A central limit theorem for pulled fronts in a random medium

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Abstract

We consider solutions to a nonlinear reaction diffusion equation when the reaction term varies randomly with respect to the spatial coordinate. The nonlinearity is the KPP type nonlinearity. For a stationary and ergodic medium, and for certain initial condition, the solution develops a moving front that has a deterministic asymptotic speed in the large time limit. The main result of this article is a central limit theorem for the position of the front, in the supercritical regime, if the medium satisfies a mixing condition. [1]

1 Introduction and main results

We consider the scalar reaction-diffusion equation

$$v_t = v_{xx} + g(x,\omega)v(1-v), \quad x \in \mathbb{R}, \ \ t > 0, \tag{1.1}$$

with random reaction rate $g(x,\omega) : \mathbb{R} \times \Omega \to (0,\infty)$, defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We will make assumptions about g and about the initial condition at $t = 0$ so that, with probability one, there is a classical solution to (1.1) satisfying $0 < v < 1$ and $\lim_{x\to-\infty} v = 1$, $\lim_{x\to\infty} v = 0$. This ensemble of solutions behaves like a traveling wave or front propagating through the random environment. We wish to understand the statistical fluctuations of the ensemble at large times.

When the reaction rate g is a constant, this equation is often called the KPP-Fisher equation, and it has been known for a long time that there is a family of traveling wave solutions moving with constant speed [12, 7]. There is a minimal speed $c^* > 0$ such that for each $c \geq c^*$ there is a traveling wave of the form $v(t,x) = \tilde{v}(x - ct)$ where $0 < \tilde{v} < 1$ and $\tilde{v}(-\infty) = 1$, $\tilde{v}(+\infty) = 0$. The equilibrium states $v = 0$ and $v = 1$ are unstable and stable, respectively. So, traveling waves describe the propagation of the stable state. When g varies with x, there may not be traveling wave solutions in this classical sense, although solutions may still exhibit some wave-like behavior. For some examples of such behavior in periodic, almost-periodic, random media, or general disordered media we refer to [2, 23, 19, 3, 26].

In this paper, we impose a statistical structure on g that is stationary and ergodic with respect to shifts in x. This means that there is a group of measure-preserving transformations $\{\pi_k\}_{k\in\mathbb{R}}$, acting ergodically on $(\Omega, \mathcal{F})$, such that for almost every $\omega \in \Omega$, $g(x + k, \omega) = g(x, \pi_k\omega)$ holds for all $x \in \mathbb{R}$, $k \in \mathbb{R}$. We also assume that there are constants g_{min} and g_{max} such that $0 < g_{min} \leq g(x,\omega) \leq g_{max}$ holds for all x with probability one, so that the state $v = 1$ is everywhere stable and the state $v = 0$ is everywhere unstable. Although the statistics of g are translation invariant, each realization $x \to g(x,\omega)$ will vary with x. We assume that the function $x \to g(x,\omega)$ is almost surely uniformly

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Lipschitz continuous. Under these assumptions and with suitable initial condition $v_0 = v(0, x, \omega)$ to be defined later, (1.1) admits a unique classical solution $v(t, x, \omega)$ for almost every $\omega \in \Omega$.

The initial condition will satisfy $0 < v_0 < 1$, $\lim_{x \to -\infty} v_0 = 1$, and $\lim_{x \to \infty} v_0 = 0$. One can show that $v \nearrow 1$ locally uniformly, as $t \to \infty$. We define the position of the wave for $t \geq 0$ to be the random process

$$X(t, \omega) = \sup \{x \in \mathbb{R} \mid \ v(t, x, \omega) = 1/2\}.$$

This process will diverge as $t \to \infty$, but because the environment is statistically stationary one expects some averaging to occur in the large time limit. Indeed, Freidlin and Gärtner [8, 9] have proved that for suitable initial conditions the limit

$$\lim_{t \to \infty} \frac{X(t, \omega)}{t} = c > 0$$

exists with probability one. This may be regarded as a law of large numbers for the random wave position. The purpose of the present analysis is to understand the fluctuations around this average behavior $X(t, \omega) \sim ct$. Specifically, are the fluctuations Gaussian? Under what conditions will the central limit theorem hold

$$\lim_{t \to \infty} \mathbb{P} \left(\frac{X(t, \omega) - ct}{\sqrt{t}} > \alpha \right) = \Phi(\alpha/\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha/\sigma} e^{-y^2/2} \, dy$$

for some $\sigma > 0$?

Equations like (1.1) arise in several physical and biological applications in which a front or phase boundary develops and invades an unstable phase. The nonlinear term $v(1 - v)$ is a prototype model that leads to "pulled fronts". For an extensive review of such applications, see [25]. From the point of view of applications, the choice of a coefficient g which varies with x is natural, given that g represents a physical or biological parameter like a reaction rate or birth rate. Because the underlying environment may vary in a way that is best described statistically, these parameters may be described as a random field, and it is interesting to consider the statistical behavior of fronts moving through such a random environment. For work on some other models of noisy pulled fronts see [22, 24, 17, 5].

1.1 The linearized equation

The dynamics of pulled fronts depends sensitively on the behavior at the leading edge where v takes values close to 0, the unstable state. For this reason, it is natural to first consider the linearized equation

$$\varphi_t = \varphi_{xx} + g(x, \omega)\varphi, \tag{1.2}$$

and then try to compare the solution of the nonlinear equation (1.1) to solutions of this linearized equation. To this end, we will study a special family of solutions for $x \in [0, \infty)$ having the form

$$\varphi(t, x, \omega; \gamma) = e^{\gamma t} u(x, \omega; \gamma),$$

where for sufficiently large values of the parameter $\gamma > 0$, the random function $u(\cdot, \omega; \gamma)$ is defined by the following theorem.

Theorem 1.1 *There is a real number $\bar{\gamma} \in [g_{min}, g_{max}]$ such that the following hold, $\mathbb{P}$ almost surely: For every $\gamma > \bar{\gamma}$ there exists a unique function $u = u(\cdot, \omega; \gamma) \in C^2((0, \infty)) \cap C([0, \infty))$ which solves the linear boundary value problem*

$$u_{xx} + (g(x, \omega) - \gamma)u = 0, \quad x > 0 \tag{1.3}$$

and satisfies $u(x,\omega,\gamma) > 0$ *for all* $x > 0$*;* $u(0) = 1$*; and* $\lim_{x\to+\infty} u(x) = 0$*. If* $\gamma < \bar{\gamma}$*, then no such solution exists. For* $\gamma > \bar{\gamma}$*, the limit*

$$\mu(\gamma) = \lim_{x\to+\infty} -\frac{1}{x}\log u(x,\omega;\gamma) \tag{1.4}$$

holds with probability one. The decay rate $\mu(\gamma)$ *is deterministic, and it is concave and increasing in* γ*.*

For $\gamma > \bar{\gamma}$, $\varphi(t,x,\omega;\gamma)$ solves the linearized equation (1.2) on the half-line $x \in [0,\infty)$ with boundary condition $\varphi(t,0,\omega;\gamma) = e^{\gamma t}$ for $t \geq 0$ and initial condition $\varphi(0,x,\omega;\gamma) = u(x,\omega;\gamma)$ for all $x \geq 0$. We define

$$Y(t,\omega;\gamma) = \sup\{x \geq 0 \mid \ \varphi(t,x,\omega;\gamma) = 1/2\},$$

and we refer to this stochastic process as the position of the wave φ at time t. This process is nonnegative and non-decreasing in t. However, because u may not be monotone decreasing, $Y(t)$ may not be continuous. Using the fact that $\log(u(x,\omega;\gamma)) \sim -\mu(\gamma)x$ as $x \to \infty$, we see that the limit

$$\lim_{t\to\infty} \frac{Y(t,\omega;\gamma)}{t} = \frac{\gamma}{\mu(\gamma)} = c(\gamma) \tag{1.5}$$

holds with probability one. The properties of the function $\mu(\gamma)$ (see Lemma 2.6) imply that the minimal speed

$$c^* = \inf_{\gamma>\bar{\gamma}} c(\gamma) = \inf_{\gamma>\bar{\gamma}} \frac{\gamma}{\mu(\gamma)} > 0$$

is positive. The asymptotic speed of φ depends on the exponential decay rate of the initial condition, $u(x,\omega;\gamma)$.

To analyze the fluctuations in $Y(t,\omega;\gamma)$, we must analyze fluctuations in the tail of $u(x,\omega;\gamma)$ via a refinement of (1.4). We will assume that g satisfies a mixing condition (2.39). This is a standard condition that appears in central limit theorems for sums of dependent random variables; it controls long-range dependence in the random field $g(x,\omega)$.

Theorem 1.2 *Suppose that* $g(x,\omega)$ *satisfies the* ϕ*-mixing condition (2.39) with* $\sum_k \phi(k)^{1/2} < \infty$*. Let* $\gamma > \bar{\gamma}$*. Then as* $n \to \infty$*, the random variable*

$$\frac{\log(u(n,\omega;\gamma)) + \mu(\gamma)n}{\sqrt{n}} \tag{1.6}$$

converges in distribution to a centered Gaussian with variance $\sigma^2 \geq 0$*. If* $\sigma^2 > 0$*, then for any* $M > 0$ *the family of processes* $\{V_n(x,\omega)\}_{n=1}^{\infty}$ *defined by*

$$V_n(x,\omega;\gamma) = \frac{\log(u(xn,\omega;\gamma)) + \mu(\gamma)xn}{\sigma\sqrt{n}}, \quad x \in [0,M] \tag{1.7}$$

converges weakly to a standard Brownian motion on $[0,M]$ *as* $n \to \infty$*, in the sense of weak convergence of measures on* $C([0,M])$ *with the uniform topology.*

In principle, the variance σ^2 could vanish, although we do not have a nontrivial example of this phenomenon. If $\sigma = 0$, the convergence described by the theorem means that the quotient (1.6) converges in distribution to zero. Later at Proposition 2.1, we construct a simple example for which σ is positive. From this we will obtain a central limit theorem for the fluctuations of $Y(t,\omega;\gamma)$ about its asymptotic mean behavior:

Theorem 1.3 *Suppose that $g(x,\omega)$ satisfies the ϕ-mixing condition (2.39) with $\sum_k \phi(k)^{1/2} < \infty$. Let $\gamma > \bar{\gamma}$, $\mu = \mu(\gamma)$, and $c = c(\gamma)$. Suppose that $\sigma > 0$, where σ is defined in Theorem 1.2. For any $\alpha \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{Y(t,\omega;\gamma)-ct}{\mu^{-1}c\sqrt{t}} \le \alpha\right) = \Phi(\alpha/\sigma) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha/\sigma} e^{-y^2/2}\,dy. \tag{1.8}$$

For each $M > 0$ the family of processes

$$Z_n(t,\omega;\gamma) = \frac{Y(nt,\omega;\gamma)-cnt}{\mu^{-1}c\sigma\sqrt{n}}, \quad t \in [0,M]$$

converges weakly, in the Skorohod space D, to a standard Brownian motion as $n \to \infty$.

The processes $Y(t,\omega;\gamma)$ and $Z_n(t,\omega;\gamma)$ may not be continuous. This is why we consider convergence in the Skorohod space D – the space of functions on $[0,M]$ which are right-continuous with left-hand limits, endowed with the Skorohod metric topology [4].

1.2 The nonlinear equation

Now we return to the nonlinear equation (1.1). Freidlin and Gärtner [8, 9] proved that if

$$\lim_{x\to-\infty} v_0(x,\omega) = 1, \tag{1.9}$$

and

$$\lim_{x\to\infty} -\frac{1}{x}\log v_0(x,\omega) > \mu^* = \mu(\gamma^*), \tag{1.10}$$

where

$$\gamma^* = \inf\left\{\gamma > \bar{\gamma} \mid \ c(\gamma) = c^*\right\}, \tag{1.11}$$

then $X(t,\omega)/t \to c^*$, with probability one as $t \to \infty$. The bound (1.10) means that the initial condition v_0 decays to zero more rapidly than $\varphi(0,x,\omega;\gamma^*)$, which corresponds to the minimal speed c^*. This result can be extended to more slowly decaying initial conditions. Specifically, if (1.9) and

$$\lim_{x\to\infty} -\frac{1}{x}\log v_0(x,\omega) = \mu(\gamma) < \mu^*, \tag{1.12}$$

hold for some $\gamma \in (\bar{\gamma},\gamma^*)$ (with probability one), then

$$\lim_{t\to\infty}\frac{X(t,\omega)}{t} = c(\gamma) > c^*.$$

So, the decay rate of the initial condition v_0 selects the asymptotic speed of the front.

The fluctuations of the solution are a more delicate issue. One expects that the large time behavior of $v(t,x,\omega)$ will be close to that of $\varphi(t,x,\omega;\gamma)$ if the initial condition $v_0(x,\omega)$ is sufficiently close to $u(x,\omega;\gamma) = \varphi(0,x,\omega;\gamma)$ for $x \gg 1$. Thus, we might obtain a central limit theorem for $X(t,\omega)$ by comparing v to φ and using Theorem 1.3. For technical reasons, however, our approach to estimating v by φ allows us to consider only **supercritical** waves which move faster than the minimal speed c^*. We suppose that for some $\gamma \in (\bar{\gamma},\gamma^*)$ the initial condition $v_0(x,\omega)$ satisfies (1.9) and

$$C_1(\omega)u(x,\omega;\gamma) \le v_0(x,\omega) \le C_2(\omega)u(x,\omega;\gamma), \quad \forall\, x > 0. \tag{1.13}$$

and some positive constants $C_1(\omega), C_2(\omega)$. Our main result for the nonlinear problem is the following:

Theorem 1.4 *Suppose that $g(x,\omega)$ satisfies the ϕ-mixing condition (2.39) with $\sum_k \phi(k)^{1/2} < \infty$. Suppose that $\gamma \in (\bar{\gamma}, \gamma^*)$, $c = c(\gamma)$, $\mu = \mu(\gamma)$, and that v_0 satisfies (1.9) and (1.13). Suppose that $\sigma > 0$, where σ is defined in Theorem 1.2. Then for any $\alpha \in \mathbb{R}$,*

$$\lim_{t\to\infty} \mathbb{P}\left(\frac{X(t,\omega) - tc}{\mu^{-1}c\sqrt{t}} \le \alpha\right) = \Phi(\alpha/\sigma) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha/\sigma} e^{-y^2/2}\,dy.$$

In [18] the author derived a complementary result in the case that the nonlinear term is of the bistable or ignition type. Those nonlinearities correspond to "pushed fronts", which are not as sensitive to fluctuations in the leading edge of the wave. Moreover, in that setting the asymptotic wave speed is unique. The strategy in [18] was to show that the wave is stable with respect to fluctuations in the environment that are far from the interface, so the motion of the interface depends primarily on the local environment. That approach relies on stability analysis for generalized traveling waves developed in [16], and resulted in a full invariance principle for the wave position.

For KPP fronts (i.e. pulled fronts), stability is a more delicate issue and the dynamics can be quite complex, even in the case of the homogeneous environment where g is constant. This is due to the sensitive dependence of the wave on the leading edge. For recent work on the stability of KPP fronts, see [1] and [11] for the supercritical case, and [15] for the critical case. As the reader will see later, our approach to proving Theorem 1.4 is to show that, with high probability as $t \to \infty$, the wave position $X(t,\omega)$ (associated with v) does not lag too far behind $Y(t,\omega)$ (associated with φ).

Observe that the initial condition $v_0(x)$ satisfying (1.13) is random. In particular, this assumption excludes the case where $v_0(x) = Ce^{-\lambda x}$ for $x \gg 1$. In that case $\log(u(x,\omega;\gamma)/v_0(x))$ behaves like a Brownian motion and is not bounded above or below; typical values of $u(x,\omega;\gamma)/v_0(x)$ are of the order $O(e^{\sigma\sqrt{x}})$ as $\epsilon \to \infty$. While the condition (1.12) is sufficient to select the asymptotic speed, it is not clear whether this is also sufficient to guarantee the central limit theorem for $X(t,\omega)$. To understand this point, consider the linear equation (1.2) with g being a constant (i.e. a deterministic, homogeneous medium). If the initial condition is $v_0(x) \sim e^{-\lambda x + \sigma\sqrt{x}}$ then it is not hard to show that the corresponding wave position $Y(t)$ satisfies $Y(t) \ge ct + k\sigma\sqrt{t}$ for some positive constant k, and t sufficiently large. Thus, even in the deterministic linear setting, fluctuations of order $e^{\sigma\sqrt{x}}$ in the initial condition could lead to $O(k\sigma\sqrt{t})$ fluctuations in the position of the wave. We hope to investigate this issue further in future work.

Let us also point out that our approach to analyzing the nonlinear equation does not extend to the critical case $\gamma = \gamma^*$. For critical waves in the homogeneous medium, there is a logarithmic gap between the solution of the linearized equation and that of the nonlinear equation: $Y(t) - X(t) \sim \frac{3}{c^*}\log t$ (see [15], and references therein). In the random setting, however, it is not clear whether the gap between φ and v is only logarithmic or much larger. If the gap is $o(\sqrt{t})$ then Theorem 1.4 also holds in the critical case. We hope to address this critical case in future work. Fluctuations of the interface in the multidimensional setting is another interesting and challenging topic. Although asymptotic spreading of the interface has been proved in this setting ([14, 13, 18]), little is known about the statistics of the fluctuations.

The rest of this paper is organized like this introductory section. In Section 2 we study the stationary equation (1.3). There we prove Theorem 1.1 and Theorem 1.2, and we derive some useful estimates on the function $u(x,\omega;\gamma)$. In Section 3 we prove Theorem 1.3 for fluctuations in the position of φ which solves the linearized evolution equation. In Section 4, we prove some technical estimates that are needed to bridge the gap between solutions of the linearized and the nonlinear equations. In particular, we show that the leading edge of $(\varphi(t,x,\omega;\gamma))^2$ is dominated by a slower-moving wave; this is where we use the supercritical assumption, $\gamma \in (\bar{\gamma}, \gamma^*)$. In Section 5 we prove Theorem 1.4 using the key estimate in Lemma 5.1 which shows that, with high probability, $X(t,\omega)$ (associated with v) does not lag far behind $Y(t,\omega)$ (associated with φ).

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2 The stationary equation

2.1 Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. Most aspects of the theorem are proved already in [8] (see Chapter 7.5 therein) using the Feynman-Kac formula and probabilistic estimates. For completeness and in order to establish some important estimates that we will use later, we give a proof here using different arguments.

The constant $\bar{\gamma}$ will be identified with the principal eigenvalue of the operator $u_{xx} + gu$ on $\mathbb{R}$ (in the sense described below), and in order to construct the function $u(x, \omega; \gamma)$ we will need to study properties of the eigenvalue problem on bounded intervals. For an interval $I = [a, b]$, let $\Gamma(I, \omega)$ be the principal eigenvalue and $\psi^I(x, \omega) \geq 0$ the principal eigenfunction of

$$\psi^I_{xx} + g(x, \omega)\psi^I = \Gamma(I, \omega)\psi^I, \quad x \in (a, b), \tag{2.14}$$

with $\psi^I(a) = \psi^I(b) = 0$, and $\psi^I(x) > 0$ for all $x \in (a, b)$, and normalized by $\int_I \psi^I \, dx = 1$.

Lemma 2.1 *There is a constant $\Gamma_\infty \in [g_{min}, g_{max}]$ such that the following statements hold $\mathbb{P}$-almost surely. If $I_1 \subset I_2 \subset \mathbb{R}$ are two intervals, then $\Gamma(I_1, \omega) \leq \Gamma(I_2, \omega)$. Also,*

$$\Gamma_\infty = \lim_{k\to\infty} \Gamma([-k, k], \omega) = \lim_{k\to\infty} \Gamma([0, k], \omega).$$

Proof of Lemma 2.1: If $I_1 \subset I_2$, the fact that $\Gamma(I_1, \omega) \leq \Gamma(I_2, \omega)$ follows from the variational representation

$$\Gamma(I, \omega) = \max\left\{ \int_I -(\psi_x)^2 + g(x, \omega)\psi^2 \, dx \;\middle|\; \psi \in H^1_0(I), \quad \int_I \psi^2 \, dx = 1 \right\}. \tag{2.15}$$

Since $g_{min} \leq g \leq g_{max}$, this representation also implies $\Gamma(I, \omega) \in [g_{min} - \pi^{-2}|I|^{-2}, g_{max} - \pi^{-2}|I|^{-2}]$, because $\pi^{-2}|I|^{-2}$ is the principal eigenvalue on of the Laplacian on I. It follows that the limit

$$\Gamma_\infty = \lim_{k\to\infty} \Gamma([-k, k], \omega) \tag{2.16}$$

exists and satisfies $\Gamma_\infty \in [g_{min}, g_{max}]$. We claim that Γ_∞ is a constant, independent of ω. This follows from the ergodicity assumption and the fact that Γ_∞ must be invariant with respect to the action of π_x. Specifically, the stationarity of g implies that with probability one,

$$\Gamma_\infty(\pi_x\omega) = \lim_{k\to\infty} \Gamma([-k, k], \pi_x\omega) = \lim_{k\to\infty} \Gamma([-k + x, k + x], \omega)$$

holds for all $x \in \mathbb{R}$. However, since Γ is nondecreasing in I, $\Gamma([-k+x, k+x], \omega) \geq \Gamma([-(k-|x|), k-|x|], \omega)$, and thus

$$\Gamma_\infty(\pi_x\omega) \geq \lim_{k\to\infty} \Gamma([-(k - |x|), k - |x|], \omega) = \Gamma_\infty(\omega), \quad \forall\, x \in \mathbb{R}$$

holds with probability one. Since π_x is measure-preserving, this implies $\Gamma_\infty(\pi_x\omega) = \Gamma_\infty(\omega)$, $\mathbb{P}$-almost surely. Now the ergodicity assumption implies Γ_∞ is constant, almost surely.

Since Γ is nondecreasing in I, the limit

$$\Gamma_\infty^+ = \lim_{k\to\infty} \Gamma([0,k],\omega)$$

also exists and satisfies $\Gamma_\infty^+ \le \Gamma_\infty$. Given $\delta > 0$ and $\epsilon > 0$, we may choose K large so that $\mathbb{P}\left(\Gamma([-k,k],\omega) > \Gamma_\infty - \epsilon\right) \ge 1-\delta$ holds for all $k \ge K$. Hence, because π_x is measure-preserving and g is stationary,

$$\begin{aligned} \mathbb{P}\left(\Gamma_\infty^+(\omega) > \Gamma_\infty - \epsilon\right) &\ge \mathbb{P}\left(\Gamma([0,2k],\omega) > \Gamma_\infty - \epsilon\right) \\ &= \mathbb{P}\left(\Gamma([0,2k],\pi_k\omega) > \Gamma_\infty - \epsilon\right) \\ &= \mathbb{P}\left(\Gamma([-k,k],\omega) > \Gamma_\infty - \epsilon\right) \ge 1-\delta. \end{aligned} \tag{2.17}$$

Since δ and ϵ may be chosen arbitrarily small, this implies $\Gamma_\infty^+ \ge \Gamma_\infty$ holds with probability one, so $\Gamma_\infty^+ = \Gamma_\infty$. $\square$

In the construction of $u(x,\omega;\gamma)$ and in the subsequent analysis we will make frequent use of the following estimates:

Lemma 2.2 *Let $I=[a,b]$ and $\gamma > \Gamma_\infty$. Let $\epsilon > 0$. There is a constant $C > 0$, such that if $w(\cdot,\omega)$ satisfies*

$$w_{xx} + (g(x,\omega)-\gamma)w \ge 0, \quad x \in (a,b)$$

then

$$w(x,\omega) \le \max(0,w(a,\omega)) C e^{-(x-a)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)} + \max(0,w(b,\omega)) C e^{(x-b)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)}, \quad \forall\, x \in I.$$

In particular, if $w(a,\omega) \le 0$ and $w(b,\omega) \le 0$, then $w \le 0$ for all $x \in [a,b]$. The constant C depends on ϵ, g_{max} and γ, but not on ω or I.

Proof of Lemma 2.2: We first prove the result assuming $w(a,\omega)=1$ and $w(b,\omega) \le 0$. Let $\delta > 0$, and define $z = e^{\delta(x-a)} w(x,\omega) - e^{-(x-a)}$ which satisfies

$$z_{xx} - 2\delta z_x + (g-\gamma+\delta^2) z \ge \left(-1 + 2\delta - (g-\gamma+\delta^2)\right) e^{-(x-a)}, \quad x \in (a,b)$$

and $z(a)=0$, $z(b)<0$. After multiplying this equation by $z^+ = \max(z,0) \in H_0^1(I)$ and integrating over I, we obtain

$$\int_I -(z^+)_x^2 + g(z^+)^2\,dx \ge (\gamma-\delta^2)\int_I (z^+)^2\,dx + \int_I \left(-1+2\delta-(g-\gamma+\delta^2)\right) e^{-(x-a)} z^+\,dx. \tag{2.18}$$

By the representation (2.15), the left side is bounded by $\Gamma(I,\omega)\int_I (z^+)^2\,dx$. Therefore,

$$(\gamma-\delta^2-\Gamma(I))\int_I (z^+)^2\,dx \le -\int_I \left(-1+2\delta-(g-\gamma+\delta^2)\right) e^{-(x-a)} z^+\,dx.$$

For $\beta > 0$, let $\delta^2 = \gamma - \Gamma_\infty - \beta \le \gamma - \Gamma(I,\omega) - \beta$ and apply the Cauchy-Schwarz inequality on the right side to obtain:

$$\beta\int_I (z^+)^2\,dx \le \frac{1}{2\beta}\int_I \left(-1+2\delta-(g-\gamma+\delta^2)\right)^2 e^{-2(x-a)}\,dx + \frac{\beta}{2}\int_I (z^+)^2\,dx.$$

Therefore,

$$\int_I (z^+)^2\,dx \le \beta^{-2}\int_a^\infty \left(-1+2\delta-(g-\gamma+\delta^2)\right)^2 e^{-2(x-a)}\,dx \le \frac{(1+2\delta+g_{max}+\Gamma_\infty+\beta)^2}{2\beta^2}.$$

Observe that the constant on the right side is independent of I and ω. Returning to (2.18) and applying Cauchy-Schwarz again, we conclude that there is a constant C_1 – depending only on δ, g_{max}, γ, Γ_∞, and β – such that

$$\int_I (z_x^+)^2 \le C_1.$$

Consequently, for all $x \in I$

$$z(x) \le z^+(x) = \int_a^x z_x^+(s)\,ds \le \sqrt{x-a}\left(\int_I (z_x^+)^2\,dx\right)^{1/2} \le \sqrt{C_1(x-a)}$$

and thus,

$$w(x,\omega) = e^{-\delta(x-a)}(z+e^{-(x-a)}) \le e^{-\delta(x-a)}(\sqrt{C_1(x-a)}+e^{-(x-a)}).$$

Now we let β be small so that $\delta = \sqrt{\gamma-\Gamma_\infty-\beta} \ge \sqrt{\gamma-\Gamma_\infty}-\epsilon/2$ and we have

$$\begin{aligned} w(x,\omega) &\le C_2 e^{-(x-a)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)} \\ &= \max(0,w(a,\omega))C_2 e^{-(x-a)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)} + \max(0,w(b,\omega))C_2 e^{(x-b)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)} \end{aligned}$$

since we have assumed $w(a,\omega)=1$ and $w(b,\omega)\le 0$. In the case $w(a,\omega)\le 0$ and $w(b,\omega)=1$, a very similar argument (with $z = e^{\delta(b-x)}w - e^{x-b}$) leads to the same bound with the same constant C_2, independent of I and ω. The general case then follows from the linearity of the equation. □

The following corollary will enable comparison of functions on the unbounded interval $I=[a,\infty)$.

Corollary 2.1 *Let $I=[a,\infty)$ and $\gamma > \Gamma_\infty$ and $\epsilon > 0$. There is a constant $C>0$ such that if w satisfies*

$$w_{xx} + (g(x,\omega)-\gamma)w \ge 0, \quad \forall\, x > a$$

with $\limsup_{x\to\infty} w(x,\omega) < \infty$, *then*

$$w(x,\omega) \le \max(0,w(a,\omega))Ce^{-(x-a)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)}, \quad \forall\, x \in I. \tag{2.19}$$

In particular, if $w(a,\omega)\le 0$, then $w(x,\omega)\le 0$ for all $x \ge a$.

Proof of Corollary 2.1: If w vanishes outside an interval $[a,b]$, then this is an immediate consequence of Lemma 2.2. Otherwise, since $M(\omega) = \limsup_{x\to\infty} w(x,\omega)$ is finite, we may choose a point b (depending on ω) arbitrarily large such that $w(b,\omega) \le 2M(\omega)$. Then apply Lemma 2.2 on the interval $[a,b]$ to conclude that

$$w(x,\omega) \le \max(0,w(a,\omega))Ce^{-(x-a)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)} + 2MCe^{(x-b)(\sqrt{\gamma-\Gamma_\infty}-\epsilon)}, \quad \forall\, x\in[a,b].$$

Since b may be chosen arbitrarily large, and since C is independent of b and ω, we conclude that $w(x,\omega)$ satisfies (2.19) for all $x \ge a$. □

Now we continue with the proof of Theorem 1.1. The solution $u(x,\omega;\gamma)$ is defined as the limit, as $k\to\infty$, of the function $u^{k,\gamma}(x,\omega)$ satisfying the boundary value problem:

$$u_{xx}^{k,\gamma} + (g(x,\omega)-\gamma)u^{k,\gamma} = 0, \quad x \in (0,k) \tag{2.20}$$

with $u^{k,\gamma}(0) = 1$ and $u^{k,\gamma}(0) = 0$. If $\gamma > \Gamma_\infty$, then $\gamma > \Gamma([0,k],\omega)$ and the Fredholm alternative implies that there exists a unique solution to this problem. For each k, $u^{k,\gamma}(x) \geq 0$ for all $x \in [0,k]$. This follows from Lemma 2.2 applied to the function $w = -u^{k,\gamma}(x)$. Moreover, for each $x \in (0,k)$, $u^{k,\gamma}(x,\omega)$ is increasing in k. Specifically, if $j > k$ then we may apply Lemma 2.2 to the function $w = u^{k,\gamma}(\bar{x},\omega) - u^{j,\gamma}(\bar{x},\omega)$ to conclude that $u^{j,\gamma}(x,\omega) \geq u^{k,\gamma}(x,\omega)$ for all $x \in [0,k]$ if $j > k$.

Therefore, for all $\gamma > \Gamma_\infty$ we may define

$$u(x,\omega;\gamma) = \lim_{k\to\infty} u^{k,\gamma}(x,\omega). \tag{2.21}$$

By Lemma 2.2, $u^{k,\gamma}(x,\omega) \leq Ce^{-(\sqrt{\gamma-\Gamma_\infty}-\epsilon)x}$ holds for all $x \in [0,k]$ with a constant $C = C(\epsilon,\gamma)$ that is independent of k and ω. Therefore, the limit $u(x,\omega;\gamma)$ is also finite and satisfies

$$u(x,\omega;\gamma) \leq Ce^{-(\sqrt{\gamma-\Gamma_\infty}-\epsilon)x} \tag{2.22}$$

with the same constant C, independent of ω. Because $u(x,\omega;\gamma)$ is finite, elliptic regularity implies that $u(\cdot,\omega;\gamma)$ satisfies equation (1.3) for $x > 0$, and $u(0,\omega;\gamma) = 1$ and $u(x,\omega;\gamma) > 0$ for $x > 0$. Thus we have established that if $\gamma > \Gamma_\infty$, there exists a function $u(x,\omega;\gamma)$ with the desired properties. With Corollary 2.1 it is not hard to see that the solution $u(x,\omega;\gamma)$ must be unique. Moreover, by applying Corollary 2.1 to the function $w = e^{-x\sqrt{\gamma-g_{min}}} - u(x,\omega;\gamma)$ we obtain the lower bound

$$u(x,\omega;\gamma) \geq e^{-x\sqrt{\gamma-g_{min}}}, \quad \forall\, x > 0. \tag{2.23}$$

Let us also observe that for each x, the functions $y \mapsto u(x+y,\omega;\gamma)$ and $y \mapsto u(x,\omega;\gamma)u(y,\pi_x\omega;\gamma)$ satisfy the same boundary value problem on $[0,\infty)$, because of $g(x+y,\omega) = g(y,\pi_x\omega)$. Therefore the uniqueness of u immediately implies the following useful relation:

Lemma 2.3 *With probability one, $u(x,\omega;\gamma)$ satisfies*

$$u(x+y,\omega;\gamma) = u(x,\omega;\gamma)u(y,\pi_x\omega;\gamma) \tag{2.24}$$

for all $x \geq 0$, $y \geq 0$, $\gamma > \bar{\gamma}$.

Now suppose $\gamma < \Gamma_k < \Gamma_\infty$. If $\psi^k = \psi^I$ is the eigenfunction (2.14) for $I = [0,k]$, then the function $\eta(t,x) = e^{(\Gamma_k-\gamma)t}\psi^k(x)$ satisfies $\eta_t = \eta_{xx} + (g(x)-\gamma)\eta$ and $\eta \nearrow \infty$. If there were a solution $u(x,\omega;\gamma)$ satisfying $u > 0$ for $x \geq 0$, then the maximum principle would imply that for suitable constant C, $u(x,\omega;\gamma) > C\eta(t,x)$ must hold for all t and $x \in [0,k]$. However, this cannot hold since $\eta \to \infty$ as $t \to \infty$. Therefore, for $\gamma < \Gamma_\infty$, no such solution exists since $\Gamma_k \to \Gamma_\infty$ as $k \to \infty$.

We now have established the first part of Theorem 1.1 and identified $\bar{\gamma} = \Gamma_\infty$. Next we show that the limit (1.4) exists, almost surely, and satisfies the stated bounds. For each integer $n \geq 1$, we may iterate the equality (2.24) $n-1$ times to obtain

$$\log(u(n,\omega;\gamma)) = \log\left(\prod_{k=0}^{n-1} u(1,\pi_k\omega;\gamma)\right) = \sum_{k=0}^{n-1} \log u(1,\pi_k\omega;\gamma). \tag{2.25}$$

Let $q_k(\omega) = \log u(1,\pi_k\omega;\gamma)$. The $\mathcal{F}$-measurability of q_k may be proved as in [19]. By the bounds (2.22) and (2.23), $\mathbb{E}[q_k]$ is finite, so the ergodic theorem implies that the limit

$$-\mu(\gamma) = \lim_{n\to\infty} \frac{1}{n}\log(u(n,\omega;\gamma)) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} q_k(\omega) = \mathbb{E}\left[q_0\right] = \mathbb{E}\left(\log u(1,\omega;\gamma)\right) \tag{2.26}$$

holds with probability one. The fact that μ is independent of ω follows from the ergodicity assumption, since we can show (using Lemma 2.3) that $\mu(\gamma)$ is invariant under the action of every π_k, $k \in \mathbb{R}$. Elliptic regularity implies that the limit holds along continuous x, as well. This concludes the proof of Theorem 1.1. □

2.2 Properties of $\mu(\gamma)$ and $u(x,\omega;\gamma)$

In this section we gather some useful observations about the functions $\mu(\gamma)$ and $u(x,\omega;\gamma)$.

Lemma 2.4 *The following bounds hold $\mathbb{P}$-almost surely. If $\gamma' > \gamma > \bar{\gamma}$, then*

$$u(x,\omega;\gamma) \geq u(x,\omega;\gamma')\frac{u(y,\omega;\gamma)}{u(y,\omega;\gamma')} \tag{2.27}$$

for all $0 \leq y \leq x$.

Proof: By (2.24) we know that,

$$\frac{u(x,\omega;\gamma)}{u(y,\omega;\gamma)} = u(x-y,\pi_y\omega;\gamma) \qquad \text{and} \qquad \frac{u(x,\omega;\gamma')}{u(y,\omega;\gamma')} = u(x-y,\pi_y\omega;\gamma')$$

hold for all $x \geq y$. By applying Corollary 2.1 to the function $w(x) = u(x-y,\pi_y\omega;\gamma') - u(x-y,\pi_y\omega;\gamma)$ on the interval $x \in [y,\infty)$, we see that $u(x-y,\pi_y\omega;\gamma) \geq u(x-y,\pi_y\omega;\gamma')$ for all $x \geq y$, so (2.27) must hold for all $x \geq y$. □

Lemma 2.5 *Let $\gamma' > \Gamma_\infty$, $\gamma > \Gamma_\infty$ and $\sigma = 2\gamma - \gamma' > \Gamma_\infty$. Then*

$$(u(x,\omega;\gamma))^2 \leq u(x,\omega;\gamma')u(x,\omega;\sigma) \tag{2.28}$$

holds for all $x \geq 0$. Also, for any $\epsilon > 0$, there is a constant C such that

$$(u(x,\omega;\gamma))^2 \leq u(x,\omega;\gamma')Ce^{-(\sqrt{\sigma-\Gamma_\infty}-\epsilon)x} \tag{2.29}$$

holds for all $x \geq 0$. The constant C depends on σ and ϵ, but not on ω.

Proof: Let $\sigma = 2\gamma - \gamma' > \Gamma_\infty$. Since $\sigma > \Gamma_\infty$, the function $u(x,\omega;\sigma)$ is well-defined. To make the notation simpler in what follows, we let $\theta(x,\omega) = u(x,\omega;\gamma')$ and $\eta(x,\omega) = u(x,\omega;\sigma)$. Now consider the function $z(x,\omega) = \sqrt{\theta(x)\eta(x)}$. A simple computation of z_{xx}/z shows that z satisfies the equation

$$z_{xx} + (g(x,\omega)-\gamma)z = -\frac{z}{4}\left(\frac{\theta_x}{\theta} - \frac{\eta_x}{\eta}\right)^2. \tag{2.30}$$

Because the right side is nonpositive and because $z(0) = 1$ and $z(x) > 0$ for $x > 0$, we may apply Corollary 2.1 to the function $w(x) = u(x,\omega;\gamma) - z$ to conclude that $w \leq 0$ for all $x \geq 0$. Since $u(x,\omega;\gamma) > 0$ this means that

$$(u(x,\omega;\gamma))^2 \leq z(x)^2 = u(x,\omega;\gamma)u(x,\omega;\sigma) \tag{2.31}$$

holds for all $x > 0$. The bound (2.29) now follows immediately from (2.19) applied to the function $w = u(x,\omega;\sigma)$.□

Lemma 2.6 *For $\gamma > \bar{\gamma}$, the function $\gamma \mapsto \mu(\gamma)$ is concave. Also, $(\gamma-\bar{\gamma})^{1/2} \leq \mu(\gamma) \leq (\gamma - g_{min})^{1/2}$.*

Proof: Let $\gamma_1 < \gamma_2$ be such that $\Gamma_\infty < \gamma_1$ and $\gamma_2 \leq 2\gamma_1 + \Gamma_\infty$. Then we may apply Lemma 2.5 with $\gamma = (\gamma_1+\gamma_2)/2$ and $\gamma' = \gamma_2$. In this case, $\sigma = \gamma_1$, and (2.28) implies

$$2\mu(\frac{\gamma_1+\gamma_2}{2}) = 2\mu(\gamma) \geq \mu(\gamma') + \mu(\sigma) = \mu(\gamma_1) + \mu(\gamma_2)$$

Since this holds for all such γ_1 and γ_2, μ must be concave. The upper and lower bound on μ follow from the bounds (2.22) and (2.23) on u. □

Lemma 2.7 *Let $\gamma' > \gamma > \bar{\gamma}$ and $\delta \in [0, \sqrt{\gamma' - g_{min}} - \sqrt{\gamma - g_{min}}\ ]$. Then*

$$u(x,\omega;\gamma') \le e^{-\delta x} u(x,\omega;\gamma), \quad \forall\, x \ge 0 \tag{2.32}$$

holds with probability one. Thus $\mu(\gamma') \ge \mu(\gamma) + \sqrt{\gamma' - g_{min}} - \sqrt{\gamma - g_{min}} > \mu(\gamma)$.

Proof: Consider the function $z = e^{-\delta x} u(x,\omega;\gamma)$ which satisfies

$$z_{xx} + (g(x,\omega) - \gamma')z = e^{-\delta x}\left(\delta^2 u(x,\omega;\gamma) - 2\delta u_x(x,\omega;\gamma) - (\gamma' - \gamma)u(x,\omega;\gamma)\right) \tag{2.33}$$

The function $\eta = \log u(x,\omega;\gamma)$ satisfies $\eta_{xx} + (\eta_x)^2 + (g(x) - \gamma) = 0$. If η_x attains a negative local minimum at a point $\bar{x}$, this implies that $g(\bar{x}) < \gamma$ and $\eta_x(\bar{x}) = -\sqrt{\gamma - g(\bar{x})} \ge -\sqrt{\gamma - g_{min}}$. Therefore,

$$u_x(x,\omega;\gamma) \ge -\sqrt{\gamma - g_{min}}\, u(x,\omega;\gamma) \tag{2.34}$$

holds for all x. Therefore,

$$z_{xx} + (g(x,\omega) - \gamma')z \le e^{-\delta x}\left(\delta^2 u(x,\omega;\gamma) + 2\delta\sqrt{\gamma - g_{min}}\, u(x,\omega;\gamma) - (\gamma' - \gamma)u(x,\omega;\gamma)\right) \tag{2.35}$$

If $\delta \in [0, \sqrt{\gamma' - g_{min}} - \sqrt{\gamma - g_{min}}\ ]$, the right hand side is nonpositive. Now by applying Corollary 2.1 to the function $w = u(x,\omega;\gamma') - z$, we conclude that $u(x,\omega;\gamma') \le z$. □

2.3 Fluctuations in the tail of u

We are ready to prove Theorem 1.2 which describes fluctuations in the tail of $u(x,\omega;\gamma)$. We have shown that the functions $u(x,\omega;\gamma)$ decay exponentially with rate $-\mu(\gamma)$. Now consider the partial sums

$$S_N(\omega;\gamma) = \log u(N,\omega;\gamma) = \sum_{k=0}^{N-1} q_k(\omega),$$

where $q_k(\omega) = \log u(1, \pi_k\omega;\gamma)$. We will show that for fixed $\gamma \in (\bar{\gamma}, \infty)$, S_N satisfies the central limit theorem, meaning that the quotient $(S_N + N\mu(\gamma))/\sqrt{N}$ is asymptotically normally distributed as $N \to \infty$. One approach to obtaining a central limit theorem (CLT) for S_N is to use the method of martingale approximation. This strategy can be made to work via the following theorem. Let $\mathcal{F}_k^- \subset \mathcal{F}$ denote the σ-algebra generated by $g(x,\omega)$ for $x \le k$. Also, let $\mathcal{F}_j^+$ denote the σ-algebra generated by $g(x,\omega)$ for $x \ge j$.

Theorem 2.1 (See Hall and Heyde [10], Section 5.4) *Suppose that a stationary sequence $\{\eta_k\}_k \subset L^2(\Omega,\mathcal{F},\mathbb{P})$ satisfies $\mathbb{E}[\eta_k] = 0$, and that the two series*

$$\sum_{k=1}^{\infty}\left(\eta_0 - \mathbb{E}[\eta_0|\mathcal{F}_k^-]\right) \qquad \text{and} \qquad \sum_{k=1}^{\infty}\mathbb{E}[\eta_k|\mathcal{F}_0^-] \tag{2.36}$$

converge in $L^2(\Omega,\mathcal{F},\mathbb{P})$. Then, the limit

$$\sigma^2 = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[\left|\sum_{k=0}^{N-1}\eta_k\right|^2\right] \tag{2.37}$$

exists and is finite. If $\sigma^2 > 0$ and $S_k = \sum_{j=0}^{k-1}\eta_j$, then as $n \to \infty$ the family of processes

$$\bar{Z}_n(x) = \frac{1}{\sigma\sqrt{n}}\left(S_k + (nx - k)\eta_k\right), \quad k \le nx \le k+1, \quad k = 0, 1, \ldots, n-1 \tag{2.38}$$

converges weakly to a standard Brownian motion on $[0,1]$, in the sense of $C([0,1])$ with the topology of uniform convergence.

In order to apply the theorem with $\eta_k = q_k + \mu = \log u(1, \pi_k\omega; \gamma) + \mu$, we will require a mixing condition on the random field $g(x, \omega)$. Suppose that $\phi : [0, \infty) \to [0, \infty)$ is a continuous decreasing function such that $\phi(+\infty) = 0$. We say that the random field $g(x, \omega)$ is **ϕ-mixing** if the following holds: for all $j \geq k$ and any $\xi \in L^2(\Omega, \mathcal{F}_k^-, \mathbb{P})$ and $\eta \in L^2(\Omega, \mathcal{F}_j^+, \mathbb{P})$,

$$|\mathbb{E}[\xi\eta] - \mathbb{E}[\xi]\mathbb{E}[\eta]| \leq (\phi(j-k))^{1/2} \left(\mathbb{E}[\xi^2]\mathbb{E}[\eta^2]\right)^{1/2}. \tag{2.39}$$

Let us suppose that $g(x, \omega)$ is ϕ-mixing for some ϕ satisfying $\sum_{k=1}^{\infty} \phi(k)^{1/2} < \infty$.

Now we consider the first series in (2.36):

$$\sum_{k=1}^{\infty} \left(\eta_0 - \mathbb{E}[\eta_0|\mathcal{F}_k^-]\right) = \sum_{k=1}^{\infty} \left(q_0 - \mathbb{E}[q_0|\mathcal{F}_k^-]\right).$$

We need to show that $\mathbb{E}\left[|q_0 - \mathbb{E}[q_0|\mathcal{F}_k^-]|^2\right]$ decays sufficiently fast as $k \to \infty$ so that the series converges in L^2. Since $\mathbb{E}[q_0|\mathcal{F}_k^-]$ is the best L^2 approximation of q_0 that is $\mathcal{F}_k^-$-measurable, we can prove the desired result by constructing an $\mathcal{F}_k^-$-measurable random variable q_0' which is very close to q_0 (error decays fast with $k \to \infty$). Since $q_0 = \log u(1, \omega; \gamma)$, a natural candidate is obtained by solving the differential equation with homogeneous Dirichlet condition at $x = k$. Let $w_k(x, \omega; \gamma)$ solve

$$w_{xx} + (g(x, \omega) - \gamma)w = 0, \quad x \in [0, k] \tag{2.40}$$

with $w_k(0, \omega; \gamma) = 1$ and $w_k(k, \omega; \gamma) = 0$. Then for all $x \in (0, k)$,

$$u(x, \omega; \gamma) = w_k(x, \omega; \gamma) + z_k(x, \omega; \gamma)u(k, \omega; \gamma)$$

where $z_k(x, \omega; \gamma)$ solves (2.40) with $z_k(0) = 0$ and $z_k(k) = 1$. Both w_k and z_k are $\mathcal{F}_k^-$-measurable. Therefore, with $q_0' = \log w_k(1, \omega; \gamma)$ we have

$$q_0 - q_0' = -\log\left(\frac{w_k(1, \omega; \gamma)}{u(1, \omega; \gamma)}\right) = -\log\left(1 - \frac{z_k(1, \omega; \gamma)u(k, \omega; \gamma)}{u(1, \omega; \gamma)}\right).$$

Let $\epsilon = (\sqrt{\gamma - \bar{\gamma}})/2 > 0$. By Lemma 2.2 there is a constant C_1 independent of k and ω such that $u(k, \omega; \gamma) \leq C_1 e^{-k\epsilon}$. By applying the same Lemma to z_k, we see that $0 \leq z_k(1, \omega; z) \leq C_1 e^{(1-k)\epsilon}$ also holds. The maximum principle implies that $u(1, \omega; \gamma)$ is bounded below by $\rho(1) > 0$ where $\rho(x)$ satisfies $\rho_{xx} + (g_{min} - \gamma)\rho$ for $x \in [0, 2]$ with $\rho(0) = 1$ and $\rho(2) = 0$. Consequently, there is a constant C_2, depending only on γ, such that

$$q_0 - q_0' \leq \log\left(1 + C_2 e^{-k\epsilon}\right)$$

holds for all $k \geq 1$, with probability one. This upper bound, and a similar lower bound, imply that

$$\mathbb{E}\left[|q_0 - \mathbb{E}[q_0|\mathcal{F}_k^-]|^2\right] \leq \mathbb{E}\left[|q_0 - q_0'|^2\right] \leq C_3 e^{-k\epsilon}$$

for some constant C_3 depending only on ϵ. So, the first series in (2.36) converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Next, we consider the series

$$\sum_{k=1}^{\infty} \mathbb{E}[\eta_k|\mathcal{F}_0^-].$$

We have to show that $\mathbb{E}[|\mathbb{E}[\eta_k|\mathcal{F}_0^-]|^2]$ decays rapidly as $k \to \infty$, and this is where we use the assumption that g is ϕ-mixing. Observe that each q_k and η_k is $\mathcal{F}_k^+$ measurable. This follows from the fact that $q_k = \log u(1, \pi_k\omega) = \log \tilde{u}^k(k+1, \omega)$, where $\tilde{u}^k$ is defined for $x \geq k$ by

$$\tilde{u}^k_{xx} + (g(x, \omega) - \gamma)\tilde{u}^k = 0, \quad x > k; \quad \tilde{u}^k(k, \omega) = 1; \quad \lim_{x \to \infty} \tilde{u}^k(x, \omega) = 0.$$

Hence, $\tilde{u}^k$ depends only on $g(x,\omega)$ for $x \geq k$. Let $\theta \in L^2(\Omega, \mathcal{F}_0^-, \mathbb{P})$ with $\mathbb{E}(\theta^2) = 1$. Because η_k is $\mathcal{F}_k^+$ measurable, θ is $\mathcal{F}_0^-$ measurable, and $\mathbb{E}[\eta_k] = 0$, we have

$$\left|\mathbb{E}\left(\theta \mathbb{E}[\eta_k|\mathcal{F}_0^-]\right)\right| = \left|\mathbb{E}\left(\theta\eta_k\right)\right| = \left|\mathbb{E}\left(\theta \mathbb{E}[\eta_k|\mathcal{F}_j^+]\right)\right| \leq \phi(k)^{1/2}\mathbb{E}[(\eta_k)^2]^{1/2} \leq C\phi(k)^{1/2}. \tag{2.41}$$

The last two inequalities follow from the mixing condition (2.39). Therefore, as $\theta \in L^2(\Omega, \mathcal{F}_0^-, \mathbb{P})$ was arbitrary, we conclude that $\mathbb{E}\left(|\mathbb{E}[\eta_k|\mathcal{F}_0^-]|^2\right)^{1/2} \leq C\phi(k)^{1/2}$. Now the triangle inequality implies

$$\mathbb{E}\left[(\sum_{k=m}^{n} \mathbb{E}[\eta_k|\mathcal{F}_0^-])^2\right]^{1/2} \leq \sum_{k=m}^{n} \mathbb{E}\left[(\mathbb{E}[\eta_k|\mathcal{F}_0^-])^2\right]^{1/2} \leq \sum_{k=m}^{n} C\phi(k)^{1/2}.$$

Since the series $\sum_{k=1}^{\infty} \phi(k)^{1/2}$ converges, it follows that $\sum_{k=1}^{\infty} \mathbb{E}[\eta_k|\mathcal{F}_0^-]$ converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We have shown that the random variables $\eta_k = q_k + \mu = \log u(1, \pi_k\omega; \gamma) + \mu$ satisfy the hypotheses of Theorem 2.1. Therefore, Theorem 1.2 follows immediately from an application of Theorem 2.1.

We can construct a class of examples for which the variance $\sigma^2 > 0$ in Theorem 1.2 is positive. Let $\{g_k(\tilde{\omega})\}_{k=-\infty}^{\infty}$ be a family of independent, identically distributed random variables on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Assume also that $g_{min} \leq g_k \leq g_{max}$ with probability one and $\mathrm{Var}(g_k) > 0$. Now we define $(\Omega, \mathcal{F}, \mathbb{P})$ to be the product space with product measure $\mathbb{P}$ defined in the usual way, and for $k \in \mathbb{Z}$ let π_k act on Ω as the shift-operator in the natural way: $\pi_k : (\ldots, \tilde{\omega}_{-1}, \tilde{\omega}_0, \tilde{\omega}_1, \ldots) \mapsto (\ldots, \tilde{\omega}_{k-1}, \tilde{\omega}_k, \tilde{\omega}_{k+1}, \ldots)$. Define the continuous, piecewise linear, random function $g(x,\omega)$ by

$$g(x,\omega) = (1 - x + k)g_k(\omega) + (x - k)g_{k+1}(\omega) \quad \forall\, x \in [k, k+1), \tag{2.42}$$

for each $k \in \mathbb{Z}$. Because the g_k are independent and idendically distributed, the family $\{\pi_k\}_{k\in\mathbb{Z}}$ is measure-preserving and ergodic in its action on $(\Omega, \mathcal{F}, \mathbb{P})$. So, Theorem 1.1 applies. Moreover, the mixing condition holds so Theorem 1.2 also applies.

Proposition 2.1 *Let $g(x,\omega)$ be defined by (2.42). For each $\gamma > \bar{\gamma}$, the constant σ defined by Theorem 1.2 is positive.*

Proof of Proposition 2.1: If $\eta_k = \log(u(1, \pi_k\omega; \gamma)) + \mu$, then

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1}\sum_{j=0}^{n-1} \mathbb{E}[\eta_k\eta_j] = \mathbb{E}[\eta_0^2] + \lim_{n\to\infty} \frac{2}{n} \sum_{k=1}^{n-1} (n-k)\mathbb{E}[\eta_0\eta_k]. \tag{2.43}$$

We claim that every term in the sum (2.43) is nonnegative, while $\mathbb{E}[\eta_0^2]$ must be positive. It is useful to think of the variables $\eta_j(\omega)$ as functions of the random sequence $\{g_k\}_{k=j}^{\infty}$. For each positive integer K, we may approximate η_0 by $\eta_0'(g_0, g_1, \ldots, g_K)$ where $\eta_0' = \log(w_K(1,\omega;\gamma)) + \mu$, and w_K was defined at (2.40). Similarly, we may approximate η_j by $\eta_j'(g_j, g_{j+1}, \ldots, g_{j+K})$ where $\eta_j' = \log w_K(1, \pi_j\omega; \gamma) + \mu$. Both η_0' and η_j' are increasing functions of their arguments, so Lemma 2.8 implies that $Cov(\eta_0'\eta_j') \geq 0$. Since $\mathbb{E}[\eta_0\eta_j] = \lim_{K\to\infty} Cov[\eta_0'\eta_j']$, it follows that $\mathbb{E}[\eta_0\eta_j] \geq 0$ for all $j \geq 0$. Because η_0 is an increasing function of each of the variables $\{g_k\}_{k=0}^{\infty}$ and $\mathrm{Var}(g_k) > 0$, η_0 cannot be constant. Thus $\mathbb{E}[\eta_0^2] > 0$. $\square$

Lemma 2.8 *Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$ be a vector of independent random variables. Suppose that X_j takes values in the interval $I_j \subset \mathbb{R}$: $\mathbb{P}(X_j \in I_j) = 1$ for $j = 1, \ldots, n$. Let $I = \prod_{j=1}^{n} I_j$. Suppose* *that* $f(x): I \to \mathbb{R}$ *and* $g(x): I \to \mathbb{R}$ *are such that* $\mathbb{E}[f(X)^2] < \infty$ *and* $\mathbb{E}[g(X)^2] < \infty$ *and for each* $j = 1, \dots, n$ *either*

$$\frac{\partial f(x)}{\partial x_j} \geq 0 \quad and \quad \frac{\partial g(x)}{\partial x_j} \geq 0$$

holds for all $x \in I$ *or*

$$\frac{\partial f(x)}{\partial x_j} \leq 0 \quad and \quad \frac{\partial g(x)}{\partial x_j} \leq 0$$

holds for all $x \in I$. *Then* $Cov(f(X), g(X)) \geq 0$.

Proof: This follows from Lemma 2.3 of [6]. If $X' = (X_1', \dots, X_n')$ is an independent copy of the random vector X, that lemma gives the representation

$$Cov(f(X), g(X)) = \frac{1}{2} \sum_{A \subsetneq [n]} \frac{1}{\binom{n}{|A|}(n - |A|)} \sum_{j \notin A} \mathbb{E}[\Delta_j g(X) \Delta_j f(X^A)]$$

where $\Delta_j g(X) = g(X_1, \dots, X_{j-1}, X_j', X_{j+1}, \dots, X_n) - g(X)$ and X^A is the random vector given by $X_i^A = X_i'$ if $i \in A$ and $X_i^A = X_i$ if $i \notin A$. By our assumptions on f and g, both $\Delta_j g(X)$ and $\Delta_j f(X^A)$ have the same sign, almost surely. Thus, each term in the sum satisfies $\mathbb{E}[\Delta_j g(X) \Delta_j f(X^A)] \geq 0$. □

3 CLT for traveling front solutions of the linear equation

In this section we prove Theorem 1.3. For $\gamma > \bar{\gamma}$ fixed, let $c = c(\gamma)$ and $\mu = \mu(\gamma)$. The function $\varphi(t, x, \omega; \gamma) = e^{\gamma t} u(x, \omega; \gamma)$ solves the linear boundary value problem

$$\begin{aligned} \partial_t \varphi = \varphi_{xx} + g(x, \omega)\varphi, \quad x > 0, t \in \mathbb{R} \\ \varphi(t, 0) = e^{\gamma t}, \quad t > 0, \\ \varphi(0, x) = u(x, \omega; \gamma), \quad x > 0, \end{aligned} \tag{3.44}$$

For $\delta \in (0, 1]$, let us define

$$Y_\delta(t, \omega; \gamma) = \sup\{x \geq 0 \mid \varphi(t, x, \omega; \gamma) \geq \delta\}. \tag{3.45}$$

Sometimes we abbreviate: $Y_\delta(t)$. In Theorem 1.3, $Y(t, \omega; \gamma) = Y_\delta(t)$ with $\delta = 1/2$. This stochastic process is nonnegative and non-decreasing in t. However, because u may not be monotone decreasing, $Y_\delta(t)$ may have jumps. The $\mathcal{F}$-measurability of $Y_\delta(t, \omega; \gamma)$ may be proved as in [19].

We define the random function $R(x, \omega; \gamma) : [0, \infty) \times \Omega \to \mathbb{R}$ by

$$R(x, \omega; \gamma) = \log u(x, \omega; \gamma) + \mu(\gamma)x \tag{3.46}$$

so that

$$u(x, \omega; \gamma) = e^{-\mu x + R(x, \omega; \gamma)},$$

and $R(x, \omega; \gamma)/x \to 0$ with probability one as $x \to \infty$. For suitable hypotheses on g, Proposition 2.1 shows that $R(x, \omega; \gamma)$ behaves like a Brownian motion for large x. Specifically, for any $M > 0$, the family of processes $V_n(x, \omega; \gamma) = R(xn, \omega; \gamma)/(\sigma\sqrt{n})$ converges weakly in $C([0, M])$ to a Brownian motion, as $n \to \infty$. For $\delta \in (0, 1]$ fixed, the wave's position is

$$Y_\delta(t, \omega) = \sup\{x \geq 0 \mid \; -x + \mu^{-1} R(x, \omega; \gamma) + ct = \mu^{-1} \log(\delta)\}.$$

Therefore, if we define $h_t = Y_\delta(t) - ct$ we have

$$h_t(\omega) = \sup\{h \geq -ct \mid \ -h + \mu^{-1}R(h+ct,\omega;\gamma) = \mu^{-1}\log(\delta)\} \tag{3.47}$$

Theorem 1.3 now follows immediately from the next lemma applied to h_t, with $W = \mu^{-1}R$ and $\delta = 1/2$.

Lemma 3.1 *Let $\kappa > 0$. Suppose that $W(x,\omega) : [0,\infty) \times \Omega \to \mathbb{R}$ is a continuous, random process on $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying*

$$\lim_{x\to\infty} \frac{W(x,\omega)}{x} = 0$$

and $W(0,\omega) = 0$, $\mathbb{P}$-almost surely. Also, suppose that the family of processes $\{n^{-1/2}W(nx,\omega)\}_{n=1}^{\infty}$ satisfies $W(0,\omega) = 0$, $\mathbb{P}$-a.s., and converges weakly as $n \to \infty$ to $\kappa B(x)$ where $B(x)$ is a standard Brownian motion on the interval $[0,M]$, for any M (in the sense of weak convergence of measures on $C([0,1])$ with the topology of uniform convergence). Let $r < 0$ and $c > 0$. Define the random process h_t by

$$h_t(\omega) = \sup\{h \geq -ct \mid \ W(h+ct,\omega) = h + r\}. \tag{3.48}$$

Then as $t \to \infty$, $(h_t)/\sqrt{t}$ converges in distribution to a Gaussian random variable with zero mean and variance $\kappa^2 c^2$. If $\kappa^2 > 0$, then the family of processes

$$H_n(t) = \frac{1}{\kappa c\sqrt{n}} h_{nt} \tag{3.49}$$

converges weakly (as $n \to \infty$) to a standard Brownian motion on $[0,T]$, in the Skorohod space D.

Proof: First we show that the finite dimensional distributions of $H_n(t)$ converge to those of a Brownian motion. Then we show that the induced family of measures is tight in the Skorohod space D. These two conditions imply the weak convergence stated in the lemma (see Chapter 3, Section 15 of [4]).

For $0 \leq t_1 < t_2 < \cdots < t_k \leq T$, we show that the finite dimensional distributions of $\{h_{t_i n}/\sqrt{n}\}_{i=1}^{k}$ converge to those of $\{\kappa B(ct_i)\}_{i=1}^{k}$. Since $c > 0$ and $r < 0$, the assumptions on W imply that with probability one h_t is well-defined and finite for all $t > 0$. Let $y_i = h_{t_i n} + ct_i n$, $i = 1,\ldots,k$. Observe that y_i is defined as the largest point of intersection between the line $y \mapsto y - ct_i n + r$ and the function $W(y)$ which behaves like Brownian motion:

$$W(y_i) = y_i - ct_i n + r. \tag{3.50}$$

Let us now define a subset of Ω on which we can control the possible location of the intersection points. For $\epsilon \in (0,1/2)$ and $\hat{y} \in \mathbb{R}$, let

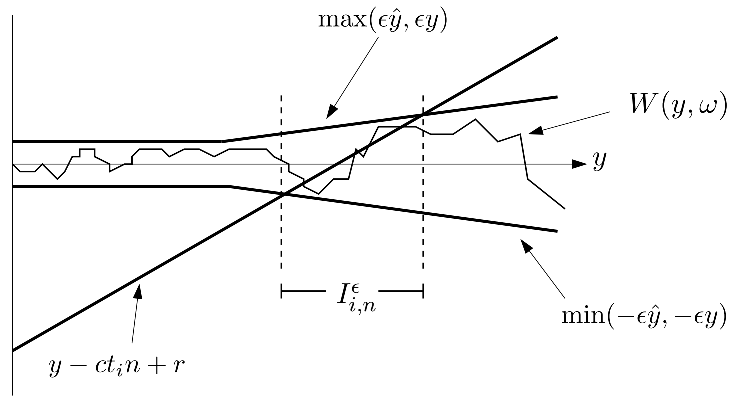
$$S(\epsilon,\hat{y}) = \{\omega \in \Omega \mid \ \min(-\epsilon\hat{y}, -\epsilon y) \leq W(y,\omega) \leq \max(\epsilon\hat{y}, \epsilon y), \quad \forall y \geq 0\}.$$

Because $W(\cdot,\omega)$ is continuous and grows at most sublinearly, we may choose a constant $\hat{y} = \hat{y}(\epsilon)$ sufficiently large so that $\mathbb{P}\left(S(\epsilon,\hat{y}(\epsilon))\right) \geq 1 - \epsilon$. For $\hat{y}$ defined in this way, set $S_\epsilon = S(\epsilon,\hat{y}(\epsilon))$. By considering the intersection of the line $y \mapsto y - ct_i n + r$ with the functions $y \mapsto \max(\epsilon\hat{y}, \epsilon y)$ and $y \mapsto \min(-\epsilon\hat{y}, -\epsilon y)$, we see that if $\omega \in S_\epsilon$, then any solution of (3.50) must satisfy

$$\min\left((1-2\epsilon)(ct_i n - r)\ ,\ -\epsilon\tilde{y} + ct_i n - r\right) \leq y_i \leq \max\left((1+2\epsilon)(ct_i n - r)\ ,\ \epsilon\tilde{y} + ct_i n - r\right)$$

Therefore, if $\omega \in S_\epsilon$, then y_i must lie in the interval $I_{i,n}^{\epsilon}$:

$$I_{i,n}^{\epsilon} = \{y \in \mathbb{R} \mid |y - ct_i n| \leq \max\left(2\epsilon ct_i n + (1+2\epsilon)|r|\ ,\ \epsilon\tilde{y} + |r|\right)\}.$$



See the figure for an illustration of this point.

Let $\alpha_i \in \mathbb{R}$. Since $\mathbb{P}(S_\epsilon^C) \le \epsilon$ and $h_{t_i n} = W(y_i) - r$, we have

$$\begin{aligned}
\mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\}\right) &= \mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\} \cap S_\epsilon\right) + \mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\} \cap S_\epsilon^C\right) \\
&\le \mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\} \cap S_\epsilon\right) + \epsilon \\
&= \mathbb{P}\left(\{n^{-1/2}W(y_i) > rn^{-1/2} + \alpha_i,\ \forall\ i\} \cap S_\epsilon\}\right) + \epsilon \\
&\le \mathbb{P}\left(\{\omega \mid \sup_{y \in I_{i,n}^\epsilon} n^{-1/2}W(y) > rn^{-1/2} + \alpha_i,\ \forall\ i\}\right) + \epsilon.
\end{aligned}$$

The family of processes $\{n^{-1/2}W(nt,\omega)\}_{n=1}^\infty$ converges weakly in $C([0, 2cT])$ to $\kappa B(t)$, so it is tight in $C([0, 2cT])$. Therefore, as $n \to \infty$ the last quantity converges to

$$\lim_{n\to\infty} \mathbb{P}\left(\{\omega \mid \sup_{y \in I_{i,n}^\epsilon} n^{-1/2}W(y) > rn^{-1/2} + \alpha_i,\ \forall\ i\}\right) = \mathbb{P}\left(\sup_{\substack{s \ge 0 \\ |s - ct_i| \le 2\epsilon ct_i}} \kappa B(s) > \alpha_i,\ \forall\ i\right). \quad (3.51)$$

Since $\epsilon > 0$ is arbitrary and $B(s)$ is almost-surely continuous, we have

$$\limsup_{n\to\infty} \mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\}\right) \le \mathbb{P}\left(\{\kappa B(ct_i) > \alpha_i,\ \forall\ i\}\right).$$

A lower bound is proved in a similar manner:

$$\begin{aligned}
\mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\}\right) &\ge \mathbb{P}\left(\{n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\} \cap S_\epsilon\right) \\
&\ge \mathbb{P}\left(\{\omega \mid (W(ct_i n) - r)n^{-1/2} > \alpha_i - n^{-1/2} \sup_{y \in I_{i,n}^\epsilon} |W(y) - W(ct_i n)|,\ \forall\ i\}\right).
\end{aligned}$$

Because $\{n^{-1/2}W(nt,\omega)\}_{n=1}^\infty$ is tight in $C([0, 2cT])$, for any $\delta_1, \delta_2 > 0$ we may choose ϵ smaller and n sufficiently large so that

$$\mathbb{P}\left(\{\omega \mid n^{-1/2} \sup_{y \in I_{i,n}^\epsilon} |W(y) - W(ct_i n)| < \delta_1, \quad \forall\ i\}\right) \ge 1 - \delta_2.$$

Therefore,

$$\mathbb{P}\left(\{\omega \mid n^{-1/2}h_{t_i n} > \alpha_i,\ \forall\ i\}\right) \ \ge\ \mathbb{P}\left(\{(W(ct_i n) - r)n^{-1/2} > \alpha_i - \delta_1,\ \forall\ i\}\right) - \delta_2 \quad (3.52)$$

for n sufficiently large. Thus, since δ_1 and δ_2 may be chosen arbitrarily small,

$$\liminf_{n\to\infty} \mathbb{P}\left(\{n^{-1/2}h_{t_in} > \alpha_i,\ \forall\ i\}\right) \geq \mathbb{P}\left(\kappa B(ct_i) \geq \alpha_i,\ \forall\ i\right) \tag{3.53}$$

holds, as well. This proves convergence of the finite dimensional distributions.

Now we prove tightness in D. We will show that for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$,

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\substack{s,t\in[0,T]\\|s-t|\leq\delta}} |H_n(t) - H_n(s)| > \epsilon_1\right) \leq \epsilon_2 \tag{3.54}$$

holds if $\delta > 0$ is sufficiently small, and

$$\sup_{n\geq 1} \mathbb{P}\left(|H_n(0)| > \alpha\right) \leq \epsilon_2 \tag{3.55}$$

holds if α is large enough. These two conditions imply tightness in D (and in $C([0,T])$, if the processes $H_n(t)$ were continuous [4]). For given n and any $s, t \in [0,T]$, let $y_s = h_{sn} + csn$ and $y_t = h_{st} + cst$. Then observe that

$$|H_n(t) - H_n(s)| = \frac{1}{\kappa c\sqrt{n}}|W(y_s) - W(y_t)|.$$

Just as before, we see that for all $\omega \in S_\epsilon$

$$|y_s - csn| \leq \max\left(2\epsilon csn + (1+2\epsilon)|r|\ ,\ \epsilon\tilde{y} + |r|\right),$$

and

$$|y_t - ctn| \leq \max\left(2\epsilon ctn + (1+2\epsilon)|r|\ ,\ \epsilon\tilde{y} + |r|\right).$$

Therefore, if $s, t \in [0,T]$, $|s-t| \leq \delta$, and $\omega \in S_\epsilon$, we must have $n^{-1}|y_s - y_t| \leq c\delta + 4Tc\epsilon + 6n^{-1}|r| + 2n^{-1}\epsilon\tilde{y}(\epsilon)$. For a given δ, let $\epsilon > 0$ be sufficiently small and then $n_{\delta,\epsilon}$ sufficiently large so that $c\delta + 4Tc\epsilon + 6n^{-1}|r| + 2n^{-1}\epsilon\tilde{y}(\epsilon) \leq 2c\delta$ holds for all $n \geq n_{\delta,\epsilon}$. Now we return to (3.54). For all $n \geq n_{\delta,\epsilon}$ we have

$$\begin{aligned}
\mathbb{P}\left(\sup_{\substack{s,t\in[0,T]\\|s-t|\leq\delta}} |H_n(t) - H_n(s)| > \epsilon_1\right) &= \mathbb{P}\left(\sup_{\substack{s,t\in[0,T]\\|s-t|\leq\delta}} \frac{1}{\sqrt{n}}|W(y_s) - W(y_t)| > \kappa c\epsilon_1\right) \\
&\leq \mathbb{P}\left(\sup_{\substack{r,\tau\in[0,2cT]\\|r-\tau|\leq 2c\delta}} \frac{1}{\sqrt{n}}|W(nr) - W(n\tau)| > \kappa c\epsilon_1\right) + \mathbb{P}(S_\epsilon^C)
\end{aligned}$$

Recall that $\mathbb{P}(S_\epsilon^C) \leq \epsilon$. Now because $\epsilon \in (0, 1/2)$ is arbitrary and because

$$\limsup_{n\to\infty} \mathbb{P}\left(\sup_{\substack{r,\tau\in[0,2cT]\\|r-\tau|\leq 2c\delta}} \frac{1}{\sqrt{n}}|W(nr) - W(n\tau)| > \kappa c\epsilon_1\right) \leq \epsilon_2/2$$

if δ is sufficiently small, this proves that (3.54) holds for δ sufficiently small.

Finally, we prove that (3.55) holds. Since $W(x)/x \to 0$ with probability one, $W(h,\omega) = h + r$ can have no solution if h is sufficiently large, depending on ω and r. Therefore, for any $\epsilon > 0$, $\mathbb{P}(h_0 \geq K) \leq \epsilon$ if K is sufficiently large. Since $h_0 \geq 0$ and $H_n(0) = h_0/(\kappa c\sqrt{n})$, this implies that for any $\epsilon > 0$, $\mathbb{P}(|H_n(0)| > \epsilon) \to 0$ as $n \to \infty$, which clearly implies (3.55). This completes the proof of Lemma 3.1. □

4 Estimates for the supercritical solutions of the linear equation

Here we prove some estimates that will enable us to compare solutions of the nonlinear problem (1.1) to the functions $\varphi(t,x,\omega;\gamma)$ which solve the linearized problem (1.2). The estimates are restricted to the supercritical regime $\gamma \in (\bar{\gamma},\gamma^*)$, which corresponds to fronts moving faster than the minimal speed, $c(\lambda) > c^*$. In the following analysis we will be comparing two functions $\varphi(t,x,\omega;\gamma)$ and $\varphi(t,x,\omega;\gamma')$ corresponding to two values γ and γ'. With the parameter γ fixed, we will use the abbreviated notation $Y_1(t)$ and h_t to refer to

$$Y_1(t,\omega,\gamma) = \sup\{x \geq 0 \mid \; \varphi(t,x,\omega;\gamma) = 1\}$$

and

$$h_t(\omega;\gamma) = Y_1(t,\omega;\gamma) - c(\gamma)t,$$

always with parameter γ, not γ'.

Lemma 4.1 *Let $\gamma \in (\bar{\gamma},\gamma^*)$ and $\mu = \mu(\gamma)$. There are constants $\beta > 0$, $\gamma' \in (\gamma,\gamma^*)$, and $C_3 > 1$ such that the following holds with probability one: If for some time $\tau > 0$ and some constants $C_1(\omega), C_2(\omega) > 0$, the function $w(t,x,\omega)$ satisfies*

$$\begin{aligned} w_t &= w_{xx} + g(x,\omega)w, \quad x > 0, \quad t > \tau \\ w(t,x,\omega) &\leq C_1(\omega)\varphi(t,x,\omega;\gamma), \quad \forall \; x \geq 0, \; t \geq \tau. \\ w(\tau,x,\omega) &\leq C_2(\omega)(\varphi(\tau,x,\omega;\gamma))^2, \quad \forall \; x \geq 0, \end{aligned}$$

then,

$$w(t,x+Y_1(t),\omega) \leq \max(C_1(\omega),C_2(\omega)C_3)\,\frac{\varphi(t,x+Y_1(t),\omega;\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')} \tag{4.56}$$

and

$$w(t,x+Y_1(t),\omega) \leq \max(C_1(\omega),C_2(\omega)C_3)u(x,\pi_{Y_1(t)}\omega,\gamma')e^{-\beta(t-\tau)}E(t,\tau,\omega,\gamma,\gamma') \tag{4.57}$$

hold for all $t \geq \tau$ and $x \geq -Y_1(t)$, where $\mu' := \mu(\gamma') > \mu$ and

$$E(t,\tau,\omega,\gamma,\gamma') = \exp\left(-\mu'(h_t(\omega;\gamma) - h_\tau(\omega;\gamma)) + R(Y_1(t),\omega,\gamma') - R(Y_1(\tau),\omega,\gamma')\right)$$

The constant C_3 does not depend on ω or τ.

The significance of the bounds in Lemma 4.1 lies in the fact that $\gamma' > \gamma$ corresponds to a wave moving more slowly than $\varphi(t,x,\omega;\gamma)$. Observe that γ' appears in (4.56) and (4.57), although $Y_t(t)$ corresponds to γ. Another important point is that the terms in the exponent defining E grow at most sublinearly in t and τ. Consequently, we have the following estimates on E which we use later:

Proposition 4.1 *Let $E(t,\tau,\gamma,\gamma',\omega)$ be defined as in Lemma 4.1. For any $\delta > 0$, the random variable*

$$\xi(\omega) = \sup_{t\geq 0} e^{-\delta t}E(t,0,\gamma,\gamma',\omega)$$

is finite with probability one, and for any $\epsilon > 0$,

$$\lim_{t\to\infty}\mathbb{P}\left(\int_0^t e^{-\delta(t-\tau)}E(t,\tau,\gamma,\gamma',\omega)\,d\tau > e^{\epsilon\sqrt{t}}\right) = 0. \tag{4.58}$$

Proof of Lemma 4.1: Using the upper bounds on w, we will show that we can fit above w a wave that moves more slowly than $\varphi(t,x,\omega;\gamma)$. Let $\epsilon = \gamma - \bar{\gamma} > 0$. Because we have assumed $\gamma \in (\bar{\gamma}, \gamma^*)$, we may choose $\gamma' \in (\gamma, \min(\gamma^*, \gamma+\epsilon))$ such that $c' := c(\gamma') < c(\gamma)$ and $\mu' = \mu(\gamma') > \mu(\gamma)$. Since $\varphi(\tau, Y_1(\tau,\omega,\gamma),\omega;\gamma) = 1$, we have, with probability one,

$$\begin{aligned} w(\tau,x,\omega) &\leq C_2(\omega)(\varphi(\tau,x,\omega,\gamma))^2 \\ &= C_2(\omega)\frac{\varphi(\tau,x,\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')}\frac{(\varphi(\tau,x,\omega;\gamma))^2}{(\varphi(\tau,Y_1(\tau),\omega,\gamma))^2}\frac{\varphi(\tau,Y_1(\tau),\omega;\gamma')}{\varphi(\tau,x,\omega;\gamma')} \\ &= C_2(\omega)\frac{\varphi(\tau,x,\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')}\frac{(u(x-Y_1(\tau),\pi_{Y_1(\tau)}\omega;\gamma))^2}{u(x-Y_1(\tau),\pi_{Y_1(\tau)}\omega;\gamma')} \end{aligned}$$

for all $x \geq Y_1(\tau) = Y_1(\tau,\omega,\gamma)$. Since $2\gamma - \gamma' > \bar{\gamma}$, we see from Lemma 2.5 that this is bounded by

$$w(\tau,x,\omega) \leq C_2(\omega)C_3\frac{\varphi(\tau,x,\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')} \tag{4.59}$$

for a constant C_3 that depends on γ and γ', but not on ω or τ.

For $0 \leq x \leq Y_1(\tau,\omega,\gamma)$, and $t \geq \tau$, Lemma 2.4 implies that

$$\begin{aligned} w(t,x,\omega) \leq C_1(\omega)\varphi(t,x,\omega;\gamma) &\leq C_1(\omega)\varphi(t,x,\omega,\gamma')\frac{\varphi(t,Y_1(\tau),\omega;\gamma)}{\varphi(t,Y_1(\tau),\omega;\gamma')} \\ &= C_1(\omega)\varphi(t,x,\omega,\gamma')\frac{e^{\gamma(t-\tau)}\varphi(\tau,Y_1(\tau),\omega;\gamma)}{e^{\gamma'(t-\tau)}\varphi(\tau,Y_1(\tau),\omega;\gamma')} \\ &\leq C_1(\omega)\frac{\varphi(t,x,\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')}. \end{aligned} \tag{4.60}$$

The last inequality follows from the fact that $\varphi(\tau,Y_1(\tau),\omega;\gamma) = 1$. Combining this with (4.59) and applying the maximum principle, we conclude that

$$w(t,x,\omega) \leq \max(C_1(\omega), C_2(\omega)C_3)\frac{\varphi(t,x,\omega;\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')} \tag{4.61}$$

holds for all $t \geq \tau$ and all $x \geq 0$. This proves (4.56). The second bound (4.57) now follows from this and Lemma 2.3:

$$\frac{\varphi(t,x+Y_1(t),\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')} = \frac{e^{\gamma'(t-\tau)}u(x,\pi_{Y_1(t)}\omega,\gamma')u(Y_1(t),\omega;\gamma')}{u(Y_1(\tau),\omega,\gamma')}. \tag{4.62}$$

Using $Y_1(\tau) = c\tau + h_\tau(\omega)$ in the last quotient, we obtain

$$\frac{\varphi(t,x+Y_1(t),\omega,\gamma')}{\varphi(\tau,Y_1(\tau),\omega;\gamma')} = u(x,\pi_{Y_1(t)}\omega,\gamma')e^{-\beta(t-\tau)}E(t,\tau,\omega,\gamma,\gamma') \tag{4.63}$$

with $\beta = \mu'(c-c') > 0$ and

$$E(t,\tau,\omega,\gamma,\gamma') = \exp\left(-\mu'(h_t(\omega)-h_\tau(\omega)) + R(Y_1(t),\omega,\gamma') - R(Y_1(\tau),\omega,\gamma')\right).$$

This proves (4.57). □

Proof of Proposition 4.1: To see that the random variable $\xi(\omega)$ is almost surely finite, observe that the terms in the exponent defining $E(t,0,\gamma,\gamma',\omega)$ grow at most sublinearly in t. Specifically, we have already established that

$$\lim_{t\to\infty}\frac{h_t(\omega)}{t} = 0 \tag{4.64}$$

holds almost surely. Also, $R(y,\omega,\gamma)/y \to 0$ as $y \to \infty$ and $Y_1(t,\omega,\gamma)/t \to c(\gamma)$ as $t \to \infty$, so that

$$\lim_{t\to\infty} \frac{R(Y_1(t),\omega,\gamma')}{t} = \lim_{t\to\infty} \frac{R(Y_1(t),\omega,\gamma')}{Y_1(t)} \frac{Y_1(t)}{t} = c \lim_{t\to\infty} \frac{R(Y_1(t),\omega,\gamma')}{Y_1(t)} = 0 \tag{4.65}$$

holds almost surely. Thus $\xi(\omega) < \infty$ holds almost surely.

Now we prove (4.58). Let $N > 0$ and let $\tau_k = kt/N$ for $k = 0, 1, 2, \ldots, N$. The integral we wish to bound is:

$$\begin{aligned}\int_0^t e^{-\delta(t-\tau)} E(t,\tau,\gamma,\gamma',\omega)\,d\tau &= \sum_{k=1}^N \int_{\tau_{k-1}}^{\tau_k} e^{-\delta(t-\tau)} E(t,\tau,\gamma,\gamma',\omega)\,d\tau \\ &\le \sum_{k=1}^N e^{-\mu'(h_t - h_{\tau_k}) + R(Y(t)) - R(Y(\tau_k))} e^{M(k,t,\omega)} \int_{\tau_{k-1}}^{\tau_k} e^{-\delta(t-\tau)}\,d\tau\end{aligned}$$

where

$$M(k,t,\omega) = \max_{\tau\in[\tau_{k-1},\tau_k]} \mu'(h_\tau - h_{\tau_k}) + \max_{\tau\in[\tau_{k-1},\tau_k]} R(Y_1(\tau_k)) - R(Y_1(\tau)).$$

(Recall that μ' denotes the constant $\mu(\gamma')$.) We claim that for any $\epsilon_1 > 0$ and $\epsilon_2 > 0$, we may take N sufficiently large, so that

$$\mathbb{P}\left(M(k,t,\omega) \le \epsilon_1 \sqrt{t}, \ \ \forall\ \ k = 1,\ldots,N \right) \ge 1 - \epsilon_2 \tag{4.66}$$

holds if t is sufficiently large. Therefore, with probability at least $1-\epsilon_2$, we have

$$\begin{aligned}\int_0^t e^{-\delta(t-\tau)} E(t,\tau,\gamma,\gamma',\omega)\,d\tau &\le \delta^{-1} e^{\epsilon_1\sqrt{t}} \sum_{k=1}^N e^{-\mu'(h_t - h_{\tau_k}) + R(Y(t)) - R(Y(\tau_k))} e^{-\delta(t-\tau_k)} \\ &= \delta^{-1} e^{\epsilon_1\sqrt{t}} + \delta^{-1} e^{\epsilon_1\sqrt{t}} \sum_{k=1}^{N-1} e^{-\mu'(h_t - h_{\tau_k}) + R(Y(t)) - R(Y(\tau_k))} e^{-\delta(t-\tau_k)}\end{aligned}$$

By taking $\alpha > 0$ sufficiently large and then t sufficiently large, we also have

$$\mathbb{P}\left(\frac{\mu'|h_t - h_{\tau_k}|}{\sqrt{t}} \ge \alpha, \quad \frac{|R(Y_1(t)) - R(Y_1(\tau_k))|}{\sqrt{t}} \ge \alpha \ \ \forall\ k = 1,\ldots,N-1 \right) \le \epsilon_2.$$

Therefore, with probability at least $1 - 2\epsilon_2$, we have

$$\begin{aligned}\int_0^t e^{-\delta(t-\tau)} E(t,\tau,\gamma,\gamma',\omega)\,d\tau &\le \delta^{-1} e^{\epsilon_1\sqrt{t}} + \delta^{-1} e^{\epsilon_1\sqrt{t}} \sum_{k=1}^{N-1} e^{2\alpha\sqrt{t} - \delta(t-\tau_k)} \\ &\le \delta^{-1} e^{\epsilon_1\sqrt{t}} + (N-1)\delta^{-1} e^{\epsilon_1\sqrt{t}} e^{2\alpha\sqrt{t} - \delta t/N}\end{aligned}$$

if t is sufficiently large, depending on ϵ_1 and ϵ_2. Hence,

$$\mathbb{P}\left(\int_0^t e^{-\delta(t-\tau)} E(t,\tau,\gamma,\gamma',\omega)\,d\tau \ge 2\delta^{-1} e^{\epsilon_1\sqrt{t}} \right) \le 2\epsilon_2 \tag{4.67}$$

if t is sufficiently large. Since ϵ_1 and ϵ_2 were chosen arbitrarily, this implies (4.58).

Let us verify the claim (4.66). Observe that

$$\frac{R(Y_1(\tau_k)) - R(Y_1(\tau))}{\sqrt{t}} = \frac{R(c\tau_k) - R(c\tau)}{\sqrt{t}} + \frac{R(Y_1(\tau_k)) - R(c\tau_k)}{\sqrt{t}} + \frac{R(c\tau) - R(Y_1(\tau))}{\sqrt{t}} \tag{4.68}$$

Obviously $(c\tau_k - c\tau)/t \leq c/N$ for $\tau \in [\tau_{k-1}, \tau_k]$. Also, Theorem 1.3 for $Y_1(t)$ and (3.54) imply that

$$\max_{\tau \in [\tau_{k-1}, \tau_k]} |(Y_1(\tau))/t - (c\tau)/t| \leq \max_{\tau \in [0,t]} |(h_\tau)/t| \leq 2c/N$$

with probability at least $1 - \epsilon_3$ if t is sufficiently large. Therefore, since the family of processes $x \mapsto R(tx)/\sqrt{t}$ is tight in $C([0,T])$ for any T, we see that

$$\mathbb{P}\left(\max_{\tau \in [\tau_{k-1}, \tau_k]} R(Y_1(\tau_k)) - R(Y_1(\tau)) \geq \epsilon_1 \sqrt{t}\right) \leq 1 - \epsilon_2 \tag{4.69}$$

holds if N is large, and then t is sufficiently large. Similar estimates hold for $h_\tau - h_{\tau_k}$, establishing the claim. $\square$

5 CLT for solutions of the nonlinear equation

Finally, we shift attention to solutions of the nonlinear problem (1.1), and we prove Theorem 1.4. We suppose that the initial condition $v(0,x,\omega) = v_0(x,\omega)$ satisfies (1.9) and (1.13) for some $\gamma \in (\bar{\gamma}, \gamma^*)$. For $r \in (0,1)$, let $X_r(t,\omega)$ be the interface position associated with $v(t,x,\omega)$:

$$X_r(t,\omega) = \sup\left\{x \in \mathbb{R} \mid \ v(t,x,\omega) = r\right\}.$$

We have already proved Theorem 1.3 for solutions of the linearized equation. So, if for almost every ω we could show that $X_r(t,\omega)$ stays sufficiently close to $Y_r(t,\omega)$ as $t \to \infty$, the CLT would follow for the solutions of the nonlinear equation (recall Y_r defined at (3.45)).

For simplicity, we now suppose that $v_0(x,\omega) = \min(1, \varphi(0,x,\omega;\gamma))$ for $x > 0$ and that $v_0(x,\omega) = 1$ for $x < 0$. The more general case (1.13) – where v_0 is trapped between two fronts – follows readily from this. The maximum principle implies that $v(t,x,\omega) \leq \varphi(t,x,\omega;\gamma)$ for all $t \geq 0$ and $x \geq 0$. So, for any $r \in (0,1)$ we have

$$X_r(t,\omega) \leq Y_r(t,\omega) \tag{5.70}$$

for all $t \geq 0$. However, since $v(t,x,\omega)$ is merely a subsolution of the linearized equation, $X_r(t,\omega)$ might lag behind $Y_r(t,\omega)$. So, in order to obtain a CLT for $X_r(t)$ by comparing with $Y_r(t)$, we need to show that $X_r(t)$ stays sufficiently close to $Y_r(t)$ with high probability. Actually, we'll need to introduce a time delay: $X_r(t+h)$ stays sufficiently close to $Y_r(t)$ with high probability, if h is sufficiently large.

Lemma 5.1 *Let $\gamma \in (\bar{\gamma}, \gamma^*)$. Suppose that $v_0(x,\omega) = \min(1, \varphi(0,x,\omega;\gamma))$ for $x > 0$ and $v_0(x) = 1$ for $x < 0$. For any $\epsilon > 0$,*

$$\lim_{t \to \infty} \mathbb{P}\left(X_{1/2}(t + \epsilon\sqrt{t}, \omega) - Y_{1/2}(t)(\omega) \geq 0\right) = 1. \tag{5.71}$$

Before proving this lemma, let us show how (5.71) and (5.70) imply Theorem 1.4. For $h > 0$, let $G(t,h) \subset \Omega$ be the set $G(t,h) = \{\omega \in \Omega \mid X_{1/2}(t+h,\omega) \geq Y_{1/2}(t)(\omega)\}$. Then for any $\alpha \in \mathbb{R}$

$$\begin{aligned}
\mathbb{P}\left(\frac{X_{1/2}(t+h,\omega) - (t+h)c}{\sqrt{t+h}} > \alpha\right) &\geq \mathbb{P}\left(\frac{Y_{1/2}(t)(\omega) - (t+h)c}{\sqrt{t+h}} > \alpha\right) - \mathbb{P}(G(t,h)^C) \\
&= \mathbb{P}\left(\frac{Y_{1/2}(t)(\omega) - tc}{\sqrt{t}} > \alpha\frac{\sqrt{t+h}}{\sqrt{t}} + \frac{hc}{\sqrt{t}}\right) - \mathbb{P}(G(t,h)^C).
\end{aligned}$$

Therefore, if we set $h = \epsilon\sqrt{t}$ for some $\epsilon > 0$, Theorem 1.3 implies

$$\liminf_{s\to\infty} \mathbb{P}\left(\frac{X_{1/2}(s,\omega) - sc}{\sqrt{s}} > \alpha\right) \geq 1 - \Phi(\alpha/\kappa + \epsilon c/\kappa) - \limsup_{t\to\infty} \mathbb{P}(G(t,\epsilon\sqrt{t})^C)$$

with $\kappa = \sigma\mu^{-1}c$. However, by (5.71), $\mathbb{P}(G(t,\epsilon\sqrt{t})^C) \to 0$ as $t \to \infty$, so we must have

$$\liminf_{s\to\infty} \mathbb{P}\left(\frac{X_{1/2}(s,\omega) - sc}{\sqrt{s}} > \alpha\right) \geq 1 - \Phi(\alpha/\kappa)$$

since ϵ is arbitrary.

An upper bound on this probability follows easily from (5.70):

$$\mathbb{P}\left(\frac{X_{1/2}(s,\omega) - sc}{\sqrt{s}} > \alpha\right) < \mathbb{P}\left(\frac{Y_{1/2}(s,\omega) - sc}{\sqrt{s}} > \alpha\right)$$

and the latter converges to $1 - \Phi(\alpha/\kappa)$ as $s \to \infty$. This proves Theorem 1.4.

In the proof of Lemma 5.1, the following estimate plays a key role. It gives a lower bound on the leading edge of the nonlinear wave in terms of the leading edge of the linear wave. The proof relies on Lemma 4.1 and works only for the supercritical regime.

Lemma 5.2 *Let $\gamma \in (\bar{\gamma}, \gamma^*)$. Suppose that $v_0(x,\omega) = \min(1, \varphi(0,x,\omega;\gamma))$ for $x > 0$ and $v_0(x) = 1$ for $x < 0$. Then there is a constant $\delta > 0$ and a random variable $\theta_t(\omega) > 0$ such that*

$$v(t, x + Y_1(t), \omega) \geq \varphi(t, x + Y_1(t), \omega; \gamma)(1 - e^{-\delta x}\theta_t(\omega)) \tag{5.72}$$

holds for all $x \geq 0$ and $t \geq 0$, and for any $\epsilon > 0$,

$$\lim_{t\to\infty} \mathbb{P}(\theta_t(\omega) \leq e^{\epsilon\sqrt{t}}) = 1. \tag{5.73}$$

Proof of Lemma 5.2: The strategy here is inspired by [1]. The idea is to think of the nonlinear equation as an inhomogeneous linear equation. The nonlinear term is then controlled through the Duhamel expansion and the estimates of Section 4. Let us define the difference $\psi(t,x,\omega) := \varphi - v$, which satisfies the equation

$$\partial_t\psi - \psi_{xx} - g(x,\omega)\psi = g(x,\omega)v^2 \leq g_{max}v^2, \quad x > 0, \ \ t > 0.$$

By the maximum principle, $v(t,x) \leq \min(1, \varphi(t,x,\omega;\gamma))$ for all $t \geq 0$, $x \geq 0$. Therefore, $g_{max}v^2 \leq g_{max}\min(1,\varphi^2) = K(t,x,\omega)$. So, for $x > 0$, we have $\psi \leq \psi_1 + \psi_2$ where $\psi_1(t,x)$ solves

$$\begin{array}{c} (\psi_1)_t = (\psi_1)_{xx} + g(x,\omega)\psi_1, \quad x > 0, \ \ t > 0 \\ \psi_1(t,0) = e^{\gamma t}, \ \ t > 0; \qquad \psi_1(0,x) = \max(0, \varphi(0,x,\omega;\gamma) - 1), \ \ x > 0 \end{array} \tag{5.74}$$

and $\psi_2(t,x)$ solves

$$\begin{array}{c} (\psi_2)_t = (\psi_2)_{xx} + g(x,\omega)\psi_2 + K(t,x), \quad x > 0, \ \ t > 0 \\ \psi_2(t,0) = 0, \ \ t > 0; \qquad \psi_2(0,x) = 0, \ \ x > 0. \end{array} \tag{5.75}$$

We will apply Lemma 4.1 to both ψ_1 and ψ_2. For ψ_1 we apply the lemma with $\tau = 0$, $C_1 = 1$, and $C_2 = 1$ to obtain the bound

$$\psi_1(t, x + Y_1(t)) \ \ \leq \ \ C_3 u(x, \pi_{Y_1(t)}\omega, \gamma')e^{-\beta t}E(t, 0, \omega, \gamma, \gamma').$$

By Proposition 4.1, the quantity

$$\xi(\omega) = \sup_{t\geq 0} e^{-\beta t/2} E(t, 0, \omega, \gamma, \gamma') \tag{5.76}$$

is finite with probability one. Therefore, for all $t > 0$ and $x \geq 0$,

$$\psi_1(t, x + Y_1(t), \omega) \leq C_3 u(x, \pi_{Y_1(t)}\omega, \gamma') e^{-\beta t/2}\xi(\omega) \leq C_3 e^{-\delta x} u(x, \pi_{Y_1(t)}\omega, \gamma) e^{-\beta t/2}\xi(\omega).$$

The last bound follows from Lemma 2.7 with $\delta = \sqrt{\gamma' - g_{min}} - \sqrt{\gamma - g_{min}}$.

Now we bound ψ_2. For each $\tau \in [0, t)$, let $\rho(s, x, \omega; \tau) : [\tau, \infty) \times [0, \infty) \times \Omega \to \mathbb{R}$ solve

$$\begin{aligned} \rho_s &= \rho_{xx} + g(x, \omega)\rho, \quad x > 0, s \geq \tau \\ \rho(s, 0) &= 0, \quad s \geq \tau \\ \rho(\tau, x) &= K(\tau, x, \omega), \quad x > 0. \end{aligned} \tag{5.77}$$

Then ψ_2 is given by the integral

$$\psi_2(t, x, \omega) = \int_0^t \rho(t, x, \omega; \tau)\, d\tau. \tag{5.78}$$

Since $K(\tau, x, \omega) \leq g_{max} \min(1, \varphi^2(\tau, x, \omega; \gamma))$ and $K(t, x, \omega) \leq g_{max}\varphi(t, x, \omega; \gamma)$, we apply Lemma 4.1 to ρ (with $C_1 = g_{max}$ and $C_2 = g_{max}$) and obtain:

$$\rho(t, x + Y_1(t); \tau) \leq C_3 g_{max} u(x, \pi_{Y_1(t)}, \omega; \gamma') e^{-\beta(t-\tau)} E(t, \tau, \omega, \gamma, \gamma'). \tag{5.79}$$

Consequently, using Lemma 2.7 we see that

$$\begin{aligned} \psi_2(t, x + Y_1(t), \omega) &\leq C_3 g_{max} u(x, \pi_{Y_1(t)}, \omega; \gamma') \int_0^t e^{-\beta(t-\tau)} E(t, \tau, \omega, \gamma, \gamma')\, d\tau \\ &\leq C_3 g_{max} e^{-\delta x} u(x, \pi_{Y_1(t)}, \omega; \gamma) \int_0^t e^{-\beta(t-\tau)} E(t, \tau, \omega, \gamma, \gamma')\, d\tau \end{aligned} \tag{5.80}$$

holds for all $x \geq 0$ and $t \geq 0$, where

$$E(t, \tau, \omega, \gamma, \gamma') = \exp\left(-\mu'(h_t(\omega; \gamma) - h_\tau(\omega; \gamma)) + R(Y_1(t), \omega, \gamma') - R(Y_1(\tau), \omega, \gamma')\right).$$

Recall that μ' refers to the constant $\mu(\gamma')$.

Combining the estimates for ψ_1 and ψ_2 we conclude that for $x > 0$,

$$\begin{aligned} v(t, x + Y_1(t), \omega) &= \varphi(t, x + Y_1(t), \omega; \gamma) - \psi(t, x + Y_1(t), \omega) \\ &\geq \varphi(t, x + Y_1(t), \omega; \gamma) - C_3 e^{-\delta x} u(x, \pi_{Y_1(t)}\omega, \gamma) e^{-\beta t/2}\xi(\omega) \\ &\quad -C_3 g_{max} e^{-\delta x} u(x, \pi_{Y_1(t)}, \omega; \gamma) \int_0^t e^{-\beta(t-\tau)} E(t, \tau, \omega, \gamma, \gamma')\, d\tau \\ &= u(x, \pi_{Y_1(t)}\omega; \gamma) \left(1 - e^{-\delta x}\theta_t(\omega)\right) \\ &= \varphi(t, x + Y_1(t), \omega; \gamma) \left(1 - e^{-\delta x}\theta_t(\omega)\right) \end{aligned} \tag{5.81}$$

where

$$\theta_t(\omega) = C_3 e^{-\beta t/2}\xi(\omega) + C_3 g_{max} \int_0^t e^{-\beta(t-\tau)} E(t, \tau, \omega, \gamma, \gamma')\, d\tau.$$

Finally, the fact that $\lim_{t\to\infty} \mathbb{P}(\theta_t(\omega) \le e^{\epsilon\sqrt{t}}) = 1$ holds for any $\epsilon > 0$ follows immediately from Proposition 4.1 and the definition of $\theta_t(\omega)$. □

Proof of Lemma 5.1:. Here is the strategy for proving Lemma 5.1. Lemma 5.2 implies that if $\bar{x}$ is sufficiently large, $v(t, \bar{x} + Y_1(t), \omega) \ge \frac{1}{2}\varphi(t, \bar{x} + Y_1(t), \omega; \gamma)$ holds with high probability. So for $\ell = \frac{1}{2}\varphi(t, \bar{x} + Y_1(t), \omega; \gamma)$, we have a lower bound $X_\ell(t) \ge \bar{x} + Y_1(t)$. If this level ℓ is not too small, then v will be larger than $1/2$ at this point $\bar{x} + Y_1(t)$ if we wait only a little longer. This would give us a bound of the form $X_{1/2}(t+h) \ge \bar{x} + Y_1(t)$. Choosing $\bar{x}$ larger, if necessary, the latter is bounded below by $Y_{1/2}(t)$. However, the necessary lag time h is random since it depends on ℓ, which depends on the behavior of φ ahead of the point $Y_1(t)$.

With $\theta_t(\omega)$ defined by Lemma 5.2, define

$$\bar{x} = \bar{x}_t(\omega) = \max\left(\delta^{-1}\log(2\theta_t), Y_{1/2}(t) - Y_1(t)\right)$$

and

$$\ell_t(\omega) = \varphi(t, \bar{x} + Y_1(t), \omega; \gamma)(1 - e^{-\delta\bar{x}}\theta_t(\omega)).$$

Observe that if $\delta^{-1}\log(2\theta_t) \le Y_{1/2}(t) - Y_1(t)$, then $\bar{x} = Y_{1/2}(t) - Y_1(t)$ so that $\ell_t(\omega) \ge 1/4$. Otherwise ℓ_t may be small. By Lemma 5.2 and the definition of $\bar{x}$,

$$X_{\ell_t}(t) \ge \bar{x} + Y_1(t) \ge Y_{1/2}(t). \tag{5.82}$$

So, we have a bound on the position of the ℓ_t-level set of v in terms of $Y_{1/2}(t)$. Since $v(t, \bar{x} + Y_1(t), \omega) \ge \ell_t$, the Harnack inequality implies that there is a constant $\kappa > 0$ such that

$$v(t+1, x, \omega) \ge \kappa\ell_t(\omega), \quad \forall x \in [\bar{x} + Y_1(t) - 1, \bar{x} + Y_1(t) + 1].$$

This constant may be chosen independently of ω and t. Now we wish to bound the first time $s \ge t$ at which $v(s, \bar{x} + Y_1(t), \omega) \ge 1/2$. To this end, define $\eta(s, x)$ which satisfies

$$\eta_s = \eta_{xx} + f_{1/2}\eta \quad x \in \mathbb{R}, \quad s > 0$$

with $\eta(0, x) = 1$ for $|x| \le 1$ and $\eta(0, x) = 0$ for $|x| > 1$. Here we choose $f_{1/2} = \frac{1}{2}f(1/2)$ so that $f(v) = g_{min}v(1-v) \ge f_{1/2}v$ for $v \in [0, 1/2]$. The maximum principle implies that

$$v(t+1+s, x, \omega) \ge \kappa\ell_t(\omega)\eta\left(s, x - (\bar{x} + Y_1(t))\right)$$

holds for $x \in \mathbb{R}$ and $s \ge 0$ as long as $\kappa\ell_t(\omega)\eta\left(s, x - (\bar{x} + Y_1(t))\right) \le 1/2$ (before this time occurs, $(s, x) \mapsto \kappa\ell_t\eta(s, x)$ is a subsolution of the nonlinear equation). By symmetry, η has a global maximum at $x = 0$. Therefore, if we define the function $T : (0, 1) \to \mathbb{R}$ by

$$T(\ell) = \inf\{s \ge 0 \mid \eta(s, 0) \ge 1/(2\kappa\ell)\},$$

we have $v(t+1+s, \bar{x} + Y_1(t), \omega) \ge 1/2$ for all $s \ge T(\ell_t(\omega))$. So, for all $h \ge 1 + T(\ell_t(\omega))$, we have

$$X_{1/2}(t+h, \omega) \ge \bar{x} + Y_1(t) \ge Y_{1/2}(t).$$

We now have shown that

$$\mathbb{P}\left(X_{1/2}(t + \epsilon\sqrt{t}, \omega) \ge Y_{1/2}(t)(\omega)\right) \ge \mathbb{P}\left(T(\ell_t(\omega)) \le \epsilon\sqrt{t} - 1\right).$$

Therefore, to finish the proof we must bound the distribution of $T(\ell_t(\omega))$. It is not difficult to show that η grows exponentially so that for any $\ell > 0$, $T(\ell) \le k_1 + k_2|\log(\ell)|$ for some constants k_1, k_2 depending only on κ and $f_{1/2}$. Therefore,

$$\mathbb{P}\left(X_{1/2}(t+\epsilon\sqrt{t},\omega) \le Y_{1/2}(t)(\omega)\right) \le \mathbb{P}\left(\ell_t(\omega) \le \exp(-\epsilon\sqrt{t}/k_2 - (1+k_1)/k_2)\right).$$

Lemma 5.1 now follows immediately from Lemma 5.3 below, which shows that the level ℓ_t cannot vanish too quickly as $t \to \infty$. □

Lemma 5.3 *With $\ell_t(\omega)$ defined as above,*

$$\lim_{t\to\infty} \mathbb{P}\left(\ell_t(\omega) \le \exp(-\epsilon\sqrt{t})\right) = 0$$

holds for any $\epsilon > 0$.

Proof: By definition,

$$\begin{aligned}\ell_t(\omega) = \varphi(t,\bar{x}+Y_1(t),\omega;\gamma)(1-e^{-\delta\bar{x}}\theta_t(\omega)) &= u(\bar{x},\pi_{Y_1(t)}\omega;\gamma)(1-e^{-\delta\bar{x}}\theta_t(\omega)) \\ &\ge \frac{1}{2}u(\bar{x},\pi_{Y_1(t)}\omega;\gamma) \end{aligned} \tag{5.83}$$

with $\bar{x} = \bar{x}_t(\omega) = \max\left(\delta^{-1}\log(2\theta_t), Y_{1/2}(t) - Y_1(t)\right)$. If $Y_{1/2}(t) - Y_1(t) \ge \delta^{-1}\log(2\theta_t)$, then $\bar{x} = Y_{1/2}(t) - Y_1(t)$ so that $\ell_t(\omega) \ge 1/2\varphi(t, Y_{1/2}(t),\omega;\gamma) = 1/4$. So, it suffices to show that

$$\lim_{t\to\infty} \mathbb{P}\left(\{\omega \mid\ u(\delta^{-1}\log(2\theta_t),\pi_{Y_1(t)}\omega;\gamma) < \exp(-\epsilon\sqrt{t}),\quad \theta_t(\omega) \ge 1/2\}\right) = 0 \tag{5.84}$$

holds. For $\epsilon \in (0,1)$, $u(\delta^{-1}\log(2\theta_t),\pi_{Y_1(t)}\omega;\gamma) < \exp(-\epsilon\sqrt{t})$ holds if and only if

$$\mu\delta^{-1}\log(2\theta_t) - R\left(\delta^{-1}\log(2\theta_t),\pi_{Y_1(t)}\omega;\gamma\right) > \epsilon\sqrt{t}.$$

On the other hand, from the bounds (2.22) and (2.23) on u, we know that there is a constant M such that $|R(x,\omega;\gamma)| \le M(x+1)$ holds with probability one. Therefore, if $\theta_t \ge 1/2$,

$$\mu\delta^{-1}\log(2\theta_t) - R\left(\delta^{-1}\log(2\theta_t),\pi_{Y_1(t)}\omega;\gamma\right) \le (\mu+M)\delta^{-1}\log(2\theta_t) + M < \epsilon\sqrt{t}$$

holds for t sufficiently large, if $\theta \le \frac{1}{2}\exp(\epsilon\sqrt{t}/p)$ with $p = 2(\mu+M)\delta^{-1}$. Therefore, by (5.73),

$$\lim_{t\to\infty} \mathbb{P}\left(\{\omega \mid\ u(\delta^{-1}\log(2\theta_t),\pi_{Y_1(t)}\omega;\gamma) < \exp(-\epsilon\sqrt{t}),\quad \theta_t(\omega) \ge 1/2\}\right) \le \lim_{t\to\infty} \mathbb{P}\left(\theta_t(\omega) \ge \frac{1}{2}\exp(\epsilon\sqrt{t}/p)\right) = 0.$$

□

Remark 5.1 *Let us point out that if the statement*

$$\mathbb{P}\left(\liminf_{t\to\infty} X_{1/2}(t+\epsilon\sqrt{t},\omega) - Y_{1/2}(t)(\omega) \ge 0\right) = 1, \tag{5.85}$$

holds (which is stronger than Lemma 5.1), then tightness of the renormalized process $(X(nt) - cnt)/\sqrt{n}$ would follow from that of $Z_n(t,\omega;\gamma)$. Thus, we would have weak convergence of the process $(X(nt)-cnt)/(\mu^{-1}c\sqrt{n})$ to Brownian motion as $n \to \infty$. The main issue is whether the estimate (4.58) can be improved to the statement that, almost surely,

$$\int_0^t e^{-\delta(t-\tau)}E(t,\tau,\gamma,\gamma',\omega)\,d\tau = o(e^{\epsilon\sqrt{t}}), \quad \textit{as } t\to\infty.$$

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