Q\left[\log(q_I(\lambda,0,t))\right]}{t}=\mu_I. \tag{33}$$

Also, by (32), $\mu_I\leq\frac{|\lambda|^2}{2}+\sqrt{2}|\lambda|^2p_1$. Because b is ergodic with respect to translation in t, $\mu_I(\lambda)$ is constant on a set of full measure (with respect to Q).

Now we show that $(1/t)\log(q_S(\lambda,0,t)$ is integrable. A lower bound on the expectation follows from Jensen's inequality:

$$\begin{aligned}E_Q\left[\log q_S(\lambda,s,t)\right] &\geq \sup_z E_QE_{z,t}[-\lambda\cdot(Z^{z,t}(t-s)-z)] \\ &= \sup_z E_{z,t}E_Q[-\lambda\cdot(Z^{z,t}(t-s)-z)]=0. \end{aligned}\tag{34}$$

Now we derive an upper bound. The Borell inequality and Theorem 3.2 of [1] (p. 63, let $\alpha=1$) imply that there is a finite constant $K_0>0$ such that

$$E_Qe^{\sup_z-\int_0^{t_i-t_j}\lambda_1b(W_2(\tau)+z,t_i-\tau)\,d\tau}<K_0<\infty \tag{35}$$

if $\hat\sigma^2<\frac{1}{2}$, where $\hat\sigma^2$ is the variance of the integral $-\int_0^{t_i-t_j}\lambda_1b(W_2(\tau)+z,t_i-\tau)\,d\tau$ with respect to Q. This variance is small, when $|t_i-t_j|$ is small. Thus by (29), there is a constant $K_1>0$ such that (35) holds when $|t_i-t_j|\leq K_1$. Now for any $s<t$, let N be the smallest integer greater than $|t-s|/K_1$ and $s=t_0<t_1<t_2<\cdots<t_N=t$ with $|t_{i+1}-t_i|=\Delta t=|t-s|/N\leq K_1$ for all $i=0,\dots,N-1$. Jensen's inequality implies that

$$\begin{aligned}&E_Q\log(q_S(\lambda,t_i,t_{i+1}))\leq \\ &\quad\leq\log E_P\left[e^{-\lambda\cdot W(t_{i+1}-t_i)}E_Q[e^{\sup_z-\lambda_1\int_0^{t_{i+1}-t_i}b(W_2(s)+z,t-s)\,ds}]\right] \\ &\quad\leq\log E_P\left[e^{-\lambda\cdot W(t_{i+1}-t_i)}K_0\right]=\frac{|\lambda|^2(t_{i+1}-t_i)}{2}+\log K_0.\end{aligned}\tag{36}$$

Combining this with the subadditivity of $\log(q_S(\lambda, s, t))$, we derive the upper bound

$$\begin{aligned} E_Q \log(q_S(\lambda, s, t)) &\leq \sum_{i=0}^{N-1} E_Q \log(q_S(\lambda, t_i, t_{i+1})) \\ &\leq \frac{|\lambda|^2(t-s)}{2} + N(\log K_0) \\ &\leq \frac{|\lambda|^2(t-s)}{2} + (\frac{|t-s|}{K_1} + 1)(\log K_0). \end{aligned}$$

The last inequality follows from our definition of N. Moreover,

$$E_P[e^{-|\lambda|\|B\||t-s|-\lambda\cdot W(t-s)}] \leq q_S(\lambda, s, t) \leq E_P[e^{|\lambda|\|B\||t-s|-\lambda\cdot W(t-s)}],$$

so that

$$\sup_{s,t\in[0,K_1]} |\log q_S(\lambda, s, t)| \leq K_1(|\lambda|\|B\| + \frac{\lambda^2}{2}) \tag{37}$$

where $\|B\|$ denotes $\sup_{t\in[0,K_1],z\in D}|B(z,t)|$. The right side of (37) is integrable. So, we can apply the sub-additive ergodic theorem to conclude that the limit

$$\lim_{t\to\infty} \frac{\log(q_S(\lambda, 0, t))}{t} = \inf_t \frac{E_Q\left[\log(q_S(\lambda, 0, t))\right]}{t} = \mu_S, \tag{38}$$

holds almost surely with μ_S a constant, $\mu_S \in [0, \infty)$. The convergence along continuous time follows from (37), the continuity of $q_S(\lambda, 0, t)$, and Theorem 2.5 of [2]. As with μ_I, μ_S is constant on a set of full measure, because of the ergodicity of b with respect to translation in t.

Clearly $\mu_I \leq \mu_S$. To show that $\mu_I = \mu_S$, we will need a kind of Harnack inequality to compare the quantities $q_I(\lambda, 0, t)$ and $q_S(\lambda, 0, t)$. Such a result has been obtained in [12] in the case that $b(y, t)$ is Gaussian in both space and time, with a white-noise temporal dependence. Under the assumptions A1-A6, however, the arguments of [12] imply that the following estimate also holds in the present case.

Theorem 3.1 *(Cranston and Mountford [12]) For any fixed $M > 0$, there are positive constants c_1, c_2 such that outside an event of $Q-$probability $e^{-\frac{1}{4}n^{5/6}}$, one has*

$$q_I(\lambda, 0, n) \geq c_1 e^{-c_2 n^{11/12}} \left(q_S(\lambda, 0, n) - e^{-\frac{1}{4}n^{7/6}}\right).$$

From this result it follows immediately that

$$\lim_{n\to\infty} \frac{1}{n} \log E\left[e^{-\lambda\cdot(Z^{z,n}(n)-z)}\right] = \mu_I(\lambda) = \mu_S(\lambda),$$

uniformly in z. By (33) and (38), we see that this extends to continuous time

$$\lim_{t\to\infty} \frac{1}{t} \log E\left[e^{-\lambda\cdot(Z^{z,t}(t)-z)}\right] = \mu_I(\lambda) = \mu_S(\lambda) = \mu(\lambda). \tag{39}$$

We have now shown that for each $\lambda \in \mathbb{R}^n$, $\mu(\lambda)$ is well-defined, independent of $z \in \mathbb{R}^n$, almost surely with respect to Q. This means that for each $\lambda \in \mathbb{R}^n$, there is a set $\hat{\Omega}_\lambda \subset \hat{\Omega}$ such that $Q(\hat{\Omega}_\lambda) = 1$ and (39) holds for all $\hat{\omega} \in \hat{\Omega}_\lambda$. We claim that the set

$$\hat{\Omega}_0 = \bigcap_{\lambda\in\mathbb{R}^n} \hat{\Omega}_\lambda \tag{40}$$

has full measure: $Q(\hat{\Omega}_0) = 1$. It is clear that the set $\hat{\Omega}_0' = \bigcap_{\lambda\in\mathbb{Q}^n} \hat{\Omega}_\lambda$ has full measure, since $\mathbb{Q}^n$ is a countable set.

Lemma 3.1 $\hat{\Omega}_0 = \hat{\Omega}_0'$. *So,* $Q(\hat{\Omega}_0) = 1$.

Proof: Clearly $\hat{\Omega}_0 \subset \hat{\Omega}_0'$. For $\lambda \in \mathbb{R}^n$, $t > 0$, $z \in D$, we define the quantities

$$\begin{aligned}
\mu^z(\lambda, t) &= \frac{1}{t} \log E_{z,t}[e^{-\lambda\cdot Z(t)}] = \frac{1}{t} \log q^z(\lambda, 0, t) \\
\mu^+(\lambda, t) &= \inf_{z\in D} \mu^z(\lambda, t) \\
\mu^-(\lambda, t) &= \sup_{z\in D} \mu^z(\lambda, t).
\end{aligned}$$

So, for all $\lambda \in \mathbb{Q}^n$ and $\hat{\omega} \in \hat{\Omega}_0'$, $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ as $t \to \infty$.

We claim that $\mu^z(\lambda, t)$ and $\mu^+(\lambda, t)$ are convex in λ, for each $t > 0$. Let $r \in [0, 1]$, $\lambda_1, \lambda_2 \in \mathbb{R}^n$. By Hölder's inequality,

$$E[e^{-r\lambda_1\cdot Z^{z,t}(t)-(1-r)\lambda_2\cdot Z^{z,t}(t)}] \le E[e^{-\lambda_1\cdot Z^{z,t}(t)}]^r E[e^{-\lambda_2\cdot Z^{z,t}(t)}]^{1-r}. \tag{41}$$

Upon taking a logarithm and dividing by t, this inequality implies that

$$\mu^z(r\lambda_1 + (1-r)\lambda_2, t) \le r\mu^z(\lambda_1, t) + (1-r)\mu^z(\lambda_2, t).$$

Hence, $\mu^z(\lambda, t)$ is convex. Since $\mu^+(\lambda, t)$ is a supremum of convex functions (μ^z), it must also be convex. It follows that the function $\mu(\lambda)$ is continuous in λ, since for each $\lambda \in \Omega_0'$, $\mu(\lambda)$ is the finite, pointwise limit of continuous convex functions $\mu^+(\lambda, t)$ (pointwise for $\lambda \in \mathbb{Q}^n$, a dense subset of $\mathbb{R}^n$). Moreover, $\mu(\lambda)$ must be uniformly continuous on compact sets.

Next, we claim that for each $\hat{\omega} \in \hat{\Omega}_0'$ fixed, $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ locally uniformly in $\lambda \in \mathbb{R}^n$. Let $\Lambda_\delta \subset \mathbb{R}^n$ be a closed ball of radius δ. For $\epsilon > 0$, let $k \in \mathbb{Z}$ be large enough so that $|\mu(\lambda_1) - \mu(\lambda_2)| < \epsilon$ whenever $|\lambda_1 - \lambda_2| \le n2^{-k}$ and $\lambda_1, \lambda_2 \in \Lambda_{2\delta}$. Then, let $t_0 > 0$ be large enough so that $|\mu^+(\lambda, t) - \mu(\lambda)| \le \epsilon$ for all $\lambda \in \Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$ and $t \ge t_0$. Such a t_0 exists since $\Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$ is a finite subset of $\subset \mathbb{Q}^n$. For k sufficiently large, depending on δ, any $\lambda \in \Lambda_\delta$ can be expressed as a convex combination of points in the set $\{\lambda_j\}_j = \Lambda_{2\delta} \cap 2^{-k}\mathbb{Z}^n$:

$$\lambda = \sum_j w_j \lambda_j \tag{42}$$

such that $w_j \ge 0$, $\sum_j w_j = 1$. Moreover, we can require that $w_j = 0$ if $|\lambda_j - \lambda| \ge n2^{-k}$. Therefore, for t sufficiently large,

$$\begin{aligned}
\mu^+(\lambda, t) &= \mu^+(\sum_j w_j \lambda_j, t) \\
&\le \sum_j w_j \mu^+(\lambda_j, t) \quad \text{(by convexity)} \\
&\le \epsilon + \sum_j w_j \mu(\lambda_j) \quad \text{(by choice of } t_0\text{)} \\
&\le \epsilon + \sum_j w_j (\epsilon + \mu(\lambda)) = 2\epsilon + \mu(\lambda).
\end{aligned}$$

This implies that

$$\lim_{t\to\infty} \sup_{\lambda \in \Lambda_\delta} \mu^+(\lambda, t) - \mu(\lambda) \le 0. \tag{43}$$

Now suppose that for some $\epsilon > 0$, there are sequences $z_k \to z_0 \in D$, $t_k \to \infty$, and $\lambda_k \to \lambda_0 \in \Lambda_\delta$ such that

$$\mu^{z_k}(\lambda_k, t_k) < \mu(\lambda_0) - \epsilon, \quad \text{for } k = 1, 2, 3, \ldots$$

Then for any $\lambda' \in \mathbb{Q}^n$,

$$\frac{\mu^{z_k}(\lambda', t_k) - \mu^{z_k}(\lambda_k, t_k)}{|\lambda_k - \lambda'|} \ge \frac{\mu^{z_k}(\lambda', t_k) - \mu(\lambda_0) + \epsilon}{|\lambda_k - \lambda'|} \tag{44}$$

Since $\lambda' \in \mathbb{Q}^n$, $\mu^{z_k}(\lambda', t_k) \to \mu(\lambda')$ as $k \to \infty$. Therefore, by choosing λ' sufficiently close to λ_0, we can make $|\mu^{z_k}(\lambda', t_k) - \mu(\lambda_0)| < \epsilon/2$ for k sufficiently large, since $\mu(\lambda)$ is continuous in λ. Hence, the right hand side of (44) can be made arbitrarily large. That is, the slopes of the secant lines through the points $(\lambda', \mu^{z_k}(\lambda', t_k))$ and $(\lambda_k, \mu^{z_k}(\lambda_k, t_k))$ can be made arbitrarily large for k large, since $\lambda_k \to \lambda_0$. However, because the functions $\mu^{z_k}(\lambda, t_k)$ are convex in λ and bounded above by $\mu^+(\lambda, t_k)$, this contradicts (43). Hence,

$$\lim_{t\to\infty} \inf_{\lambda\in\Lambda_\delta} \mu(\lambda) - \mu^-(\lambda, t) \le 0. \tag{45}$$

Equations (43) and (45) imply the claim that $\mu^+(\lambda, t) \to \mu(\lambda)$ and $\mu^-(\lambda, t) \to \mu(\lambda)$ locally uniformly in $\lambda \in \mathbb{R}^n$ for each $\hat{\omega} \in \hat{\Omega}'_0$. Therefore, $\hat{\Omega}'_0 \subset \hat{\Omega}_0$.
□

To complete the proof of Proposition 1.1, we now show that $\mu(\lambda)$ is super-linear in λ. Clearly $\mu(0) = f'(0)$. Let $\lambda = (\lambda_1, 0)$, $\lambda_1 \in \mathbb{R}$, so that the nonzero component of λ is in the x-direction. Then ϕ in (27) can be chosen to depend only on the y variable: $\phi = \phi(y, t)$. Thus, the problem (27) reduces to

$$\phi_t = \frac{1}{2}\Delta_y \phi - \lambda_1 b(y, t)\phi. \tag{46}$$
$$\phi(y, 0) \equiv 1.$$

Since $b(y, t)$ has the same distribution as $-b(y, t)$, we conclude that $\rho(-(\lambda_1, 0)) = \rho((\lambda_1, 0))$ for all $\lambda_1 \in \mathbb{R}$. Hence $\rho((\lambda_1, 0))$ and $\mu((\lambda_1, 0))$ are even functions of λ_1. Using the Feynman-Kac representation for $\rho((\lambda_1, 0))$ as in (41), we find that $\rho((\lambda_1, 0))$ is convex in λ_1. Since $\rho(0) = 0$, we conclude that

$$\rho((\lambda_1, 0)) \ge 0 \quad \text{and} \quad \mu((\lambda_1, 0)) \ge \lambda_1^2/2, \quad \forall \lambda_1 \in \mathbb{R}. \tag{47}$$

If we choose $\lambda = (0, \lambda_2)$, $\lambda_2 \in \mathbb{R}$, so that the nonzero component of λ is in the y-direction, then (27) ϕ can be chosen to be constant $\phi \equiv 1$, for all time. Hence

$$\rho((0, \lambda_2)) = 0 \quad \text{and} \quad \mu((0, \lambda_2)) = \lambda_2^2/2, \quad \forall \lambda_2 \in \mathbb{R}. \tag{48}$$

Combining (47), (48), and the convexity of $\mu(\lambda)$, we conclude that $\mu(\lambda)$ is super-linear. This completes the proof of Proposition 1.1.

4 Proof of Theorem 1.1

The proof of Theorem 1.1 is based on the assumption that $f(u) \le f'(0)u$. This allows us to construct a super-solution to equation (1) as follows. Consider the solution to the auxiliary initial value problem

$$\Phi_t = \frac{1}{2}\Delta_z \Phi + (B - \lambda) \cdot \nabla \Phi + (|\lambda|^2/2 + f'(0) - \lambda \cdot B(z,t))\Phi, \Phi(z,0) \equiv 1.$$

where $\Phi = \Phi(z,t) > 0$ is periodic in z. As shown at the beginning of section 3, we can express $\mu(\lambda)$ in terms of Φ:

$$\mu(\lambda) \quad = \quad -f'(0) + \lim_{t\to\infty} \frac{1}{t} \log \Phi(z,t). \tag{49}$$

Now suppose $S(c) - f'(0) > 0$. Then for $\epsilon > 0$ sufficiently small, there exists $\lambda > 0$ such that $\lambda \cdot c > \mu(\lambda) + f'(0) + 2\epsilon$. By Proposition 1.1, $\mu(\lambda)$ is finite, and there is a function $R = R(z,t)$ such that $|R| \to 0$ as $t \to \infty$, uniformly in z, and

$$\Phi(y,t) = e^{\mu(\lambda)t + f'(0)t + R(z,t)t}. \tag{50}$$

If $\delta > 0$ is sufficiently small, then $\lambda \cdot c' > \mu(\lambda) + f'(0) + \epsilon$, whenever $|c' - c| < \delta$. Then for any $\alpha > 1$, we also have $\lambda \cdot \alpha c' > \mu(\lambda) + f'(0) + \epsilon$. If we define the function $\psi(z,t) = e^{-\lambda \cdot z}\Phi(z,t)$, then ψ solves the equation

$$\psi_t = \frac{1}{2}\Delta_z \psi + B \cdot \nabla \psi + f'(0)\psi \tag{51}$$
$$\psi(z,0) = e^{-\lambda \cdot z},$$

and ψ is a super solution to the original nonlinear equation (1), since $f(u) \le f'(0)u$. Combining (50) and (51), we see that

$$\begin{aligned}
\limsup_{t\to\infty} \psi(\alpha c' t, t) &= \limsup_{t\to\infty} e^{-\lambda \cdot \alpha c' t} \Phi(\alpha c' t, t) \\
&\le \limsup_{t\to\infty} e^{-\epsilon t + R(\alpha c' t, t)t} \\
&= 0,
\end{aligned}$$

since $|R(z,t)| < \epsilon$ for t sufficiently large. By definition of $R(z,t)$, the limit is uniform in α and δ, for $\alpha \ge 1$ and δ small. After multiplying ψ by a constant, if necessary, the maximum principle implies that $u(z,t) \le \psi(z,t)$

for all z and t. The function u is therefore trapped below ψ which moves with velocity c'. Now we piece together a collection of such super solutions.

If F is bounded, then it is compact, since it is closed. From the above analysis, we see that we can pick finite sets $\{c_j\} \subset F$ and $\{\lambda_j\}$ such that $F \subset \bigcup_j U_{\delta_j}(c_j) \subset \{S(c) - f'(0) > 0\}$ and $\lambda_j \cdot c' > \mu(\lambda_j) + \epsilon$ whenever $c \in U_{\delta_j}(c_j)$. If we define ψ_j according to (51) with $\lambda = \lambda_j$, and set

$$\hat{\psi}(z,t) = \inf_j \psi_j(z,t),$$

we see that

$$\lim_{t\to\infty} \sup_{c\in F, \alpha>1} u(\alpha ct, t) \le \lim_{t\to\infty} \sup_{c\in F,\alpha>1} \hat{\psi}(\alpha ct, t) = 0$$

Since $\{S(c) - f'(0) \le 0\}$ is bounded, then general result follow from the fact that the limit is uniform in $\alpha > 1$. This completes the proof of Theorem 1.1. □

Note that the function $R(z,t) = R(z,t,\hat{\omega})$ depends on the realization $\hat{\omega} \in \hat{\Omega}$. However, such a function exists almost surely with respect to Q, according to Proposition 1.1.

5 Proof of Theorem 1.2

Proving Theorem 1.2 requires more estimates on the random solutions $u(z,t)$ and the processes $Z^{z,t}(s)$. The following estimate is a lower bound analogous to Lemma 7.3.3 of [16]. It estimates the exponential decay rate of u in terms of the function S:

Lemma 5.1 *For any compact set $K \subset \{c \in \mathbb{R}^n |\ S(c) - f'(0) > 0\}$,*

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c\in K} u(ct,t) \ge -\max_{c\in K}(S(c) - f'(0)). \tag{52}$$

This lemma and the estimates of sections 2 and 3 represent the main technical difficulty in extending the work of [16] to the present case with a stochastic time dependence in $b(y,t)$. For the moment, however, we delay the proof of this lemma and show how the bound leads to Theorem 1.2. Lemma 5.1 is proved in the next section. The proof of Theorem 1.2 is based on the observation that when $u < h < 1$, the reaction rate can be bounded below. For each $u \in (0, 1]$, define the reaction rate ζ by

$$\zeta(u) = \frac{f(u)}{u}$$

and $\zeta(0) = f'(0)$. Now equation (1) can be written

$$u_t = \frac{1}{2}\Delta_z u + b(y,t)u_x + \zeta(u)u. \tag{53}$$

By the properties of $f(u)$ we see that $\zeta(u) > 0$ for $u \in [0, 1)$, $\zeta(u)$ is continuous for $u \in [0, 1]$, and $\zeta(0) \geq \zeta(u)$ for any $u \in [0, 1]$. If $h \in (0, 1)$ we define a lower bound on ζ:

$$\zeta_h = \inf_{u \in (0,h)} \zeta(u) > 0.$$

So, in regions where u is bounded away from one, the reaction rate can be bounded below by $\zeta_h > 0$.

For a fixed $\hat{\omega} \in \hat{\Omega}$, we can estimate $u(z, t)$ using the Feynman-Kac formula for the solution of (53):

$$u(z,t) = E\left[e^{\int_0^t \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u_0(Z^{z,t}(t))\right], \tag{54}$$

where the expectation is with respect to measure $P^{z,t}$. If τ is any stopping time, we also have

$$u(z,t) = E\left[e^{\int_0^{t\wedge\tau} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u(Z^{z,t}(t-(t\wedge\tau)), t-(t\wedge\tau))\right], \tag{55}$$

where $t \wedge \tau = \min(t, \tau)$. Therefore, we can obtain estimates on u by carefully choosing stopping times and restricting the expectation to certain sets of paths. The exponential term inside the expectation will be large when the path $Z^{z,t}(s)$ passes through regions where u is small and the reaction rate is large. On the other hand, if $u(Z^{z,t}(t - (t \wedge \tau)), t - (t \wedge \tau))$ is too small, then the expectation as a whole may be small.

Now we follow the ideas of Freidlin [16] (see p. 494). For $s \in \mathbb{R}$, define the set

$$\Psi(s) = \{c \in \mathbb{R}^n | \ S(c) - f'(0) = s\} \ \text{ and } \ \underline{\Psi}(s) = \{c \in \mathbb{R}^n | \ S(c) - f'(0) \leq s\}.$$

For any $\delta > 0$ and $T > 1$, define

$$\Gamma_T = \left([\underline{\Psi}(\delta) \times \{1\}] \cup [\bigcup_{1 \le t \le T} (t\Psi(\delta)) \times \{t\}] \right).$$

This defines the boundary of a region that spreads outward in z, linearly in t. Outside this region u may be close to zero, but on the boundary of this region, we have the crucial lower bound from Lemma 5.1:

$$u(z,s) \ge e^{-2\delta t} \quad \text{for all } \ (z,s) \in \Gamma_t \tag{56}$$

for t sufficiently large.

Let K be a compact set $K \subset \{c \in \mathbb{R}^n | \ S(c) - f'(0) < 0\}$ and $z = ct$ for some $c \in K$. For $h \in (0,1)$, $t, \eta > 0$, define the Markov times

$$\begin{aligned}
\sigma_h(t) &= \min\{s \in [0,t] | \ u(Z^{z,t}(s), t-s) \ge h\}, \\
\sigma_\Gamma(t) &= \min\{s \in [0,t] | \ (Z^{z,t}(s), t-s) \in \Gamma_t\}, \\
\tau_\eta(t) &= \min\{s \in [0,t] | \ |Z^{z,t}(s) - z| > \eta t\}, \\
\hat{\sigma}(t) &= \sigma_h(t) \wedge \sigma_\Gamma(t).
\end{aligned}$$

(We set these variables equal to $+\infty$ if the sets on the right are empty.) Using (55) with the stopping time $\hat{\sigma}(t)$ we express $u(z,t)$ as

$$\begin{aligned}
u(z,t) = E[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s, u(Z^{z,t}(s), t-s))ds} \ \times \\
\times\, u\left(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma})\right)(\chi_{A_1} + \chi_{A_2} + \chi_{A_3})],
\end{aligned} \tag{57}$$

where A_1, A_2, and A_3 are the disjoint sets

$$\begin{aligned}
A_1 &= \{\omega | \ \sigma_h(t) \le t\}, \\
A_2 &= \{\omega | \ \sigma_h(t) > t, \, \sigma_\Gamma(t) \ge rt\}, \\
A_3 &= \{\omega | \ \sigma_h(t) > t, \, \sigma_\Gamma(t) < rt\}
\end{aligned}$$

for some $r \in (0,1)$ so be chosen. Note that $P(A_1) = P^{z,t}(\sigma_h(t) \le t)$ is the probability that a particle starting at $z = ct$ will encounter the "hot region", $u \ge h$, at or before time t.

Because the sets A_1, A_2, and A_3 are disjoint, the expectation (57) splits into three integrals. The first integral, over A_1, can be bounded below by

$$\begin{aligned}
E\left[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s, u(Z^{z,t}(s), t-s))ds} u(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma}))\chi_{A_1}\right] \ge \\
\ge hP(A_1).
\end{aligned} \tag{58}$$

since $\zeta \geq 0$. The second integral, over A_2, can be bounded below by

$$E\left[e^{\int_0^{t\wedge\hat{\sigma}} \zeta(t-s,u(Z^{z,t}(s),t-s))ds} u(Z^{z,t}(t-(t\wedge\hat{\sigma})), t-(t\wedge\hat{\sigma}))\chi_{A_2}\right] \geq$$
$$\geq e^{-2\delta t} e^{\zeta_h r t} P(A_2). \tag{59}$$

Combining (58) and (59) we have

$$u(z,t) \geq hP(A_1) + e^{-2\delta t + \zeta_h r t} P(A_2). \tag{60}$$

If we chose δ to be small, depending on h and r, then $-2\delta t + \zeta_h r t > 0$. Since $u(z,t) \leq 1$ for all (z,t), (60) then implies that $P(A_2) \to 0$ exponentially fast, if δ is small. Therefore, if we can also show that $P(A_3) \to 0$, then $P(A_1) \to 1$, and (60) implies the desired result (9), since h can be chosen arbitrarily close to 1.

The compact set K is bounded away from the boundary of $\Psi(0) \subset \Psi(\delta)$, so we can choose η small and then $r \in (0,1)$ sufficiently small so that

$$rt < \sigma_\Gamma(t) \leq t - 1$$

whenever $\tau_\eta(t) > t$. In other words, the trajectory $Z^{z,t}(s)$ stays in the set Γ_t for at least some fixed proportion of the interval $[0,t]$. Therefore,

$$P(A_3) \leq P(\sigma_\Gamma(t) < rt) \leq P(\tau_\eta(t) \leq t).$$

By Corollary 2.1,

$$\sup_{z\in\mathbb{R}^n} P^{z,t}\left(\tau_\eta(t) < t\right) \to 0 \tag{61}$$

as $t \to \infty$, for all $\eta > 0$, except on a set of $Q-$measure zero. Hence $P(A_3) \to 0$, uniformly over $c \in K$. This completes the proof of Theorem 1.2. □

We note that the main difficulty in extending the argument of [16] for the periodic case is the manner in which the estimates (56) and (61) are obtained. In [16], estimate (61) followed from the uniform boundedness of the field B and the independence of B with respect to time, properties that we do not have in the present case.

6 Proof of Lemma 5.1

The main issue in proving the estimate of Lemma 5.1 (and thus the lower bound (56)) is whether the random variable

$$\eta_z^t(\kappa t) = \frac{z - Z^{z,t}(\kappa t)}{\kappa t} \tag{62}$$

satisfies a large deviation principle with a convex rate function that can be characterized by $\mu(\lambda)$, almost surely with respect to Q. The variable $\eta_z^t(\kappa t)$ is the average speed of a trajectory over time interval $[0, \kappa t]$.

Definition 6.1 *For fixed $\hat{\omega} \in \hat{\Omega}$, the random variables $\eta_z^t(\kappa t)$ satisfy a* **large deviation principle** *with a convex rate function $S(c)$ if there exists a convex function $S(c)$, independent of $z \in \mathbb{R}^n$, such that*

(i) For each $s \geq 0$, the set $\Phi(s) = \{c \in \mathbb{R} | \; S(c) \leq s\}$ is compact.

(ii) For any $\delta, h > 0$, there exists $t_0 > 0$ such that for all $t > t_0$

$$P\left(d(\eta_z^t(\kappa t), \Phi(s)) > \delta\right) \leq e^{-\kappa t(s-h)}.$$

(iii) For any $\delta, h > 0$, there exists $t_0 > 0$ such that for all $t > t_0$

$$P\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) \geq e^{-\kappa t(S(c)+h)}. \tag{63}$$

If such a function $S(c)$ exists, it might depend on the parameter $\kappa \in (0, 1]$, and it might depend on $\hat{\omega} \in \hat{\Omega}$. However, we will show that

Theorem 6.1 *Almost surely with respect to Q, the random variables $\eta_z^t(\kappa t)$ satisfy a large deviation principle (with respect to $P^{z,t}$) with a convex rate function $S(c)$ that is independent of κ and $\hat{\omega} \in \hat{\Omega}$.*

We postpone the proof for the moment while we finish the proof of Lemma 5.1. By Proposition 1.1, the quantity $\mu(\lambda)$ is well defined and is almost surely constant with respect to Q for $\lambda \in \mathbb{R}^n$, independently of κ. Since, by our assumption of Theorem 6.1, the variables $\eta_z^t(\kappa t)$ have a convex rate function, it follows (see Section 5.1 of [17]) that the rate function $S(c)$ is the same convex function defined by (8):

$$S(c) = \sup_{\lambda \in \mathbb{R}^n} [c \cdot \lambda - \mu(\lambda)]. \tag{64}$$

Thus, our use of the notation $S(c)$ in Definition 6.1, Theorem 6.1, and (8) anticipates this equivalence. The characterization (64) does not hold if the rate function is not convex, in which case the Legendre transform of μ is equal to the convex envelope of the rate function. Let us emphasize that $S(c)$ is independent of $\kappa \in (0,1]$ and $\hat{\omega} \in \hat{\Omega}$, although the constants t_0 in Definition 6.1 may depend on $\kappa, \hat{\omega}$.

Now, by definition of $S(c)$,

$$\liminf_{t\to\infty} \frac{1}{\kappa t} \log \inf_{z\in\mathbb{R}^n} P^{z,t}\left\{\eta_z^t(\kappa t) \in U_\delta(c)\right\} \geq -S(c) > -\infty, \tag{65}$$

and the lower bound (52) of Lemma 5.1 is

$$\liminf_{t\to\infty} \frac{1}{t} \log \inf_{c\in K} u(ct,t) \geq f'(0) - \max_{c\in K} S(c). \tag{66}$$

To prove the lower bound we now use the Feynman-Kac formula to relate (65) to (66), as in the arguments of Freidlin in Lemma 7.3.2 in [16]. The compactness of K implies that it suffices to show that given any $\epsilon > 0$, and any c for which $S(c) - f'(0) > 0$,

$$\liminf_{t\to\infty} \left(\frac{1}{t} \log \inf_{\tilde{c}\in U_\delta(c)} u(\tilde{c}t,t)\right) \geq f'(0) - S(c) - \epsilon \tag{67}$$

for $\delta > 0$ sufficiently small. Without loss of generality, we assume that the initial data is the characteristic function of a small ball centered at the origin:

$$u_0(z) \geq \chi_{U_\delta(0)}(z) \tag{68}$$

for some $\delta > 0$. We define q to be the limit on the left hand side of (67):

$$q = \liminf_{t\to\infty} \left(\frac{1}{t} \log \inf_{z\in U_{\delta t}(ct)} u(z,t)\right).$$

We also assume that $S(c) - f'(0) > 0$.

Step 1: The first step is essentially the same as in [16]. Suppose for the moment that q is finite. By the representation (55) we have for any $\kappa \in (0,1]$

$$\inf_{\tilde{c}\in U_\delta(c)} u(t\tilde{c},t) \geq \inf_{\tilde{c}\in U_\delta(c)} E\left[e^{\int_0^{\kappa t} \zeta(t-s,u(Z(s),t-s))ds} u(Z(t-\kappa t), t-\kappa t)\chi_A\right] \tag{69}$$

for any set $\mathcal{F}_{s\le t}$-measurable set A. Recall that when $u \le h$, the reaction rate $\zeta(u)$ is bounded below by $\zeta_h > 0$. If we choose A to be the set of paths satisfying both

$$Z^{z,t}(\kappa t) \in U_{(1-\kappa)\delta t}((1-\kappa)tc) \tag{70}$$

and

$$u(Z^{z,t}(s), t-s) \le h \ \text{ for all } \ s \in [0, \kappa t], \tag{71}$$

then from (69) and the assumption that q is finite we have a lower bound

$$q \ \ \ge \ \ \zeta_h + \liminf_{t\to\infty} \frac{1}{\kappa t} \log \inf_{\tilde{c}\in U_\delta(c)} P(A), \tag{72}$$

provided that the limit on the right also exists and is finite.

Step 2: Now we bound the right hand side of (72) and show how it relates to (65). Since we have assumed that $S(c) - f'(0) > 0$, Theorem 1.1 implies that there is δ sufficiently small so that for any $h \in (0,1)$ there is a constant $t_0 > 0$, depending on h, such that

$$u(c't, t) \le h \quad \text{for all} \ \ c' \in U_{6\delta}(c), \ t \ge t_0.$$

Now if $\kappa < 1/2$ and

$$\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - (t-s)c| \le 3\delta t, \tag{73}$$

then (71) is achieved along such paths when $t > 2t_0$. Next, if $\tilde{c} \in U_\delta(c)$ is written $\tilde{c} = c + \delta e_1$ for some $e_1 \in \mathbb{R}^n$ with $|e_1| < 1$, then define $\hat{c} = c + 2\delta e_1$. Then for any $|e_2| < 1$

$$\tilde{c}t - \kappa t\hat{c} + \kappa t\delta e_2 \in U_{(1-\kappa)\delta t}((1-\kappa)ct). \tag{74}$$

It follows that for each $\tilde{c} \in U_\delta(c)$ there is a $\hat{c} \in U_{2\delta}(c)$ such that (70) is achieved whenever $\eta_z^t(\kappa t) \in U_\delta(\hat{c})$, where η is defined by (6). This gives us a lower bound on $P(A)$ in terms of the $\eta_z^t(\kappa t)$, the average speed of a trajectory over $[0, \kappa t]$:

$$\inf_{\tilde{c}\in U_\delta(c), z=ct} P(A) \ge \tag{75}$$
$$\inf_{\hat{c}\in U_{2\delta}(c), z=\hat{c}t} P\left(\sup_{s\in[0,\kappa t]} |Z^{z,t}(s) - (t-s)c| \le 3\delta t, \ \ \eta_z^t(\kappa t) \in U_\delta(\hat{c}) \right)$$

For κ sufficiently small, $\kappa < (2\delta)/(3\max(1,|c|))$, we see that

$$\sup_{\hat{c}\in U_{2\delta}(c),z=\hat{c}t} P\left(\sup_{s\in[0,\kappa t]}|Z^{z,t}(s)-(t-s)c|\geq 3\delta t\right)\leq$$
$$\leq \sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\kappa t]}|Z^{z,t}(s)-z|\geq \delta t/3\right).$$

By Corollary 2.1 there are constants $K_1, K_2 > 0$ independent of κ such that (except possibly on a set of Q-measure zero)

$$\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\kappa t]}|Z^{z,t}(s)-z|\geq \delta t/3\right)\leq K_1 e^{-K_2\delta^2 t/\kappa} \tag{76}$$

for t sufficiently large, depending on $\hat{\omega}$. Therefore, for any $M > 0$, by choosing κ arbitrarily small, we can make $K_2\delta^2/\kappa^2 > M$, so that

$$\limsup_{t\to\infty}\frac{1}{\kappa t}\log(\sup_{z\in\mathbb{R}^n} P\left(\sup_{s\in[0,\kappa t]}|Z^{z,t}(s)-(t-s)c|\geq 2\delta t\right))\leq$$
$$\leq \lim_{t\to\infty}\frac{1}{\kappa t}\log(K_1 e^{-K_2\delta^2 t/\kappa})\leq -M.$$

Therefore, from (72) and (75) we now see that for κ sufficiently small,

$$q \;\; \geq \;\; \zeta_h + \liminf_{t\to\infty}\frac{1}{\kappa t}\inf_{\hat{c}\in U_{2\delta}(c),z\in\mathbb{R}^n} P\left(\eta_z^t(\kappa t)\in U_\delta(\hat{c})\right) \tag{77}$$

provided that the limit on the right is finite and bounded below, independently of κ. However, this follows immediately from Theorem 6.1 and the lower bound (65), since $S(c)$ is independent of κ. Then (67) follows by letting $h \to 0$ so that $\zeta_h \to f'(0)$.

Step 3: It remains to establish the initial claim that $q > -\infty$, almost surely with respect to Q. To see this, define for any $c \in \mathbb{R}^n$

$$\hat{q}_\delta(c,t) = \inf_{z\in U_\delta(tc)} P^{z,t}\left(Z^{z,t}(t)\in U_\delta(0)\right), \tag{78}$$

a random variable over $\hat{\Omega}$. We will show that for any bounded set $\Lambda \subset \mathbb{R}^n$, there is a finite constant $K_1 > 0$ such that the limit

$$\liminf_{t\to\infty}\frac{1}{t}\log\hat{q}_\delta(c,t)\geq -K_1 \tag{79}$$

holds uniformly over $c \in \Lambda$. This immediately implies that $q > -\infty$.

For $z \in U_\delta(ct)$, let us write $X^{z,t}(s)$ as

$$X^{z,t}(s) = x + I^{z,t}(s) + W_1^0(s)$$

where $I^{z,t}$ is the first integral term in (5) and $W_1^0(0) = 0$. Note that the integral $I^{z,t}$ is independent of W_1, due to the shear structure of the flow. For simplicity of notation, we will write $W_2(t) \in U_\delta(0)$ to mean that $|W_2(t)| < \delta$, even though $U_\delta(0)$ generally denotes an n-dimensional ball. First, we claim that for $z \in U_\delta(tc)$

$$P\left(Z_t^{z,t} \in U_\delta(0) |\ W_2^y(t) \in U_{\delta/2}(0)\right) \geq e^{-(3|c|^2+1)t} \tag{80}$$

for t sufficiently large. For $\hat{\omega} \in \Omega$ fixed, let $M > 0$ and define the set

$$A_M = A_M(t) = \{w \in \Omega \,|\, \sup_{z \in D, s \in [0,t]} |I^{z,t}(s)| \leq Mt\}.$$

Using the fact that $W_1(s)$ and $I^{z,t}(s)$ are independent, we see that for $z \in U_\delta(tc)$,

$$\begin{aligned}
&P(Z_t^{z,t} \in U_\delta(0) |\ W_2^y(t) \in U_{\delta/2}(0)) \geq \\
&\quad \geq P(W_1^0(t) \in U_{\delta/2}(0) - I^{z,t}(t) - x |\ W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq P(W_1^0(t) \in U_{\delta/2}(0) - I^{z,t}(t) - x,\ A_M |\ W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq \inf_{|\hat{e}_1|, |\hat{e}_2| \leq 1} P(W_1^0(t) \in U_{\delta/2}(0) + \hat{e}Mt + ct + \delta \hat{e}_2) P(A_M |\ W_2^y(t) \in U_{\delta/2}(0)) \\
&\quad \geq \frac{\delta}{\sqrt{2\pi t}} e^{-\frac{(Mt+|c|t+\delta)^2}{2t}} P(A_M |\ W_2^y(t) \in U_{\delta/2}(0)).
\end{aligned} \tag{81}$$

By Lemma 2.5, $P(A_M |\ W_2^y(t) \in U_{\delta/2}(0)) \geq 1/2$ for t sufficiently large, depending on $\hat{\omega}$ and M. Moreover, if we chose $M = \max(1, |c|)$, then $P(A_M |\ W_2^y(t) \in U_{\delta/2}(0)) \geq 1/2$ for t sufficiently large, independent of c. Using this in (81) establishes (80), for t sufficiently large. Now, we can bound $\hat{q}_\delta$:

$$\begin{aligned}
\hat{q}_\delta(c,t) &= \inf_{z\in U_\delta(tc)} P\left(Z_t^{z,t}\in U_\delta(0)\right)\\
&\geq \inf_{z\in U_\delta(tc)} P\left(Z_t^{z,t}\in U_\delta(0),\ W_2(t)\in U_{\delta/2}(0)\right)\\
&= \inf_{z\in U_\delta(tc)} E\left[\chi_{W_2(t)\in U_{\delta/2}(0)} P(Z_t^{z,t}\in U_\delta(0)|\ W_2(t)\in U_{\delta/2}(0))\right]\\
&\geq \inf_{z\in U_\delta(tc)} E\left[\chi_{W_2(t)\in U_{\delta/2}(0)} e^{-(3|c|^2+1)t}\right]\\
&\geq e^{-(3|c|^2+1)t} \inf_{z\in U_\delta(tc)} P\left(W_2(t)\in U_{\delta/2}(0)\right)\geq e^{-(4|c|^2+1)t} \qquad (82)
\end{aligned}$$

for t sufficiently large. Therefore,

$$\liminf_{t\to\infty}\frac{1}{t}\log\hat{q}_\delta(c,t)\geq -(4|c|^2+1) \qquad (83)$$

is finite almost surely with respect to Q. For any bounded set $\Lambda\subset\mathbb{R}^n$, we can choose K_1 to be

$$K_1 = 1+\sup_{c\in\Lambda} 4|c|^2<\infty. \qquad (84)$$

This establishes the claim (79). Having shown that q is finite, we have completed the proof of Lemma 5.1. □

For use in the next section, we now show that for all $t>0$, $\log(\hat{q}_\delta(c,t))$ is integrable with respect to Q. Note that bound (82) holds for t sufficiently large, depending on $\hat{\omega}$, so more work is needed in order to establish the integrability of $\log(\hat{q}_\delta(c,t))$.

Lemma 6.1 *For each $c\in\mathbb{R}^n$,*

$$\sup_{t>1} E_Q[|\frac{1}{t}\log\hat{q}_\delta(c,t)|]<\infty. \qquad (85)$$

Proof: Using (81) we see that

$$\begin{aligned}
&\frac{1}{t}\log\hat{q}_\delta(c,t)\geq\frac{1}{t}\log P\left(Z_t^{z,t}\in U_\delta(0),\ W_2(t)\in U_{\delta/2}(0)\right)\geq\\
&\quad\geq\frac{1}{t}\log\left(P\left(W_2(t)\in U_{\delta/2}(0)\right)\frac{\delta}{\sqrt{2\pi t}}e^{-\frac{((1+2|c|)t+\delta)^2}{t}}P(A_M|\ W_2^y(t)\in U_{\delta/2}(0))\right)\\
&\quad\geq -C_1+\frac{1}{t}\log\left(e^{-\frac{((1+2|c|)t+2\delta)^2}{t}}P(A_M|\ W_2^y(t)\in U_{\delta/2}(0))\right)
\end{aligned}$$

for a constant $C_1 > 0$ depending only on c and δ, for $t \geq 1$. This constant is uniformly bounded for c in a bounded set and $\delta > 0$ fixed. Let $\hat{g}$ be the term inside the logarithm:

$$\hat{g} = e^{-\frac{(Mt+|c|t+2\delta)^2}{2t}} P(A_M|\, W_2^y(t) \in U_{\delta/2}(0)),$$

a random variable with respect to Q. Then for $\alpha \geq 2C_1$,

$$\begin{aligned} Q\left(\frac{1}{t}\log \hat{q}_\delta(c,t) \leq -\alpha\right) &\leq Q\left(\frac{1}{t}\log \hat{g} \leq -\alpha/2\right) \\ &= Q\left(\hat{g} \leq e^{-\alpha t/2}\right) \qquad (86) \\ &= Q\left(P(A_M|\, W_2^y(t) \in U_{\delta/2}(0)) \leq e^{-\alpha t/2} e^{\frac{(Mt+|c|t+2\delta)^2}{2t}}\right). \end{aligned}$$

Also, from Lemma 2.5,

$$Q\left(P(A_M|\, W_2^y(t) \in U_{\delta/2}(0)) \leq 1 - e^{-K_2 M^2 t/2}\right) \leq K_1 e^{-K_2 M^2 t/2}. \qquad (87)$$

It is easy to see that there exist constants $K_3, K_4 > 0$ independent of t such that whenever $t \geq 1$, $M = K_3\sqrt{\alpha}$, and $\alpha \geq K_4|c|^2$, we have

$$e^{-\alpha t/2} e^{\frac{(Mt+|c|t+2\delta)^2}{2t}} \leq 1/2 \leq 1 - e^{-K_2 M^2 t/2}.$$

By combining (86) and (87), we now conclude that

$$Q\left(\frac{1}{t}\log \hat{q}_\delta(c,t) \leq -\alpha\right) \leq K_1 e^{-K_2 K_3^2 \alpha t/2} \qquad (88)$$

whenever $\alpha \geq K_4|c|^2$ and $t \geq 1$. It follows that for $t \geq 1$

$$\begin{aligned} E_Q[|\frac{1}{t}\log \hat{q}_\delta(c,t)|] &= \int_0^\infty Q\left(|\frac{1}{t}\log \hat{q}_\delta(c,t)| \geq \alpha\right) d\alpha \\ &\leq K_4|c|^2 + \int_{K_4|c|^2}^\infty K_1 e^{-K_2 K_3^2 \alpha t/2}\, d\alpha < \infty. \qquad (89) \end{aligned}$$

This is bounded uniformly in t, for $t \geq 1$. □

7 Proof of Large Deviation Estimates

In this section we prove Theorem 6.1. We work first with the case $\kappa = 1$. For $c \in \mathbb{R}^n$ and $0 \leq r < s < t$, define the probability

$$q_\delta^z(c, r, t) = P^{z,t}(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) = P^{z,t}(\eta_z^t(t-r) \in U_\delta(c))$$

and

$$\begin{aligned} q_\delta^+(c, r, t) &= \sup_{z \in D} q_\delta^z(c, r, t) \\ q_\delta^-(c, r, t) &= \inf_{z \in D} q_\delta^z(c, r, t). \end{aligned}$$

The quantity $q_\delta^z(c, r, t)$ is the probability that a trajectory should have average velocity sufficiently close to c, over a given time interval. This probability depends on the starting point z, so q_δ^+ and q_δ^- are the maximum and minimum possible probabilities. The quantities q_δ^z, q_δ^+, and q_δ^- also depend on $\hat{\omega} \in \hat{\Omega}$, but we will use the sub-additive ergodic theorem to show that $(1/t) \log q_\delta^-(c, 0, t)$ converges to a finite constant, $-S_\delta(c)$, almost surely with respect to Q. From these constants we will recover the desired rate function $S(c)$ as the limit of $S_\delta(c)$ as $\delta \to 0$. Then, we will derive a Harnack-type inequality to compare $(1/t) \log q_\delta^-(c, 0, t)$ and $(1/t) \log q_\delta^+(c, 0, t)$ and show that $S(c)$ satisfies the requirements for Definition 6.1. The same analysis will extend to the case of $\kappa < 1$.

Define the events

$$\begin{aligned} A &= \left\{\omega \in \Omega |\ z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))\right\} = \left\{\eta_z^t(t-r) \in U_\delta(c)\right\} \\ B &= \left\{\omega \in \Omega |\ z - Z^{z,t}(t-s) \in U_{\delta(t-s)}(c(t-s))\right\} = \left\{\eta_z^t(t-s) \in U_\delta(c)\right\}. \end{aligned}$$

Note that event B is $\mathcal{F}_{s' \leq \tau}^t$ measurable for any $\tau \geq t - s$. Using the Markov property of the Wiener process, we find that $\log(q_\delta^-(c, s, t))$ is super-additive for each $c \in \mathbb{R}^n$ since

$$\begin{aligned}
q_\delta^-(c,r,t) &= \inf_y P(A) \geq \inf_y P(A \cap B) \\
&\geq \inf_z P(\{-Z^{z,t}(t-r) + Z^{0,z,t}(t-s) \in U_{\delta(r-s)}(c(s-r))\} \cap B) \\
&= \inf_z E\left[\chi_B P[\{-Z^{z,t}(t-r) + Z^{z,t}(t-s) \in U_{\delta(s-r)}(c(s-r))\} \mid \mathcal{F}_{t-s}]\right] \\
&= \inf_z E\left[\chi_B P[\{-Z^{z,t}(t-r) + Z^{z,t}(t-s) \in U_{\delta(s-r)}(c(s-r))\} \mid Z^{z,t}(t-s)]\right] \\
&\geq \inf_z E\left[\chi_B \inf_z P[\{z - Z^{z,s}(s-r) \in U_{\delta(s-r)}(c(s-r))\}]\right] \\
&= \inf_z P(\{z - Z^{z,s}(s-r) \in U_{\delta(s-r)}(c(s-r))\}) \inf_z P(B) \\
&= q_\delta^-(c,r,s) q_\delta^-(c,s,t).
\end{aligned}$$

Also, due to the stationarity of B with respect to t,

$$\begin{aligned}
\tau_h q_\delta^-(c,r,t) &= \tau_h \inf_y P(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\
&= \inf_y P(z - Z^{z,t+h}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\
&= q_\delta^-(c, r+h, t+h).
\end{aligned}$$

For any $\epsilon > 0$, we can bound q below by translating in z and using (78):

$$q_\delta^-(c,r,t) \geq \tau_r \hat{q}_\epsilon(c, t-r) = \inf_{z \in U_\epsilon(ct)} P(Z^{z,t}(t-r) \in U_\epsilon(cr)) \tag{90}$$

if $\epsilon < \delta(t-r)$. Hence, $\log(q_\delta^-(c,r,t))$ is integrable by (89). Kingman's ergodic theorem [23] now implies that the limit

$$\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-(c,0,n) = \sup_{n>0} \frac{1}{n} E_Q[\log q_\delta^-(c,0,n)] = -S_\delta(c) \tag{91}$$

exists and is a finite constant, Q-a.s, because of the ergodicity assumption A2.

To extend the convergence in (91) to continuous time, we employ a technique from [2] (see the proof of Theorem 2.5 therein). Let

$$g(\hat{\omega}) = \sup_{\substack{r,t \in [0,2] \\ |r-t| \geq 1}} |\log(q_\delta^-(c,r,t))|.$$

Let $\Upsilon(\hat\omega) = \sup_{z\in D, t\in[0,2]} |B(z,t)|$. Then for all $0 \le r < t \le 2$, we can bound

$$\sup_{y\in D} \left| \int_0^{t-r} b(W_2^y(\tau), t-\tau)d\tau \right| \le \Upsilon |t-r|$$

independently of the realization of W_2^y. As in (81),

$$\begin{aligned} &P(z - Z^{z,t}(t-r) \in U_{\delta(t-r)}(c(t-r))) \\ &\quad \ge \inf_{|\hat e_1|, |\hat e_2| \le 1} P(W_1^0(t-r) \in U_{\delta(t-r)/2}(0) + \hat e_1 \Upsilon(t-r) + (c+\delta\hat e_2)(t-r)) \times \\ &\qquad \times P(W_2^0(t-r) \in U_{\delta(t-r)/2}(0)) \\ &\quad \ge \frac{\delta|t-r|}{\sqrt{2\pi|t-r|}} e^{-\frac{(t-r)^2(\Upsilon+|c|+\delta)^2}{(t-r)}} \frac{\delta|t-r|}{\sqrt{2\pi|t-r|}} e^{-\frac{(t-r)^2(|c|+\delta)^2}{(t-r)}}. \end{aligned}$$

Therefore, since $r, t \in [0,2]$ and $|r-t| \ge 1$ in the definition of $g(\hat\omega)$,

$$0 \le g(\hat\omega) \le K_1 + K_2 \Upsilon^2$$

for some constants $K_1, K_2 > 0$ that depend on δ and c. Hence $g(\hat\omega)$ is integrable with respect to Q, since Υ^2 is integrable by the Borell inequality. By the super-additivity of $\log q_\delta^-(c,r,t)$,

$$\log q_\delta^-(c,0,n-1) - \tau_{n-1} g \le \log q_\delta^-(c,0,t) \le \log q_\delta^-(c,0,n+2) + \tau_n g \quad (92)$$

whenever $t \in (n, n+1)$, $n \in Z$. The ergodic theorem implies that

$$\frac{1}{N} \sum_{n=1}^N \tau_n g \to E[g] < \infty \quad (93)$$

almost surely. Therefore, $\frac{1}{n}\tau_n g \to 0$ almost surely as $n \to \infty$. It now follows from (92) that the limit along continuous time

$$\lim_{t\to\infty} \frac{1}{t} \log q_\delta^-(c,0,t) = -S_\delta(c) \quad (94)$$

holds almost surely with respect to Q.

Now we extend this conclusion to the case of $\kappa < 1$, as well. If $\kappa \in (0,1)$ and $\delta > 0$, the stationarity of $b(y,t)$ implies that

$$\frac{1}{\kappa t} \log \inf_z P\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) = \frac{1}{\kappa t} \log q_\delta^-((1-\kappa)t, t) \to -S_\delta(c) \quad (95)$$

in distribution (with respect to Q) as $t \to \infty$, but this does not immediately imply pointwise, almost-sure convergence. However, the collection of sets $\{[(1-\kappa)t, t]\}_{t\geq 0}$ is a regular family of sets in the sense of [2], since $0 \leq |[0,t]| \leq C|[(1-\kappa)t, t]|$ for all t, with $C = \frac{1}{\kappa}$. It now follows from Theorem 2.8 of [2] that $\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-((1-\kappa)n, n)$ converges almost surely along any rational sequence. Therefore, (95) implies that, indeed, this limit is $-S_\delta(c)$

$$\lim_{n\to\infty} \frac{1}{n} \log q_\delta^-((1-\kappa)n, n) = -S_\delta(c), \quad Q - a.s., \tag{96}$$

Finally, this convergence can be extended to continuous time, using the same technique as in (92).

For each $c \in \mathbb{R}^n$, $S_\delta(c)$ can be bounded above independently of $\delta > 0$ using (79) and (90). From the definition, it is clear that $S_\delta(c) \geq 0$ for all δ, and that

$$S_{\delta_1}(c_1) \leq S_{\delta_2}(c_2) \tag{97}$$

whenever $U_{\delta_1}(c_1) \supset U_{\delta_2}(c_2)$. In particular, $S_{\delta_1}(c) \leq S_{\delta_2}(c)$ for $\delta_1 > \delta_2$, $c \in \mathbb{R}^n$. Therefore, we define for each $c \in \mathbb{R}^n$

$$S(c) = \lim_{\delta\to 0} S_\delta(c) = \sup_{\delta>0} S_\delta(c) \in [0, +\infty).$$

This will be the rate function described in the theorem.

Lemma 7.1 *For all $\delta > 0$, the functions $S_\delta(c)$ are continuous and convex in c. Also, $S(c)$ is continuous and convex in c.*

Proof: The continuity and convexity of $S(c)$ follows immediately from the fact that it is the finite, pointwise limit of the continuous, convex functions $S_\delta(c)$. The convexity of $S_\delta(c)$ follows from the Markov property of the process $Z^{z,t}$, as follows.

Let $p \in [0,1]$ and $c_0 = pc_1 + (1-p)c_2$. Let $t > 0$ and denote $t_1 = pt$, $t_2 = (1-p)t$. Then we see that

$$\begin{aligned} q_\delta^-(c_0, 0, t) &= \inf_z P\left(z - Z^{z,t}(t) \in U_{\delta t}(c_0 t)\right) \geq \\ &\geq \inf_z P\left(z - Z^{z,t}(t) \in U_{\delta t}(c_0 t)\ ,\ \ z - Z^{0,y,t}(t_1) \in U_{\delta t_1}(c_1 t_1)\right) \\ &\geq \inf_z P\left(z - Z^{z,t-t_1}(t_2) \in U_{\delta t_2}(c_2 t_2)\right) \inf_z P\left(z - Z^{z,t}(t_1) \in U_{\delta t_1}(c_1 t_1)\right) \\ &= q_\delta^-(c_2, 0, t_2) q_\delta^-(c_1, t_2, t). \end{aligned}$$

Hence

$$-\frac{1}{t}\log q_\delta^-(c_0,0,t) \leq \tag{98}$$
$$\leq -\frac{1}{t}\log q_\delta^-(c_2,0,(1-p)t) - \frac{1}{t}\log q_\delta^-(c_1,(1-p)t,t)$$

By the preceding arguments, both terms on the right converge (Q-a.s.) as $t \to \infty$ to the constants $(1-p)S_\delta(c_2)$ and $pS_\delta(c_1)$. Therefore, we infer that

$$S_\delta(c_0) \leq (1-p)S_\delta(c_2) + pS_\delta(c_1).$$

So, $S_\delta(c)$ is convex and must also be continuous in c, since it is finite for every $c \in \mathbb{R}^n$. □

This establishes the existence and convexity of the function $S(c)$. Part (iii) of the Definition 6.1 follows from the definition of $S_\delta(c)$ and the fact that $S_\delta(c) \nearrow S(c)$.

To finish the proof of Theorem 6.1, we must establish a Harnack-type inequality to relate the probabilities

$$P(z - Z^{z,t}(t) \in U_{\delta t}(ct)) \quad \text{and} \quad P(z' - Z^{z',t}(t) \in U_{\delta t}(ct))$$

corresponding to different starting points $z, z' \in D$. This will allow us to remove the $\inf_z$ in the definition of q and $S(c)$ and to establish parts (i) and (ii) of Definition 6.1. Unfortunately, the quantity $\log q_\delta^+$ is not sub-additive or super-additive, so we cannot use the ergodic theorem to show that $\frac{1}{t}\log q_\delta^+ \to -S_\delta(c)$, almost surely, as is the case with $\log q_\delta^-$.

Note that it is not true that two trajectories starting close together will remain close. For example, suppose the flow is $B(z,t) = (\sin(y), 0)$. Then if $z = (0,0)$ and $z' = (0,\pi)$, the X components of the trajectories will satisfy

$$X^{z,t}(t) - X^{z',t}(t) = \int_0^t \sin(W_2(s)) - \sin(\pi + W_2(s))\, ds = 2\int_0^t \sin(W_2(s))\, ds,$$

which we expect will grow like $\sqrt{t}$. Nevertheless, the estimate we need must only relate the distributions of the two processes, not the individual trajectories for fixed realizations of W. We prove the following lemma:

Lemma 7.2 *There are constants $K_1, K_2, K_3, K_4 > 0$ such that for all $\kappa \in (0,1]$, $c \in \mathbb{R}^n$, $\epsilon > 0$, and $\delta > 0$,*

$$\inf_z P\left(\eta_z^t(\kappa t) \in U_{(1+\epsilon)\delta}(c)\right) \geq K_4 e^{-K_3\epsilon\delta t} \sup_z P\left(\eta_z^t(\kappa t) \in U_\delta(c)\right) - K_1 e^{-K_3\epsilon^2\delta^2\kappa^2 t^2}$$

Proof of Lemma 7.2: For clarity we let $\kappa = 1$. Extension to $\kappa < 1$ is straightforward, as in the proof of Lemma 2.5. Because of the shear flow structure, $(Z^{z,t}(t) - z)$ and $\eta_z^t(t)$ are independent of x (where $z = (x, y)$), and the component $Y^{z,t}$ is just a Wiener process. These two facts will enable us to estimate the cost of switching the initial point from z to z'.

For $M > 0$ and $s \in (0, t)$ to be chosen, the Markov property of the process implies that

$$\begin{aligned} P(Z^{z,t}(t) \in U_{\delta t+2M}(z+ct), |Z^{z,t}(s) - z| \leq M) = \\ = \int_{|\hat{z}-z|\leq M} \rho(z,t,\hat{z},s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+2M}(z+ct))\, d\hat{z} \\ \geq \int_{|\hat{z}-z|\leq M} \rho(z,t,\hat{z},s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{z} \end{aligned} \tag{99}$$

where $\rho(z, t, \hat{z}, r)$ denotes the transition density of the Markov process $Z^{z,t}(r)$. The term $P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))$ inside the integral is independent of $\hat{x}$ (where $\hat{z} = (\hat{x}, \hat{y})$). Using this fact, we will bound the integral in (99) by first integrating over $\hat{x}$. Since $Y^{z,t}$ is just a Wiener process, the marginal distribution of ρ with respect to $\hat{y}$ is Gaussian:

$$\int_R \rho(z,t,\hat{z},s) d\hat{x} = \frac{1}{\sqrt{2\pi s}} e^{-\frac{|\hat{y}-y|^2}{2s}} = F(\hat{y}-y,s).$$

Therefore,

$$\int_{|x-\hat{x}|\leq M} \rho(z,t,\hat{z},s) d\hat{x} = \frac{1}{\sqrt{2\pi|s|}} F(\hat{y}-y,s) - P^{z,t}(A_M|\, Y^{z,t}(s) = \hat{y})$$

where $A_M = \{\omega|\ |X^{z,t}(s) - x| \geq M\}$. This set will turn out to be very small. For $\epsilon > 0$, let $s = 1$ and $M = \epsilon\delta t/2$ (use $M = \epsilon\delta\kappa t/2$ when $\kappa < 1$). Therefore, integrating only in $\hat{x}$, we have

$$\begin{aligned} \int_{|\hat{z}-z|\leq M} \rho(z,t,\hat{z},s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{z} \geq \\ \geq \int_{|\hat{y}-y|\leq M/2} F(\hat{y}-y,s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{y} - G_1 \end{aligned} \tag{100}$$

where

$$G_1 = M \sup_{\hat{y}} P^{z,t}(A_M|\, Y^{z,t}(s) = \hat{y}).$$

Now we switch the initial point from z to z', such that $|y-y'| \leq L$. Then, continuing from (100), we have

$$\int_{|\hat{y}-y|\leq M/2} F(\hat{y}-y,s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{y} - G_1 \tag{101}$$
$$\geq Ce^{-M/2} \int_{|\hat{y}-y'|\leq \frac{M}{2}-L} F(\hat{y}-y',s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{y} - G_1$$
$$\geq Ce^{-M/2} \int_{|\hat{z}-z'|\leq \frac{M}{2}-L} \rho(z',t,\hat{z},s) P^{\hat{z},s}(Z^{\hat{z},s}(t-s) \in U_{\delta t+M}(\hat{z}+ct))\, d\hat{z} - G_1$$
$$\geq Ce^{-M/2} P(Z^{z',t}(t) \in U_{\delta t}(z'+ct), |Z^{z',t}(s)-z'| \leq \frac{M}{2} - L) - G_1$$

Combining (99) and (101) we have

$$P(Z^{z,t}(t) \in U_{\delta t+2M}(z+ct)) \geq \tag{102}$$
$$\geq Ce^{-M/2} P(Z^{z',t}(t) \in U_{\delta t}(z'+ct), |Z^{z',t}(s)-z'| \leq \frac{M}{2} - L) - G_1$$
$$\geq Ce^{-M/2} P(Z^{z',t}(t) \in U_{\delta t}(z'+ct)) - Ce^{-M/2} G_2 - G_1$$

where

$$G_2 = \sup_{\hat{z}} P^{z,t}(A_{\frac{M}{2}-L}).$$

It follows from Lemma 2.5, that

$$G_1, G_2 \leq K_1 e^{-K_2 M^2} = K_1 e^{-K_2 \epsilon^2 \delta^2 t^2/4}$$

for t sufficiently large. Now the lemma follows from (102). □

Since $\epsilon > 0$ is arbitrary, the lemma implies that

Corollary 7.1 *For all $\epsilon, \delta > 0$, $c \in \mathbb{R}^n$, and $\kappa \in (0,1]$,*

$$\begin{aligned} -\lim_{t\to\infty} \frac{1}{\kappa t} \log q_\delta^-(c,(1-\kappa)t,t) &= S_\delta(c) \\ &\geq -\limsup_{t\to\infty} \frac{1}{\kappa t} \log q_\delta^+(c,(1-\kappa)t,t) \\ &\geq S_{(1+\epsilon)\delta}(c) \end{aligned}$$

almost surely with respect to Q.

Now, using this estimate, we can establish parts (i) and (ii) from Definition 6.1. From Lemma 2.1 there are constants $K_1, K_2 > 0$ such that for t sufficiently large,

$$P(|\eta_z^t(\kappa t)| \geq |c|) \leq K_1 e^{-K_2|c|^2 t/\kappa}. \tag{103}$$

This implies that $\lim_{|c|\to\infty} S(c)/|c| = +\infty$. Hence, $\Phi(s)$ is a bounded set, for each $s \geq 0$. By continuity of $S(c)$, $\Phi(s)$ is compact. Let A be the set

$$A = \{c \in \mathbb{R}^n | \ d(c, \Phi(s)) > \delta\}.$$

We must show that for any fixed $\delta > 0$, $h > 0$,

$$P(\eta_z^t(\kappa t) \in A) \leq e^{-\kappa t(s-h)}, \tag{104}$$

for $t > 0$ sufficiently large. Because of the bound (103), it suffices to show that (104) holds with A replaced by any compact subset A' of A (because $K_2|c|^2 > \kappa^2 s$ when $|c|$ is sufficiently large).

We have shown that for $c \in \mathbb{R}^n$, there is a set $V(c) \subset \hat{\Omega}$ such that $Q(V) = 0$ and the convergence (94) holds for all $\hat{\omega} \in \hat{\Omega} \setminus V(c)$. To obtain convergence on a set independent of c, we define the set $\hat{V} \subset \hat{\Omega}$ by

$$\hat{V} = \bigcup_{c\in\mathbb{Q}^n} V(c), \tag{105}$$

which has measure zero. Therefore, for all $c \in \mathbb{Q}^n$, (94) holds for all $\hat{\omega} \in \hat{\Omega}\backslash\hat{V}$.

Now, we claim that we can choose $\epsilon > 0$ small enough so that $\epsilon < \delta$ and

$$\inf_{c'\in A'} S_\epsilon(c') > s - \frac{h}{2}. \tag{106}$$

If this were not so, then there must be a sequence $\epsilon_k \to 0$ and $\{c_k\} \subset A'$ such that $S_{\epsilon_k}(c_k) \leq s - h/2$. Because A' is compact, there must be a subsequence c_{k_n} converging to some $c_0 \in A'$. Since $c_0 \in A'$, $S_\epsilon(c_0) > s - h/4$ whenever ϵ is less than some $\epsilon_0 > 0$. However, $U_{\epsilon_k}(c_k) \subset U_{\epsilon_0}(c_0)$ for k sufficiently large. It follows from (97) that $S_{\epsilon_k}(c_k) > s - h/4$, for k sufficiently large. This is a contradiction, so the claim must hold.

Having chosen ϵ to satisfy (106), cover A' with a finite number of balls having size $\frac{\epsilon}{2}$:

$$A' \subset \bigcup_{n=1}^{N} U_{\epsilon/2}(c_n)$$

for some finite set $\{c_n\}_{n=1}^N \subset A' \cap \mathbb{Q}^n$. Therefore,

$$\begin{aligned} P(\eta_z^t(\kappa t) \in A') &\leq \sum_{n=1}^N P(\eta_z^t(\kappa t) \in U_{\epsilon/2}(c_n)). \\ &\leq \sum_{n=1}^N q_{\epsilon/2}^+((1-\kappa), t, c_n). \end{aligned}$$

Now we apply Corollary 7.1 and inequality (106) to conclude that for all $\hat{\omega} \in \hat{\Omega} \setminus \hat{V}$

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{\kappa t} \log P(\eta_z^t(\kappa t) \in A') \leq \\ &\leq \limsup_{t\to\infty} \frac{1}{\kappa t} \log \sum_{n=1}^N q_{\epsilon/2}^+((1-\kappa)t, t, c_n) \\ &\leq - \inf_{c' \in A'} S_{2\epsilon/3}(c') < -(s - \frac{h}{2}) \end{aligned}$$

for t sufficiently large. Thus, (104) holds almost surely for t sufficiently large. This proves part (ii) of Definition 6.1 for $\kappa \in (0, 1]$ completes the proof of Theorem 6.1. □

8 Estimating c^* With the Variational Formula

In this final section, we use formula (10) to derive some analytical bounds on c^*. Throughout this section we will assume that the shear $b(y, t)$ has the form

$$b(y,t) = \sum_{j=1}^N b_1^j(y) b_2^j(t) \tag{107}$$

where $b_1^j(y)$ are Lipschitz continuous and periodic in y, and $b_2^j(t)$ are stationary centered Gaussian fields such that Assumptions A1-A6 are satisfied. Also, we assume that the initial data is independent of y: $u(z, 0, \hat{\omega}) = u_0(x)$, and we consider only front propagation in the direction $k = (1, \mathbf{0})$, which is aligned with the direction of the shear. Under these assumptions, the random solutions $u(z, t, \hat{\omega})$ will be periodic in y for all time, and the maximum principle implies that the speed in the k direction is given by

$$c^*(k) = \sup_{c \in \Gamma} c \cdot k$$

where $\Gamma = \{c \in \mathbb{R}^n | \; S(c) - f'(0) = 0\}$. Using the definition of $S(c)$ and the fact that Γ is a convex set, one can see that this supremum is achieved at a point $\hat{c} \in \mathbb{R}^n$ that satisfies

$$S(\hat{c}) - f'(0) = 0 = \sup_{\lambda_1 > 0} \hat{c} \cdot k\lambda_1 - \mu(\lambda_1 k) - f'(0).$$

Consequently, the variational formula for the front speed reduces to the one-dimensional optimization problem

$$c^*(k) = \inf_{\lambda_1 > 0} \frac{\mu(\lambda_1 k) + f'(0)}{\lambda_1},$$

where μ is determined by the limit

$$\mu(\lambda k) = \frac{\lambda^2}{2} + \lim_{t \to \infty} \frac{1}{t} \log \phi(y,t) \tag{108}$$

and

$$\phi_t = \frac{1}{2}\Delta\phi - \lambda b(y,t)\phi, \qquad \phi|_{t=0} \equiv 1. \tag{109}$$

In the case of time-independent flows, $c^*(k)$ is the minimal speed of the traveling wave in the direction k, as described in the work of Berestycki and Nirenberg [7]. Notice that (109) is the same as equation (46). As before, we will use $\rho(\lambda)$ to denote the limit on the right side of (108).

We consider the scaling $b(y,t) \mapsto \delta b(y,t)$ and the resulting enhancement of the corresponding speed $c^* = c^*(\delta)$. It is known [32] that if $b(y,t)$ is periodic in both space and time that $c^*(\delta) = c^*(0) + O(\delta^2)$ for δ small and $c^*(\delta) = c^*(0) + O(\delta)$ for δ as $\delta \to \infty$. The following theorem gives analytical upper bounds consistent with this asymptotic behavior.

Theorem 8.1 (Bounds on c^*) *For all $\delta \geq 0$, $c^*(\delta)$ satisfies the bounds*

(i) $c^*(\delta) \geq c^*(0)$.

(ii) $c^*(\delta) = c^*(0)$ *if* $b(y,t) = b(t)$.

(iii) $c^*(\delta) \leq c^*(0) + \delta \sum_{j=1}^{N} \|b_1^j\|_\infty E_Q[|b_2^j|]$.

(iv) $c^*(\delta) \le c^*(0)\sqrt{1+\delta^2 p_1}$.

From (iv), we also have

$$c^*(\delta) \le c^*(0)(1+\frac{\delta^2 p_1}{2}) + O(\delta^3)$$

when δ is small.

We also have a linear lower bound on the growth of $c^*(\delta)$ as $\delta \to \infty$.

Theorem 8.2 (Linear growth of c^*) *The non-random constant $\bar{C} \in [0,+\infty)$ defined by*

$$\liminf_{\delta\to\infty} \frac{c^*(\delta)}{\delta} = \bar{C} \tag{110}$$

is equal to zero if and only if $b(y,t) \equiv b(t)$.

Proof of Theorem 8.1: The first bound (i) follows from (47) and the formula

$$c^*(\delta) = \inf_{\lambda>0} \frac{\mu(\lambda k) + f'(0)}{\lambda} \ge \inf_{\lambda>0} \frac{\lambda}{2} + \frac{f'(0)}{\lambda} = c^*(0).$$

When ϕ satisfies (109), the function $\psi = \log(\phi)$ satisfies

$$\begin{aligned} \psi_t &= \frac{1}{2}\Delta\psi + \frac{1}{2}|\nabla_y \psi|^2 - \lambda b(y,t) \\ \psi(y,0) &\equiv 0. \end{aligned} \tag{111}$$

Integrating (111) over $D \times [0,t]$, we have

$$\frac{1}{t}\int_D \psi(y,t)\,dy = \frac{1}{2t}\int_0^t \int_D |\nabla_y \psi|^2\,dy\,dt - \frac{\lambda}{t}\int_0^t\int_D b(y,t)\,dy\,dt. \tag{112}$$

Now let $t \to \infty$:

$$\begin{aligned} \rho(\lambda) = \lim_{t\to\infty}\frac{1}{t}\int_D \psi(y,t)\,dy &\ge -\lambda \lim_{t\to\infty}\frac{1}{t}\int_0^t\int_D b(y,t)\,dy\,dt \\ &= -\lambda E_Q\left[\int_D b(y,t)\,dy\right] = 0, \end{aligned}$$

almost surely with respect to Q. If $b(y,t) = b(t)$, then the first integral on the right hand side of (112) vanishes since $|\nabla_y \psi|^2 \equiv 0$. Then taking the limit as $t \to \infty$, we have equality:

$$\rho(\lambda) = E_Q\left[\int_D \delta b(t)\,dy\right] = 0.$$

Hence $c^*(\delta) = c^*(0)$. This proves part (ii).

For the linear upper bound (iii), note that

$$\begin{aligned} \frac{1}{t}\log E[e^{\lambda\delta\int_0^t b(W(s),t-s)\,ds}] &\le \frac{1}{t}\log E[e^{\lambda\delta\sum_j \|b_1^j\|_\infty \int_0^t |b_2^j(t-s)|\,ds}] \\ &= |\lambda|\delta\sum_{j=1}^N \|b_1^j\|_\infty \frac{1}{t}\int_0^t |b_2^j(s)|\,ds. \end{aligned}$$

As $t \to \infty$, this last term converges almost surely to $|\lambda|\delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|]$. Therefore, $c^*(\delta)$ always satisfies the linear upper bound

$$\begin{aligned} c^*(\delta) = \inf_{\lambda>0}\frac{\mu(\lambda k)+f'(0)}{\lambda} &\le \inf_{\lambda>0}\frac{\lambda}{2}+\frac{f'(0)}{\lambda}+\delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|] \\ &= c^*(0)+\delta\sum_{j=1}^N \|b_1^j\| E_Q[|b_2^j|]. \end{aligned} \tag{113}$$

Finally, for upper bound (iv), observe that under the scaling $b \mapsto \lambda\delta b$, the constant p_1 defined in assumption A5 can be replaced by $p_1 \mapsto \lambda^2\delta^2 p_1$. Then by (32),

$$\rho(\lambda) \le \sqrt{2}\lambda^2\delta^2 p_1$$

and

$$\begin{aligned} c^*(\delta) = \inf_{\lambda>0}\frac{\mu(\lambda k)+f'(0)}{\lambda} &\le \inf_{\lambda>0}\frac{\lambda}{2}+\frac{f'(0)}{\lambda}+\frac{\lambda^2\delta^2 p_1}{2} = c^*(0) \\ &= 2\sqrt{(1+\delta^2 p_1)f'(0)/2} \\ &= c^*(0)\sqrt{(1+\delta^2 p_1)} \\ &= c^*(0)(1+\frac{\delta^2 p_1}{2}) + O(\delta^3). \end{aligned} \tag{114}$$

□

In proving Theorem 8.2, we will make use of the following lemma:

Lemma 8.1 *The infimum of the curve* $\frac{\mu(\lambda k)+f'(0)}{\lambda}$ *over* $(0,\infty)$ *is achieved at a unique point* $\lambda^* \in (0,\lambda_0]$ *where* $\lambda_0 = \sqrt{f'(0)/2}$. *Moreover, there are no other local minima.*

We will also make use of the following growth estimate on the principal Lyapunov exponents:

Proposition 8.1 *There is a constant $K > 0$ such that for λ sufficiently large, $\rho(\lambda) \geq K\lambda$.*

Proof of Lemma 8.1: This follows from the fact that $\mu(\lambda k) = \lambda^2/2 + \rho(\lambda)$ with ρ being convex in λ and $\rho(0) = 0$ (see discussion leading to (47)). The point λ_0 is the value of λ where the infimum of the curve $\lambda/2 + f'(0)/\lambda$ is attained. □

Proof of Proposition 8.1: In the case that $b(y,t)$ is a Gaussian field with white-noise time dependence, the authors of [12] studied the behavior of $\rho(\kappa)$ as $\kappa \to 0$, where $\kappa > 0$ is a diffusion constant (replace Δ with $\kappa\Delta$ in (1)). Here we modify their strategy in order to treat the large advection limit when b has the form (107).

For clarity of notation, we assume $y \in D = [-L/2, L/2]$. The argument generalizes to multiple dimensions in a straightforward way. For $0 \leq s < t < \infty$, let $A^k_{s,t}$ be the set of functions $g \in C^{0,1}([s,t]; R)$ such that $g(s) = g(t) = 0$, and $\|g'\|_\infty \leq k$. Define the random variable

$$I^k(s,t) = \sup_{f \in A^k_{s,t}} \int_0^{t-s} b(f(\tau), t-\tau)\, d\tau.$$

The variable $I^k(s,t)$ is super-additive, and $\tau_h I^k(s,t,\hat{\omega}) = I^k(s+h, t+h, \hat{\omega})$. Then, the sub-additive ergodic theorem implies that the limit

$$\lim_{t\to\infty} \frac{1}{t} I^k(0,t) = \zeta(k)$$

exists Q-almost surely, and that $\zeta(k)$ is a non-random constant given by the formula

$$\zeta(k) = \sup_{t>0} \frac{1}{t} E_Q[I^k(0,t)]. \tag{115}$$

We claim that $\zeta(k) > 0$. Therefore, given $\epsilon \in (0,1)$

$$Q\left(\sup_{f \in A^k_{0,t}} \int_0^t b(f(s), t-s)\, ds \geq (\zeta(k) - \epsilon)t \right) \geq 1 - \epsilon \tag{116}$$

if t is sufficiently large. That is, for any ϵ small there is a set of probability at least $(1-\epsilon)$ such that we can find $f(s) = f(s,\hat{\omega}) \in A^k_{0,t}$ satisfying

$$\int_0^t b(f(s,\hat{\omega}), t-s, \hat{\omega})\, ds \geq \zeta(k)(1-\epsilon)t, \tag{117}$$

and we expect that Brownian paths staying close to this f will make a significant contribution to the exponential in the definition of $\rho(\lambda)$. For a constant $\gamma > 0$ to be determined and $f \in A^k_{0,t}$, we let $B_t(f,\gamma)$ be the γ-neighborhood of f in $C([0,t];D)$:

$$B_t(f,\gamma) = \{X \in C([0,t],D) \mid \ \|X-f\|_{C^0} < \gamma\}.$$

Using the Girsanov transformation, one can show that there are constants K_1, K_2 independent of λ, t, and $f \in A^k_{0,t}$ such that

$$P(B_t(f,\gamma)) \geq K_1 e^{-K_2(k^2+1/\gamma^2)t}$$

for $t > 1$. Because the $\{b^j_1(y)\}$ are assumed to be Lipschitz continuous, we see that for any path $X \in B_t(f,\gamma)$,

$$|\int_0^t b(X(s), t-s)\, ds - \int_0^t b(f(s), t-s)\, ds| < \gamma M \sum_{j=1}^N \int_0^t |b^j_2(s)|\, ds \tag{118}$$

where M is the maximum of the Lipschitz constants for the functions $\{b^j_1(y)\}_{j=1}^N$. By (117) and (118) with $\epsilon > 0$ sufficiently small, there is a set of Q-probability at least $(1-\epsilon)$ such that

$$\begin{aligned} E_P\left[e^{\lambda \int_0^t b(W^y_s(s),t-s)\, ds}\right] &\geq E_P\left[e^{\lambda \int_0^t b(W^y_s(s),t-s)\, ds} \chi_{B_t(f,\gamma)}\right] \\ &= e^{\lambda\zeta(k)(1-\epsilon)t} e^{-\lambda\gamma M V} P(B_t(f,\gamma)) \\ &\geq e^{\lambda\zeta(k)(1-\epsilon)t} e^{-\lambda\gamma M V} K_1 e^{-K_2(k^2+1/\gamma^2)t}, \end{aligned} \tag{119}$$

where $V = \sum_{j=1}^N \int_0^t |b^j_2(s)|\, ds$ and $f \in A^k_{0,t}$ is chosen to satisfy (117). For t large, independently of λ, k, and γ, V can be bounded by

$$V \leq t(\sum_{j=1}^N E[|b^j_2(0|] + 1)$$

except on a set of probability less than ϵ. Therefore, we can choose γ small so that

$$\gamma \leq \frac{\epsilon\zeta(k)}{M(\sum_{j=1}^{N} E[|b_2^j(0|] + 1)}.$$

Hence $e^{\lambda\zeta(k)(1-\epsilon)t}e^{-\lambda\gamma MV} \geq e^{\lambda\zeta(k)(1-2\epsilon)t}$ for t sufficiently large. Then by choosing λ large, $\lambda \geq \frac{K_2(k_1^2/\gamma)}{\zeta(k)\epsilon}$, and we obtain from (119)

$$E_P\left[e^{\lambda\int_0^t b(W_s^y(s),t-s)\,ds}\right] \geq e^{\lambda(\zeta(k)-3\epsilon)t}$$

with Q-probability at least $(1-2\epsilon)$, for t sufficiently large, independently of λ. Since the limit defining $\rho(\lambda)$ exists Q-almost surely, this establishes the lemma with $K = \zeta(k)(1-3\epsilon)$, for any $\epsilon \in (0,1)$, $k > 0$.

It remains to establish the claim that $\zeta(k) > 0$. Note that for all $k \geq 0$, $E_Q[I^k(0,t)] \geq \sup_{f\in A_{0,t}^k} E_Q[\int_0^t b(f(s),t-s)ds] = 0$. Also, $E_Q[I^{k_2}(0,t)] \geq E_Q[I^{k_1}(0,t)]$ whenever $k_2 > k_1$, since $A_{0,t}^{k_2} \supset A_{0,t}^{k_1}$. Without loss of generality, suppose that there is an $\epsilon > 0$ such that for all $j = 1,\ldots,N$ we have $|b_1^j(y) - b_1^j(0)| \neq 0$ if $|y| < \epsilon$ and $y \neq 0$. This means that the $b^j(y)$ do not have a flat spot touching $y = 0$. Define the set $G = \{\hat{\omega}|\ b_1^j(y)b_2^j(s) > b_1^j(0)b_2^j(s), \forall s \in [0,1],\ y \in (0,\epsilon), j = 1,\ldots,N\}$. Then $Q(G) > 0$. For $k > 0$, let $\tilde{f} \in A_{0,1}^k$ such that $\tilde{f}(s) \in (0,\epsilon)$ for $s \in (0,1)$. Then we have

$$\begin{aligned}
E_Q[I^k(0,1)] &= E_Q[\sup_{f\in A_{s,t}^k}\int_0^1 b(f(s),1-s)ds\chi_G] + \\
&\qquad + E_Q[\sup_{f\in A_{s,t}^k}\int_0^1 b(f(s),1-s)ds\chi_{G^c}] \\
&\geq E_Q[\int_0^1 b(\tilde{f}(s),1-s)ds\chi_G] + E_Q[\int_0^1 b(0,1-s)ds\chi_{G^c}] \\
&> E_Q[\int_0^1 b(0,1-s)ds\chi_G] + E_Q[\int_0^1 b(0,1-s)ds\chi_{G^c}] \\
&= E_Q[\int_0^1 b(0,1-s)ds] = 0.
\end{aligned}$$

Combining this with (115) establishes the claim that $\zeta(k) > 0$ for all $k > 0$. $\square$

Proof of Theorem 8.2: The fact that $\bar{C} \in [0, +\infty)$ follows from Theorem 8.1. Also, if $b(y,t) \equiv b(t)$ then $\bar{C} = 0$ since $c^*(\delta) = c^*(0)$ for all $\delta > 0$. By Lemma 8.1 there is a unique $\lambda = \lambda_\delta \in (0, \lambda_0]$ such that

$$c^*(\delta) = \inf_{\lambda > 0} \frac{\mu(\lambda k) + f'(0)}{\lambda} = \frac{\mu(\lambda_\delta k) + f'(0)}{\lambda_\delta}.$$

Let $\delta_j \to \infty$ as $j \to \infty$ and suppose that $\limsup_{j\to\infty}(\lambda_{\delta_j}\delta_j) \leq M$. This implies that

$$\liminf_{j\to\infty} \frac{c^*(\delta_j)}{\delta_j} = \liminf_{j\to\infty} \frac{\mu(\lambda_{\delta_j} k) + f'(0)}{\lambda_{\delta_j}\delta_j} \geq \liminf_{j\to\infty} \frac{f'(0)}{\lambda_{\delta_j}\delta_j} \geq \frac{f'(0)}{M} > 0.$$

So, in this case the result holds with $\bar{C} = f'(0)/M$.

Now suppose $\lambda_{\delta_j}\delta_j$ is unbounded as $j \to \infty$. By Proposition 8.1, there is a positive constant K such that

$$\rho(\lambda_{\delta_j}\delta_j) \geq K\lambda_{\delta_j}\delta_j > 0$$

for j sufficiently large. Note that Proposition 8.1 treats the case of $\delta = 1$; this is why we use $\rho(\lambda_{\delta_j}\delta_j)$ instead of $\rho(\lambda_{\delta_j})$. Therefore,

$$\liminf_{j\to\infty} \frac{c^*(\delta_j)}{\delta_j} \quad = \quad \liminf_{j\to\infty} \frac{\lambda_{\delta_j}}{2\delta_j} + \frac{f'(0)}{\lambda_{\delta_j}\delta_j} + \frac{\rho(\lambda_{\delta_j}\delta_j)}{\lambda_{\delta_j}\delta_j} \geq K > 0$$

since $\lambda_{\delta_j} \in (0, \lambda^*]$ and $\lambda_{\delta_j}\delta_j \to \infty$. Hence $\bar{C} \geq K > 0$. □

Theorems 8.1 and 8.2 give a linear upper and lower bounds on the enhancement of c^* as the flow intensity increases. However, experiments with premixed flames have shown that increasing turbulence intensity does not lead to unlimited linear enhancement of the turbulent burning rate [36]. Denet [13] has proposed that this "bending" of the turbulent burning velocity in high-intensity flows can be explained by a rapid temporal decorrelation of the flow (see also Ashurst [3]). For the present model, the following upper bound confirms the hypothesis that rapid temporal decorrelation leads to sub-linear enhancement of the front speed. Notice that the derivation uses no information about the spatial structure of the flow other than the maximum value ($\|b_1^j\|_\infty$). As a result, it is likely that the actual speed may grow more slowly than $\delta^{1/2}$, or that c^* eventually decreases with δ, as suggested by the numerical experiments of [13] and [3] for temporally periodic flows.

Corollary 8.1 *For $\delta > 0$, let $\{b_2^j(t)\}_{j=1}^N$ be a family of stationary Gaussian fields on $[0,\infty)$ satisfying $E_Q[b_2^j(s)b_2^k(t)] \le C_1 e^{-\alpha_{j,k}|t-s|}$, where $\alpha_{j,k} > 0$ and $C_1 > 0$. Then for the scaled flow $b^\delta(y,t) = \sum_{j=1}^N \delta b_1^j(y) b_2^j(\delta t)$*

$$\limsup_{\delta\to\infty} \frac{c^*(\delta)}{\sqrt{\delta}} < +\infty. \tag{120}$$

Proof: For the flow $\sum_{j=1}^N b_1^j(y)b_2^j(\delta t)$,

$$\begin{aligned} \hat{\Gamma}(r) = \sup_{y_1,y_2} \Gamma(y_1,y_2,0,r) &\le \sum_{j,k} \|b_1^j\|_\infty \|b_1^k\|_\infty E_Q[b_2^j(0)b_2^k(\delta r)] \\ &\le \sum_{j,k} \|b_1^j\|_\infty \|b_1^k\|_\infty C_1 e^{-\alpha_{j,k}\delta|r|} \end{aligned} \tag{121}$$

Then from (3) we have

$$\int_0^\infty \hat{\Gamma}(r)\, dr \le p_1 = C_1 \sum_{j,k} \frac{\|b_1^j\|_\infty \|b_1^k\|_\infty}{\alpha_{j,k}\delta}$$

The result now follow from part (iv) of Theorem 8.1.
□

9 Conclusions

We have considered the propagation of KPP reaction fronts in temporally random shear flows with sufficiently decaying correlations. We showed that, under assumptions A1-A6 on a Gaussian shear field, the front speeds obey a variational formula that extends the known variational formula in the case of periodic media. Using this formula, we derived basic bounds on the front speeds. As a function of large shear root mean square amplitude, the front speed obeys linear growth at fixed correlation length. However, front speed growth becomes sublinear if there is sufficient temporal decorrelation. Developing methods to generalize the variational front speed formula for nonshear random flows will be left as a future work.

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