Min-Max Variational Principle and Front Speeds in Random Shear Flows

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Abstract

Speed ensemble of bistable (combustion) fronts in mean zero stationary Gaussian shear flows inside two and three dimensional channels is studied with a min-max variational principle. In the small root mean square regime of shear flows, a new class of multi-scale test functions are found to yield speed asymptotics. The quadratic speed enhancement law holds with probability arbitrarily close to one under the almost sure continuity (dimension two) and mean square Hölder regularity (dimension three) of the shear flows. Remarks are made on the conditions for the linear growth of front speed expectation in the large root mean square regime.

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1 Introduction

We consider propagation speeds of reaction-diffusion fronts in random shear flows. The model equation is:

$$u_t = \Delta_x u + \delta B \cdot \nabla_x u + f(u), \tag{1.1}$$

where $x = (x_1, \tilde{x}) \in D = R^1 \times \Omega$, $\Omega = [0, L]^{n-1}$, $n = 2, 3$; $\delta > 0$ a scaling parameter; the vector field $B = (b(\tilde{x}, \omega), 0)$, $b(\tilde{x}, \omega)$ is a scalar stationary Gaussian field with zero ensemble mean and almost surely continuous sample paths; f is either one of the following:

(1) the combustion nonlinearity with ignition temperature cutoff: $f(u) = 0$, $u \in [0, \mu]$, $\mu \in (0, 1)$; $f(u) > 0$, $u \in (\mu, 1)$, $f(1) = 0$, $f \in C^2([0, 1]$;

(2) the bistable nonlinearity: $f(u) = u(1 - u)(u - \mu)$, $\mu \in (0, 1/2)$.

The boundary condition is zero Neumann at $R^1 \times \partial\Omega$. The initial data u_0 belongs to the set I_s. The set I_s for bistable f consists of bounded continuous functions with limits one and zero at $x_1 \sim \pm\infty$ respectively. For combustion f, one requires also that u_0 decay to zero exponentially at $-\infty$. For each realization of B, there exists a traveling front solution of the form $u = \Phi(x_1 + ct, \tilde{x})$, with unique speed c and profile Φ [4]. For $u_0 \in I_s$, $u(x, t)$ converges to a traveling front at large times [14].

We are interested in the asymptotic behavior of $c = c(\delta)$ when δ is small or large. Such speed asymptotics have been recently studied for both deterministic fronts [3, 5, 8, 10, 9, 11, 13] and KPP fronts in random shears [12, 17]. See also [16] for related applied science literature. The bistable (combustion) front speed c has a min-max variational characterization [7, 8]. Following the notation of [8], define the functional:

$$\psi(v) = \psi(v(x)) \equiv \frac{L\,v + f(v)}{\partial_{x_1} v} \equiv \frac{\Delta v + \delta b(\tilde{x})\partial_{x_1} v + f(v)}{\partial_{x_1} v}. \tag{1.2}$$

The strong form of the min-max variational formula of [8] for $c(\delta)$ is:

$$\sup_{v \in K} \inf_{x \in D} \psi(v(x)) = c(\delta) = \inf_{v \in K} \sup_{x \in D} \psi(v(x)) \tag{1.3}$$

where K is the set of admissible functions:

$$K = \{v \in C^2(D) |\; \partial_{x_1} v > 0,\; 0 < v(x) < 1, v \in I_s\}.$$

The variational principle (1.3) is a powerful tool for getting tight bounds on c, if one can construct an admissible test function v that approximates well the exact traveling front solution Φ. This is the case when δ is small and the test function is a perturbation of the known traveling front solution when $\delta = 0$. In [8], a form of such test function is proposed for deterministic shear flow, and the quadratic speed enhancement law ([13] and references therein) is recovered if δ is small enough. To handle an ensemble of shear flows in the random case, one must know which norm of b controls the smallness of δ by analyzing an admissible test function v and the functional $\psi(v)$. A new form of multi-scale test function is found for this purpose, and it cures a delicate divergence problem when perturbing the traveling front at $\delta = 0$. It turns out that the smallness of δ for the quadratic speed enhancement depends on the maximum (Hölder) norm of b if $n = 2\,(3)$.

Our main results are that the quadratic speed enhancement law holds with probability arbitrarily close to one, under the almost sure continuity ($n = 2$) or mean square Hölder regularity ($n = 3$) conditions of the shear. As b can be arbitrarily large (though with small probability), our admissible test functions are not valid for the entire ensemble. We are unable to discuss the expectation of front speed. This is in contrast to the Kolmogorov-Petrovsky-Piskunov (KPP) case (e.g. when $f(u) = u(1-u)$), where there is less restriction on test functions and the minimal speed variational principle is minimization over functions on the cross section Ω [12].

The paper is organized as follows. In section 2, the new multi-scale test function is presented and used to estimate the front speed and identify the smallness of δ in terms of proper norms of b. In section 3, probabilistic estimates based on Borell's inequality and Karhunen-Loève expansion of mean zero Gaussian fields help to control the maximum (Hölder) norms of b, and lead to the quadratic speed enhancement laws. Remarks are made on the linear front speed growth law when δ is large.

2 Deterministic Front Speed Asymptotics

Let us consider b to be a given Hölder continuous deterministic function, and assume that $< b > \equiv \frac{1}{|\Omega|}\int_\Omega b(y)\,dy = 0$. Otherwise, the integral mean contributes a linear term $\delta < b >$ to the front speed. Let $U = U(x_1 + c_0 t)$ be the traveling fronts when $\delta = 0$, satisfying the equation:

$$U'' - c_0 U' + f(U) = 0,$$

$U(-\infty) = 0$, $U(+\infty) = 1$, $U' > 0$, U approaches zero and one at exponential rates.

In the small δ regime, define the test function

$$v(x) = U(\xi) + \delta^2 \tilde{u}(\xi, \tilde{x}), \tag{2.1}$$

where the variable:

$$\xi = (1 + \alpha\delta^2) x_1 + \delta\chi, \tag{2.2}$$

with α a constant to be determined and $\chi = \chi(\tilde{x})$ solution of:

$$-\Delta_{\tilde{x}} \chi = b, \quad \tilde{x} \in \Omega,$$

subject to zero Neumann boundary condition at $\partial\Omega$. We normalize χ by setting $\int_\Omega \chi \, dy = 0$. The test function (2.1) reduces to the one in [8] if $\alpha = 0$. However, we shall see that $\alpha \neq 0$ is essential in suppressing certain divergence arising in the evaluation of $\psi(v)$, similar to curing secular growth in multiple-scale perturbation expansions.

For Hölder continuous $b(\tilde{x})$, χ is C^{2+p}, $p \in (0,1)$. Assume that $\tilde{u}$ is C^2, and decays sufficiently fast at ξ infinities. A straightforward computation shows that:

$$\begin{aligned}
Lv + f(v) \;=\; & U'' + f(U) \\
& + \delta^2 \left(2\,\alpha\, U'' + \Delta_{\xi,\tilde{x}} \tilde{u} + U'' |\nabla\chi|^2 + f'(U)\tilde{u} \right) \\
& + \delta^3 (b\,\alpha\, U' + 2 \nabla_{\tilde{x}} \tilde{u}_\xi \cdot \nabla\chi) \\
& + \delta^4 (\alpha^2\, U'' + 2\,\alpha\, \tilde{u}_{\xi\xi} + \Delta_\xi\, \tilde{u}\, |\nabla\chi|^2) \\
& + f(v) - f(U) - f'(U)\, \delta^2\, \tilde{u} \\
& + \delta^5\, \alpha\, b\, \tilde{u}_\xi + \delta^6\, \alpha^2\, \tilde{u}_{\xi\xi},
\end{aligned} \tag{2.3}$$

where $f(v) - f(U) - f'(U)\delta^2 \tilde{u} = O(\delta^4 \tilde{u}^2)$ and the $O(\delta)$ terms cancel by choice of $-\Delta\chi = b$. On the other hand, up to an undetermined constant γ, we have:

$$\begin{aligned}
(c_0 + \delta^2 \gamma)\, \partial_{x_1} v \;=\; & c_0 U' + \delta^2 (c_0 \alpha U' + c_0 \tilde{u}_\xi + \gamma U') \\
& + \delta^4 (\gamma \alpha U' + c_0 \alpha \tilde{u}_\xi + \gamma \tilde{u}_\xi) \\
& + \delta^6 (\alpha \gamma \tilde{u}_\xi).
\end{aligned} \tag{2.4}$$

All terms in (2.3) and (2.4) are evaluated at $(\xi, \tilde{x})$. Let us choose $\tilde{u}$ to solve the equation:

$$\bar{L}\, \tilde{u} \;=\; \Delta_{\xi,\tilde{x}}\, \tilde{u} - c_0\, \tilde{u}_\xi + f'(U)\, \tilde{u} = -|\nabla\chi|^2\, U'' - 2\,\alpha\, U'' + \gamma\, U' + c_0\, \alpha\, U', \tag{2.5}$$

subject to zero Neumann boundary condition at $\partial\Omega$, and exponential decay at ξ infinities. The solvability of (2.5) will be discussed later. Then we would have

$$Lv + f(v) = (\partial_{x_1} v)(c_0 + \delta^2\gamma) + R_1. \tag{2.6}$$

The remainder R_1 is:

$$R_1 = \delta^3 A + \delta^4 B + \delta^5 D + \delta^6 E + O(\delta^4 \tilde{u}^2), \tag{2.7}$$

where

$$\begin{aligned} A &= b\,\alpha U' + 2\nabla_{\tilde{x}}\, \tilde{u}_\xi \cdot \nabla\chi \\ B &= \alpha^2\, U'' + 2\alpha\, \tilde{u}_{\xi\xi} + \Delta_\xi\, \tilde{u}\, |\nabla\chi|^2 - \gamma\, \alpha\, U' - c_0\, \alpha\, \tilde{u}_\xi - \gamma\, \tilde{u}_\xi \\ D &= \alpha b\, \tilde{u}_\xi \\ E &= \alpha^2\, \tilde{u}_{\xi\xi} - \alpha\, \gamma\, \tilde{u}_\xi. \end{aligned} \tag{2.8}$$

The constant γ is determined by a solvability condition for (2.5). The right hand side must be orthogonal to the function $U'(\xi)\, e^{-c_0\xi}$ which spans the kernel of the adjoint operator $\bar{L}^* v = \Delta_{\xi,\tilde{x}} v + c_0 v_\xi + f'(U) v$. Thus,

$$\begin{aligned} \gamma \int_R (U')^2 e^{-c_0\xi}\, d\xi &= \frac{1}{|\Omega|} \int_D \left((|\nabla\chi|^2 + 2\alpha) U'' U' - c_0 \alpha (U')^2 \right) e^{-c_0\xi} d\xi\, d\tilde{x} \\ &= \frac{c_0}{2} \langle |\nabla\chi|^2 + 2\alpha \rangle \int_R (U')^2 e^{-c_0\xi}\, d\xi - c_0\alpha \int_R (U')^2 e^{-c_0\xi}\, d\xi \end{aligned}$$

where the braket denotes the integral average over Ω. We have performed integration by parts once, the boundary terms decay to zero and the integral $\int_R (U')^2 e^{-c_0\xi}\, d\xi$ converges for both bistable and combustion type nonlinearities. It follows that equation (2.5) is solvable when

$$\gamma = \frac{c_0}{2|\Omega|} \int_\Omega |\nabla\chi|^2\, d\tilde{x} = \frac{c_0}{2} \langle |\nabla\chi|^2 \rangle, \tag{2.9}$$

regardless of the choice of α. For such γ, we may choose $\alpha = -\frac{1}{2}\langle |\nabla\chi|^2 \rangle$ so $\gamma + c_0\, \alpha = 0$ and the $\tilde{u}$ equation becomes:

$$\bar{L}\tilde{u} = \Delta_{\xi,\tilde{x}} \tilde{u} - c_0 \tilde{u}_\xi + f'(U)\tilde{u} = \left(\langle |\nabla\chi|^2 \rangle - |\nabla\chi|^2 \right) U''. \tag{2.10}$$

Note that the right hand side of (2.10) has zero integral average over Ω.

The solution $\tilde{u}$ of (2.10) can be expanded in eigenfunctions of the Laplacian $\Delta_{\tilde{x}}$:

$$\begin{aligned}\tilde{u}(\xi,\tilde{x}) &= \sum_{j\in Z_+^{n-1}\cup\{0\}} u_j(\xi)\phi_j(\tilde{x}),\\ |\nabla_{\tilde{x}}\chi|^2 &= \sum_{j\in Z_+^{n-1}\cup\{0\}} a_j\phi_j(\tilde{x}),\end{aligned}$$

where Z_+^{n-1} denotes nonnegative integer vectors in R^{n-1} with at least one positive component; $-\Delta_{\tilde{x}}\phi_j = \lambda_j\phi_j$ with zero Neumann boundary condition on $\partial\Omega$. The eigenfunctions ϕ_j are normalized so that $\|\phi_j\|_{L^\infty} = 1$, and the eigenvalues $\lambda_j = O(|j|^2)$ for large $|j|$. Clearly, $\phi_0 = 1$ and $\lambda_0 = 0$, $a_0 = \langle|\nabla\chi|^2\rangle$. The functions u_j solve the equations

$$L_j u_j = u_j'' - \lambda_j u_j - c_0 u_j' + f'(U)u_j = -(a_j - \delta_{j,0}\, a_0)\, U''. \tag{2.11}$$

The $j = 0$ equation has zero right hand side, and we take $u_0 \equiv 0$. If α were zero, equation (2.5) implies that the right hand side of the $j = 0$ equation would be $-a_0\, U'' + \gamma\, U'$. The general solution (zeroth-mode) for u_0 is $-\frac{a_0}{2}\xi\, U'(\xi) + \text{const.}\, U'(\xi)$. The linear factor ξ would render $v(x)$ unable to stay in the set K due to divergence at large ξ (and x_1).

Now consider the equations for $j \neq 0$. As shown in [15], the operators L_j are invertible in part due to $\lambda_j > 0$ for $j \neq 0$. As in Lemma 2.3 of [15], we have the regularity estimate:

$$\|u_j\|_{C^2(R)} \leq C_1|a_j|, \quad j \neq 0, \tag{2.12}$$

where the constant C_1 is independent of j and depends only on U and its derivatives.

Next we show that for $j \neq 0$, the ratio $\frac{u_j}{U'}$ is bounded by $C_2\,|a_j|$, where C_2 is a positive constant independent of j. It suffices to prove a uniform bound if the a_j factor is one. Consider the function $w_j = \beta U' - u_j$ for some positive constant β. For some $r_0 > 0$ sufficiently large, $f'(U) \leq 0$ whenever $|\xi| \geq r_0$. Choose β as

$$\beta \geq \beta_1 \equiv \frac{C_1}{\min_{|\xi|\leq r_0} U'(\xi)}, \tag{2.13}$$

so that by (2.12), $w_j > 0$ in the region $|\xi| \leq r_0$. By equation (2.11) for u_j and the fact that U' solves $L_j U' = -\lambda_j U' < 0$, we have

$$L_j w_j = -\beta\lambda_j U' + U''. \tag{2.14}$$

Using the fact that U'' is bounded by a constant multiple of U', and that $\inf_{j\neq 0} \lambda_j > 0$, we may further increase β if necessary to ensure that $L_j w_j < 0$ in the region $|\xi| > r_0$. Maximum principle implies that $w > 0$ for all $\xi \in R$. Repeating the argument for $-u_j$ and taking into account the a_j factor give:

$$\frac{|u_j|}{U'} \leq C_3 |a_j|, \tag{2.15}$$

where the constant C_3 depends only on U, and (2.15) holds uniformly in $j \neq 0$. Inequality (2.15) says that there is no resonance when inverting the operator L_j to find u_j, for $j \neq 0$, in other words u_j decays same as U' at infinities.

To improve the estimate above, we move the term $f'(U)\, u_j$ to the right hand side of (2.11) which is then bounded by $C_3' |a_j| U'$, for C_3' independent of j. The left hand side operator becomes $u_j'' - \lambda_j\, u_j - c_0\, u_j'$. An explicit formula can be written for u_j, and the asymptotics of λ_j imply that

$$\frac{|u_j^{(i)}(\xi, \tilde{x})|}{U'(\xi)} \leq C_4 \frac{|a_j|}{(1 + |j|^2)}, \tag{2.16}$$

for a constant C_4 depending only on U, where $i = 0, 1, 2$ denotes the order of ξ derivatives.

Suppose that b is Hölder continuous, then Schauder estimates give [6]:

$$\| |\nabla \chi|^2 \|_{1+p} \leq C\, \|b\|_p^2, \tag{2.17}$$

for some $p \in (0,1)$, $C = C(\Omega)$, $\| \cdot \|_p$ is the standard Hölder norm. For rectangular cross section Ω (dimension $n-1$), the ϕ_j's are trigonometric functions. Then

$$|a_j| \leq C_5\, \|b\|_p^2\, (1 + |j|)^{-(1+p)}, \;\; p \in (0,1), \tag{2.18}$$

for some constant C_5 [18]. Combining (2.16), (2.17), and (2.18), we see that the eigenfunction expansion of $\tilde{u}$

$$\tilde{u}(\xi, \tilde{x}) = \sum_{j \in Z_+^{n-1}} u_j(\xi)\, \phi_j(\tilde{x}) \tag{2.19}$$

converges uniformly in $(\xi, \tilde{x})$. Moreover,

$$\begin{aligned}\frac{|\tilde{u}_\xi^{(i)}(\xi,\tilde{x})|}{U'(\xi)} &\leq \sum_{j\in Z_+^{n-1}} |u_j^{(i)}|(\xi)\,|\phi_j|(\tilde{x})/U' \\ &\leq C_4\,C_5\,\|b\|_p^2 \sum_{j\in Z_+^{n-1}} (1+|j|)^{-(3+p)} = C_6\,\|b\|_p^2, \end{aligned} \tag{2.20}$$

if $n = 2, 3$. The mixed derivative term is bounded as:

$$\begin{aligned}\frac{|\nabla_{\tilde{x}}\,\tilde{u}_\xi(\xi,\tilde{x})|}{U'(\xi)} &\leq \sum_{j\in Z_+^{n-1}} \frac{|u_j^{(1)}|(\xi)}{U'}\,|j|\,|\phi_j|(\tilde{x}) \\ &\leq C_5\,\|b\|_p^2 \sum_{j\in Z_+^{n-1}} (1+|j|)^{-(2+p)} = C_7\,\|b\|_p^2, \end{aligned} \tag{2.21}$$

if $n = 2, 3$. In case $n = 2$, Ω is an interval, then $b \in L^\infty(\Omega)$ suffices, because $\nabla_{\tilde{x}}\,\chi$ is Lipschitz, so $|a_j| \leq O(\|b\|_\infty^2 |j|^{-p})$ for some $p \in (0,1)$ which replaces estimate (2.18). The exponent in (2.21) goes down to $1 + p$, yet enough for convergence on $j \in Z_+^1$. As a result, for $n = 2$, $i = 0, 1, 2$, the estimates:

$$\frac{|\tilde{u}_\xi^{(i)}(\xi,\tilde{x})|}{U'(\xi)} \leq C_8\,\|b\|_\infty^2, \quad \frac{|\nabla_{\tilde{x}}\,\tilde{u}_\xi(\xi,\tilde{x})|}{U'(\xi)} \leq C_9\,\|b\|_\infty^2, \tag{2.22}$$

hold.

As a consequence of these estimates, let us verify the admissibility of test function v. Clearly, v approaches zero (one) exponentially at $x_1 = -\infty$ $(+\infty)$. As

$$v_{x_1} = (1+\delta^2\alpha)U' + \delta^2(1+\delta^2\alpha)\tilde{u}_\xi,$$

we have for $n = 3$:

$$v_{x_1} \geq \frac{1}{2}U' - \frac{1}{2}\delta^2|\tilde{u}_\xi| \geq \left[\frac{1}{2} - \delta^2\,\frac{C_6}{2}\,\|b\|_p^2\right] U' \geq \frac{1}{4}\,U' > 0, \tag{2.23}$$

if:

$$\delta^2 \leq \min\left(\frac{1}{2|\alpha|}, \frac{1}{2}\,(C_6\,\|b\|_p^2)^{-1}\right). \tag{2.24}$$

For $n = 2$, in view of (2.22), $v_{x_1} \geq \frac{1}{4}\,U'$ if

$$\delta^2 \leq \min\left(\frac{1}{2|\alpha|}, \frac{1}{2}\,(C_8\,\|b\|_\infty^2)^{-1}\right). \tag{2.25}$$

It follows from the monotonicity of v in x_1 and its asymptotics near x_1 infinities that $0 < v < 1$ for all $x_1, \tilde{x}$.

With v being an admissible test function, we see from (2.6) that

$$\psi(v(x)) = \frac{Lv + f(v)}{\partial_{x_1} v} = c_0 + \delta^2 \gamma + \frac{R_1}{\partial_{x_1} v}, \tag{2.26}$$

with R_1 defined by (2.7). By (2.24) or (2.25), $\partial_{x_1} v \geq \frac{1}{4} U'$. Also, each term in R_1 is bounded by a multiple of U'. It follows that for δ satisfying (2.24) or (2.25), $|\frac{R_1}{\partial_{x_1} v}| \leq |4 \frac{R_1}{U'}| \leq C_{10}\, \delta^3$, where $C_{10} = C_{10}(U, \|b\|_p), p \in (0,1)$, if $n = 3$, $C_{10} = C_{10}(U, \|b\|_\infty)$ if $n = 2$. We have shown:

Proposition 2.1 *Let $n = 2$ or 3, and let the shear flow profile $b(\tilde{x})$ be Hölder continuous with exponent $p \in (0,1)$ over rectangular domain $\Omega \subset R^{n-1}$. There is a positive constant δ_0 depending only on U and the Hölder (maximum) norm of b for $n = 3$ $(n = 2)$ such that if $\delta \leq \delta_0$:*

$$c(\delta) = c_0 + \frac{c_0 \delta^2}{2|\Omega|} \int_\Omega |\nabla \chi|^2 \, dy + O(\delta^3). \tag{2.27}$$

A specific form of δ_0 follows from (2.24) and (2.25).

At large δ, numerical evidence suggested that $\lim_{\delta \to \infty} c(\delta)/\delta$ exists for bistable and combustion nonlinearities [11]. Such a limit holds for the KPP nonlinearity [3].

3 Random Front Speed Asymptotics

Let us consider $b = b(\tilde{x}, \omega)$ as a stationary mean zero Gaussian field with almost surely continuous sample paths in dimension one and two for $n = 2$, 3 respectively. We are interested in the restriction of b over Ω. In order to apply results of the previous section on each realization of b, let us write $b = \bar{b} + b_1$, where $\bar{b} = |\Omega|^{-1} \int_\Omega b(\tilde{x}, \omega)\, d\tilde{x}$. Correspondingly, $c = c(\delta, \omega) = c_0 - \delta \bar{b} + c_2(\omega)$, where c_2 depends on b_1. As the size of δ_0 depends on either the maximum norm or the Hölder norm of b_1 which is an unbounded random variable, there is no uniform way to choose a δ_0 for all realizations. On the other hand, the probability of the occurance of very large maximum (Hölder) norm of b_1 is often small, so the random speed asymptotics may hold with probibility arbitrarily close to one. Let us consider $n = 2$ first.

Theorem 3.1 (Two dimensional channel) *Let $D = R^1 \times [0, L]$, $\Omega = [0, L]$, and $b(\tilde{x})$ be the restriction on Ω of a mean zero stationary, Gaussian random process with almost surely continuous sample paths. Let f be a bistable or combustion nonlinearity. Then for each small $\epsilon \in (0, 1/4)$ and $q \in (0, 1)$, there is a deterministic constant $\delta_0 = \delta_0(\epsilon, q)$ such that if $\delta \in (0, \delta_0)$,*

$$Prob\left\{\left|c(\delta,\omega) - c_0 + \delta\bar{b} - \delta^2 \frac{1}{2}\langle\left(\int_0^{\tilde{x}} b_1(y)\,dy\right)^2\rangle\right| \geq \kappa\delta^{3-q}\right\} < \epsilon.$$

where the constant $\kappa > 0$ is independent of δ, ϵ, and q. As $\epsilon \to 0$, the constant δ_0 can be chosen to satisfy

$$\delta_0(\epsilon, q) \geq C|\log(\epsilon)|^{-\frac{2}{q}} \tag{3.1}$$

Proof: Recall the inequality (Lemma 3.1, p. 62, [2]):

$$E[\sup_\Omega b(\tilde{x})] \leq E\sup_\Omega |b(\tilde{x})| \leq E|b(0)| + 2E[\sup_\Omega b(\tilde{x})], \tag{3.2}$$

and the Borell inequality on mean zero Gaussian process with almost surely continuous sample paths (Theorem 2.1, p. 43, [2]):

$$\begin{aligned} &\mu \equiv E[\sup_\Omega b(\tilde{x})] < \infty \\ &P(|\sup_\Omega b(\tilde{x}) - \mu| > \lambda) \leq 2e^{-\lambda^2/(2\sigma^2)}, \end{aligned} \tag{3.3}$$

where $\sigma^2 = E[b^2]$. With the choice of $\lambda = \sqrt{-2\sigma^2 \log(\epsilon/4)}$, inequality (3.3) implies that

$$\begin{aligned} P(\|b\|_\infty > \lambda + \mu) &\leq P(\sup_\Omega b(\tilde{x}) - \mu > \lambda) + P(-\inf_\Omega b(\tilde{x}) - \mu > \lambda) \\ &\leq 4e^{-\lambda^2/(2\sigma^2)} \leq \epsilon. \end{aligned} \tag{3.4}$$

We now see that for $\epsilon > 0$ sufficiently small, $\lambda > 1$ and there is a constant C_{11} independent of ϵ such that if

$$\delta < \delta_0(\epsilon) = C_{11}\left(\frac{1}{\mu + \lambda}\right)^{\frac{4}{q}} \leq C_{11}\frac{1}{\mu + \lambda} \tag{3.5}$$

then (2.25) holds with probability at least $1-\epsilon$. Therefore, by Proposition 2.1, the front speed asymptotics hold with probability $1-\epsilon$. Using (2.7), (2.8), and (2.22) we see that the remainder in the expansion satisfies

$$\begin{aligned} \left|\frac{R_1}{\partial_{x_1} v}\right| \leq C_{12}\delta^3\|b\|_\infty^4 &\leq C_{12}\delta^3(\mu+\lambda)^4 \\ &\leq C_{13}\delta^{3-q}\delta^q\,(\mu+\lambda)^4 \\ &\leq C_{13}\delta^{3-q}\delta_0^q\,(\mu+\lambda)^4 \\ &\leq C_{14}\delta^{3-q} \end{aligned}$$

with probability $1-\epsilon$. The estimate on $\delta_0(\epsilon)$ follows from (3.5).

Remark 3.1 *An example of such b process is the mean zero stationary Ornstein-Uhlenbeck (O-U) process which is both Gaussian and Markov. The O-U sample paths are almost surely continuous.*

The $n=3$ case requires a probabilistic estimate of Hölder norm of b. For a mean zero Gaussian field, this is related to the structure of the covariance function $R(t,s) = E[b(t)b(s)]$, t, $s \in \Omega$. If the covariance function is continuous, positive and non-negative definite, there exists a Gaussian process with this covariance [2]. The symmetric integral operator: $\phi \to \int_\Omega R(t,s)\,\phi(t)\,dt$ generates a complete set of orthonormal eigenfunctions ϕ_j on $L^2(\Omega)$ with nonnegative eigenvalues λ_j, $j = 1, 2, \cdots$. Define:

$$p(u) = \max_{\|s-t\| \leq |u|\sqrt{2}} [E|b(s)-b(t)|^2]^{1/2}.$$

Consider the partial sum (m a positive integer):

$$X^{(m)}(t,\omega) = \sum_{j=1}^{m} \sqrt{\lambda_j}\,\phi_j(t)\theta_j(\omega), \tag{3.6}$$

where θ_j's are independent unit Gaussian random variables. The convergence of the partial sum to b is given by Garsia's Theorem (Theorem 3.3.2, p. 52, [1]). It says that if $\int_0^1 (-\log u)^{1/2}\,dp(u) < \infty$, then with probability one, $X^{(m)}(t)$ are almost surely equicontinuous and converge uniformly on Ω. The resulting infinite series is the celebrated Karhunen-Loève expansion. Moreover, the following estimate holds for all m:

$$\begin{aligned} |X^{(m)}(s) - X^{(m)}(t)| &\leq 16\sqrt{2}[\log B]^{1/2}\,p(\|s-t\|) \\ &\quad +32\sqrt{2}\int_0^{\|s-t\|} (-\log u)^{1/2}\,dp(u), \end{aligned} \tag{3.7}$$

where $B = B(\omega)$ is a positive random variable, $E[B^2] \le 32$.

Suppose that $p(u)$ is Hölder continuous with exponent $s \in (0,1)$, then b is almost surely Hölder continuous with exponent s, and the Hölder norm of b is bounded by $\alpha_1[\log B]^{1/2} + \alpha_2$, where α_1, α_2 are two positive deterministic constants. Chebyshev's inequality gives:

$$\text{Prob}\left([\log B]^{1/2} \ge \lambda\right) = \text{Prob}\left(B \ge e^{\lambda^2}\right) \le E(B^2)/e^{2\lambda^2} \le 32\, e^{-2\lambda^2}. \quad (3.8)$$

This implies that

$$\text{Prob}\ (\|b\|_s > \lambda) \le 32 e^{-2\left(\frac{\lambda-\alpha_2}{\alpha_1}\right)^2} \le C_{15} e^{-2\lambda^2} \le \epsilon, \quad (3.9)$$

if $\lambda = \sqrt{\log(\frac{\epsilon}{C_{15}})}$. For $\epsilon > 0$ sufficiently small, $\lambda > 1$ and we take

$$\delta < \delta_0(\epsilon) = C\left(\frac{1}{\lambda}\right)^{\frac{4}{q}} \le C\frac{1}{\lambda},$$

so that (2.24) holds with probability at least $1-\epsilon$. As before we use (2.7), (2.8), and (2.22) to conclude that the remainder in the expansion satisfies

$$\left|\frac{R_1}{\partial_{x_1} v}\right| \le C_{12}\delta^3 \|b\|_s^4 \le C_{16}\delta^{3-q}$$

with probability $1-\epsilon$. We now conclude from Proposition 2.1:

Theorem 3.2 (Three dimensional channel) *Let $D = R^1 \times \Omega$, $\Omega = [0, L]^2$, and b be a mean zero Gaussian process such that the function $p(u)$ is Hölder continuous. Then b has almost surely Hölder continuous sample paths. For each small $\epsilon \in (0, 1/5)$ and $q \in (0, 1)$ there is a deterministic constant $\delta_0 = \delta_0(\epsilon, q)$ such that if $\delta \in (0, \delta_0)$,*

$$Prob\left\{\left|c(\delta,\omega) - c_0 + \delta\bar{b} - \delta^2\,\frac{1}{2}\langle|\nabla_{\tilde{x}}\chi|^2\rangle\right| \ge \kappa\delta^{3-q}\right\} < \epsilon,$$

where χ satisfies: $-\Delta_{\tilde{x}}\,\chi = b_1$, subject to zero Neumann boundary condition at $\partial\Omega$. The constant $\kappa > 0$ is independent of δ, ϵ, and q. As $\epsilon \to 0$, the constant δ_0 can be chosen to satisfy

$$\delta_0(\epsilon, q) \ge C|\log(\epsilon)|^{-\frac{2}{q}}\,. \quad (3.10)$$

Remark 3.2 *Suppose that $\lim_{\delta\to\infty} c(\delta,\omega)/\delta$ exists almost surely and that $E(\|b\|_\infty)$ is finite. The dominated convergence theorem implies that $\lim_{\delta\to\infty} E[c]/\delta$ exists and is finite. This argument is same as for the KPP case, see [12] for details.*

4 Concluding Remarks

Bistable and combustion front speeds in mean zero Gaussian random shear flows have been studied with the min-max variational principle of [8]. The quadratic speed enhancement law is valid with probability arbitrarily close to one in both two and three dimensional channels under the almost sure continuity and the mean square Hölder regularity conditions of the Gaussian shear flows.

It would be interesting to extend results here to the case of general convex cross section Ω with smooth boundary, or to the case of Ω in dimension higher than two.

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References

[1] R. Adler, "The Geometry of Random Fields", John Wiley & Sons, 1980.

[2] R. Adler, "An Introduction to Continuity, Extrema and Related Topics for General Gaussian Processes", Institute of Math Stat, Lecture Notes-Monograph Series, Vol. 12, 1990.

[3] H. Berestycki, *The influence of advection on the propagation of fronts in reaction-diffusion equations*, in "Nonlinear PDEs in Condensed Matter and Reactive Flows", NATO Science Series C, 569, H. Berestycki and Y. Pomeau eds, Kluwer, Doordrecht, 2003.

[4] H. Berestycki, L. Nirenberg, *Travelling fronts in cylinders*, Ann. Inst. H. Poincare, Anal. Nonlineaire, Vol. 9, (1992), No. 5, pp. 497-572.

[5] P. Constantin, A. Kiselev, A. Oberman, L. Ryzhik, *Bulk burning rate in passive-reactive diffusion*, Arch Rat. Mech Analysis, 154, (2000), pp. 53-91.

[6] D. Gilbarg, N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, 2nd edition, 1983.

[7] F. Hamel, *Formules min-max pour les vitesses d'ondes progressives multidimensionnelles*, Ann. Fac. Sci. Toulouse, 8 (1999), pp. 259-280.

[8] S. Heinze, G. Papanicolaou, A. Stevens, *Variational principles for propagation speeds in inhomogeneous media*, SIAM J. Applied Math, 62, no. 1, (2001), pp. 129 - 148.

[9] A. Kiselev, L. Ryzhik, *Enhancement of the traveling front speeds in reaction-diffusion equations with advection*, Ann. de l'Inst. Henri Poincaré, Analyse Nonlinéaire, 18, (2001), pp. 309–358.

[10] B. Khouider, A. Bourlioux, A. Majda, *Parameterizing turbulent flame speed-Part I: unsteady shears, flame residence time and bending*, Combustion Theory and Modeling, 5 (2001), pp. 295-318.

[11] J. Nolen, J. Xin, *Reaction diffusion front speeds in spacially-temporally periodic shear flows*, SIAM J. Multiscale Modeling and Simulation, Vol. 1, (2003), No. 4, pp. 554-570.

[12] J. Nolen and J. Xin, *A Variational Principle Based Study of KPP Minimal Front Speeds in Random Shears*, math.AP/0410564, www.arxiv.org, 2004.

[13] G. Papanicolaou, J. Xin, *Reaction-diffusion fronts in periodically layered media*, J. Stat. Physics, 63 (1991), pp. 915-931.

[14] J-M Roquejoffre, *Eventual monotonicity and convergence to traveling fronts for the solutions of parabolic equations in cylinders*, Ann Inst. H. Poincaré, Analyse Nonlineare, 14 (1997), 499-552.

[15] J. Xin, *Existence and Stability of Travelling Waves in Periodic Media Governed by a Bistable Nonlinearity*, J. Dynamics Diff. Eqs. 3, (1991), pp. 541-573.

[16] J. Xin, *Front propagation in heterogeneous media,* SIAM Review, Vol. 42, No. 2, June 2000, pp. 161-230.

[17] J. Xin, *KPP front speeds in random shears and the parabolic Anderson problem*, Methods and Applications of Analysis, Vol. 10, No. 2, (2003), pp. 191-198.

[18] A. Zygmund, *Trigonometric Series*, Third Edition, Cambridge University Press, 2002.