Normal approximation for a random elliptic equation

James Nolen*

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Abstract

We consider solutions of an elliptic partial differential equation in $\mathbb{R}^d$ with a stationary, random conductivity coefficient that is also periodic with period L. Boundary conditions on a square domain of width L are arranged so that the solution has a macroscopic unit gradient. We then consider the average flux that results from this imposed boundary condition. It is known that in the limit $L \to \infty$, this quantity converges to a deterministic constant, almost surely. Our main result is that the law of this random variable is very close to that of a normal random variable, if the domain size L is large. We quantify this approximation by an error estimate in total variation. The error estimate relies on a second order Poincaré inequality developed recently by S. Chatterjee.

1 Introduction

Elliptic partial differential equations of the form

$$-\nabla \cdot (a(x)\nabla u) = f$$

arise in many physical applications where the coefficient $a(x)$ may be modeled best as a random field, due to inherent uncertainty and complexity of the physical medium [23]. In this situation, the solutions u are also random objects. Homogenization theory for these equations [20, 14] shows that, although the coefficient $a(x)$ may be highly irregular, a solution u may be approximated well by the solution of an "effective" elliptic equation having a more regular coefficient, perhaps a deterministic coefficient. Quantifying the error in such an approximation and understanding the statistical structure of the random solution is very important.

Here we consider solutions of the elliptic equation

$$-\nabla \cdot (a(x)(\nabla\phi(x) + e_1)) + \beta\phi(x) = 0, \quad x \in D_L \subset \mathbb{R}^d, \tag{1.1}$$

where the scalar function $a(x) \in L^\infty(\mathbb{R}^d)$ is a stationary random field satisfying the uniform ellipticity condition $0 < a_* \le a(x) \le a^*$, with a_* and a^* being deterministic constants. The parameter $\beta \ge 0$ is deterministic. The set $D_L = [0, L)^d$ is the domain, and we require that ϕ satisfies periodic boundary conditions on the boundary of D_L. If we interpret (1.1) in terms of electrical conductivity, then ϕ is a potential, $a(x)$ is the conductivity, and the vector field $-a(x)(\nabla\phi + e_1)$ is a current density. The unit vector e_1 is deterministic, and it is the gradient of the linear potential $x \cdot e_1$.

The equation (1.1) plays an important role in the homogenization theory for the random elliptic operator $u \mapsto -\nabla \cdot (a(x/\epsilon)\nabla u)$ in the limit $\epsilon \to 0$ [20, 14]. It is well-known that the homogenized conductivity tensor $\bar{a}$ for that operator can be expressed in terms of functions ϕ, called "correctors",

*Department of Mathematics, Duke University, Durham, North Carolina, USA. (nolen@math.duke.edu).

which solve (1.1) with e_1 being one of the d standard basis vectors and which have stationary gradient. On the other hand, in a numerical computation of $\bar{a}$ one must approximate the true correctors by solving (1.1) in a bounded domain D_L with suitable boundary condition. The periodic boundary condition that we impose here is one choice that allows accurate approximation of the effective coefficient $\bar{a}$ in the limit $L \to \infty$ [6, 19].

The focus of this paper is on the statistical behavior of the quantity

$$\Gamma_{L,\beta} = \frac{1}{|D_L|} \int_{D_L} a(x)|\nabla \phi + e_1|^2 + \beta \phi^2 \, dx$$

for large L. Using (1.1) and the periodicity of ϕ we see that $\Gamma_{L,\beta}$ may also be written as

$$\Gamma_{L,\beta} = \frac{1}{|D_L|} \int_{D_L} e_1 \cdot a(x)(\nabla \phi(x) + e_1) \, dx.$$

This is a random variable since $a(x)$ and the solution ϕ are random. In terms of conductivity, $\Gamma_{L,\beta}$ may be interpreted as an average flux in the direction e_1 that results from a macroscopic potential gradient imposed in the direction of e_1. The results of [6, 19] imply that for $\beta \geq 0$ fixed, $\Gamma_{L,\beta}$ converges almost surely, as $L \to \infty$, to a deterministic constant $\bar{\Gamma}_\beta > 0$. For $\beta = 0$, the limit $\bar{\Gamma}_0$ is one of the diagonal entries of the homogenized tensor $\bar{a}$. For finite L, it is interesting to understand how $\Gamma_{L,\beta}$ and ϕ fluctuate around their means. However, not much is known about the distribution of $\Gamma_{L,\beta}$ or the distribution of ϕ. Our main result is an estimate showing that for $L \gg 1$, the distribution of $\Gamma_{L,\beta}$ is very close to that of a normal random variable.

Before we present the main result and explain its relation to other works, let us define the problem precisely and establish notation. For $L \in \mathbb{Z}^+$, let $D_L = [0, L)^d \subset \mathbb{R}^d$ and let $L^\infty_{per}(D_L)$ denote the set of functions in $L^\infty(\mathbb{R}^d)$ which are periodic with period L in each direction. That is, for all $a \in L^\infty_{per}(D_L)$, $a(x + Lk) = a(x)$ holds for all $k \in \mathbb{Z}^d$ and almost every $x \in \mathbb{R}^d$. The coefficient $a(x)$ in (1.1) will be a random function in $L^\infty_{per}(D_L)$. We also require that $a(x)$ is stationary with respect to integer shifts: for every $k \in \mathbb{Z}^d$ and $a(\cdot + k)$ is equal in law to $a(\cdot)$.

We suppose that the random nature of $a(x)$ comes from its dependence on a random vector $\zeta = (\zeta_k)_{k \in \mathbb{Z}^d \cap D_L}$ whose L^d components are independent and identically distributed real-valued random variables, defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\Omega = \mathbb{R}^{L^d}$ and $\mathbb{P}$ is a product measure on Ω. We often will write $a(x)$ for the random function $a(x, \zeta)$, the dependence on ζ being understood. Let $\mathbb{E}[Y]$ denote expectation with respect to the probability measure $\mathbb{P}$ defining the law of ζ (there will be another assumption about the law of ζ below). We will make the following structural assumptions about the random function $a(x, \zeta)$. First, we suppose that the map $\zeta \to a(\cdot, \zeta)$ from $\mathbb{R}^{L^d} \to L^\infty_{per}(D_L)$ is a twice Fréchet differentiable map. For each $k \in \mathbb{Z}^d$, let $Q_k = k + [0, 1)^d \subset \mathbb{R}^d$ denote the cube of size 1 with a corner at k. We suppose there are positive constants $\tau, C_1, C_2, C_3, a^*, a_* > 0$ such that the following hold $\mathbb{P}$-almost surely:

$$a_* \leq a(x) \leq a^*, \quad a.e. \;\; x \in D_L, \tag{1.2}$$

$$C_1 \mathbb{I}_{Q_k}(x) \leq \frac{\partial a}{\partial \zeta_k}(x) \leq C_2 \mathbb{I}_{B_\tau(k)}(x), \quad a.e. \;\; x \in D_L, \quad \forall\, k \in \mathbb{Z}^d \cap D_L, \tag{1.3}$$

$$\left| \frac{\partial^2 a}{\partial \zeta_k \partial \zeta_j}(x) \right| \leq C_3 \mathbb{I}_{B_\tau(k) \cap B_\tau(j)}(x), \quad a.e. \;\; x \in D_L, \quad \forall\, k, j \in \mathbb{Z}^d \cap D_L. \tag{1.4}$$

The function $\mathbb{I}_S(x)$ is the indicator of the set S, and $B_\tau(y)$ is the ball of radius τ centered at y, in the metric topology of the torus on D_L. That is, $x \in B_\tau(y)$ if and only if $x = y + Lk + z$ for some $k \in \mathbb{Z}^d$

and $|z| < \tau$. We suppose the constants τ, C_1, C_2, C_3, a^*, and a_* are independent of L. Notice that the bound (1.3) implies that $a(x, \zeta)$ depends only on those ζ_k for k in a τ-neighborhood of x (fixed with respect to L). Thus, $a(x, \zeta)$ and $a(y, \zeta)$ are independent if $|x - y| > 2\tau$. Also, (1.2) and (1.3) imply that there are constants $\zeta_{min} < \zeta_{max}$ such that $\zeta_k \in [\zeta_{min}, \zeta_{\max}]$ holds with probability one.

For clarity, let us highlight a simple example for which these assumptions hold. Suppose that $0 < \zeta_{min} \leq \zeta_k \leq \zeta_{max}$ for all $k \in \mathbb{Z}^d \cap D_L$ holds with probability one. For $x \in \mathbb{R}^d$, define the piecewise constant function

$$a(x) = a(x, \zeta) = \sum_{k \in \mathbb{Z}^d} \zeta_{(k \bmod L)} \mathbb{I}_{M_k}(x), \tag{1.5}$$

where the sets $M_k = k + M_0$ are translates of a given bounded and measureable set M_0 satisfying $Q_0 \subset M_0 \subset B_\tau(0)$. The notation $(k \bmod L)$ refers to the point $(k_1 \bmod L, \ldots, k_d \bmod L) \in D_L \cap \mathbb{Z}^d$. It is easy to see that $a(x) \in L^\infty_{per}(D_L)$ with probability one. Conditions (1.2), (1.3), and (1.4) hold with $a_* = \zeta_{min}$, $a^* = \zeta_{max} O(\tau^d)$, and $C_1 = C_2 = C_3 = 1$. Moreover, for each $k \in \mathbb{Z}^d$, $a(\cdot + k)$ has the same law as $a(\cdot)$, since $a(x + k, \zeta) = a(x, \hat{\zeta})$ where $\hat{\zeta}_j = \zeta_{(j+k) \bmod L}$.

Let $H^1_{per}(D_L)$ denote the set of L-periodic functions in $H^1_{loc}(\mathbb{R}^d)$. That is, $\phi \in H^1_{per}(D_L)$ if $\phi \in H^1_{loc}(\mathbb{R}^d)$ and $\phi(x + Lk) = \phi(x)$ a.e. $\mathbb{R}^d$ for every $k \in \mathbb{Z}^d$. If $a(x) \in L^\infty(D_L)$ and satisfies $0 < a_* \leq a(x) \leq a^*$ almost everywhere, then there exists a weak solution $\phi \in H^1_{per}(D_L)$ to (1.1):

$$\int_{D_L} \nabla v \cdot a(x)(\nabla \phi + e_1) + \beta \phi v \, dx = 0, \quad \forall \, v \in H^1_{per}(D_L). \tag{1.6}$$

For $\beta > 0$, the solution is unique. For $\beta = 0$, the solution is not unique, but any two solutions in $H^1_{per}(D_L)$ must differ by a constant. So, under the normalization condition

$$\int_{D_L} \phi(x) \, dx = 0, \tag{1.7}$$

and for fixed L, the solution is unique in $H^1_{per}(D_L)$ for all $\beta \geq 0$. With $a(x) = a(x, \zeta)$ satisfying the conditions above, this unique solution $\phi(x) = \phi(x, a, L, \beta)$ depends on the parameters L and β, on $x \in D_L$, and on the random variables $(\zeta_j)_{j \in D_L \cap \mathbb{Z}^d}$ which determine a. The uniqueness of the solution implies that $\phi(x)$ is statistically stationary with respect to integer shifts: the law of $\phi(x)$ is the same as that of $\phi(x + k)$ for any $k \in \mathbb{Z}^d$, since the variables ζ_j are identically distributed.

Having defined both $a(x)$ and $\phi(x)$, we now define the random variable

$$\Gamma_{L,\beta} = \frac{1}{|D_L|} \int_{D_L} a(x) |\nabla \phi + e_1|^2 + \beta \phi^2 \, dx = \frac{1}{|D_L|} \int_{D_L} e_1 \cdot a(x)(\nabla \phi(x) + e_1) \, dx,$$

which also is a function of the L^d random variables $\{\zeta_j \mid j \in \mathbb{Z}^d \cap D_L\}$. It is known that $\Gamma_{L,\beta}$ has a variational representation:

$$\Gamma_{L,\beta} = \min_{v \in H^1_{per}(D_L)} \frac{1}{|D_L|} \int_{D_L} a(x) |\nabla v + e_1|^2 + \beta v^2 \, dx. \tag{1.8}$$

The Euler-Lagrange equation for this variational problem is (1.1), and ϕ is the unique minimizer (unique up to addition of a constant if $\beta = 0$).

We make one more technical assumption about the variables ζ_k. We suppose that the law of ζ_k is that of $h(Z_k)$ where Z_k is a standard normal random variable and $h : \mathbb{R} \to \mathbb{R}$ is twice differentiable and satisfies $|h'(z)| \leq c_1$ and $|h''(z)| \leq c_2$. While this assumption excludes some interesting choices for the law of ζ_k, it does not imply that the law of ζ_k has a density with respect to Lebesgue measure on $\mathbb{R}$. We suppose that h is not a constant, so that $\mathrm{Var}(\zeta_k) > 0$.

Our main result is the following theorem. Recall that the total variation distance $d_{TV}(X,Y)$ between the laws of two real-valued random variables X and Y is defined as

$$d_{TV}(X,Y) = \sup |\mathbb{P}(X \in A) - \mathbb{P}(Y \in A)|,$$

where the supremum is over all Borel sets $A \subset \mathbb{R}$. This quantity is invariant under centering and scaling: $d_{TV}(X,Y) = d_{TV}((X-\mu)/\sigma, (Y-\mu)/\sigma)$ for all $\mu \in \mathbb{R}$, $\sigma > 0$.

Theorem 1.1 *Let $d \geq 1$. There is a constant $C > 0$ and $q > 4$ such that*

$$d_{TV}(\Gamma_{L,\beta}, W_{L,\beta}) \leq C L^{-d/2} \frac{\mathbb{E}[\Phi_0^q]^{2/q}}{\mathbb{E}[\Phi_0]^2} \tag{1.9}$$

holds for all $L > 1$ and $\beta \geq 0$, where $W_{L,\beta}$ is a normal random variable having the same mean and variance as $\Gamma_{L,\beta}$, and

$$\Phi_0 = \int_{Q_0} |\nabla \phi(x) + e_1|^2 \, dx. \tag{1.10}$$

Observe that the random variable Φ_0 which appears in Theorem 1.1 depends on both L and β. It is easy to see that $\mathbb{E}[\Phi_0] \geq 1$ holds for all $L > 1$ and $\beta \geq 0$ and all $d \geq 1$ (see (2.27)). So, when it is also true that $\mathbb{E}[\Phi_0^q]$ is bounded by a constant, independent of $L > 1$, then the right side of (1.9) is bounded by $O(L^{-d/2})$; in particular, the distribution of $\Gamma_{L,\beta}$ approaches that of a normal random variable. As explained below, $O(L^{-d/2})$ is the optimal bound on $d_{TV}(\Gamma_{L,\beta}, W_{L,\beta})$, in the sense that this is the expected bound if $\Gamma_{L,\beta}$ behaves like the average of $O(L^d)$ independent random variables. For all dimensions $d \geq 1$, if $\beta \geq \beta_0 > 0$ is bounded away from zero independently of L, then all moments $\mathbb{E}[\Phi_0^q]$ are bounded independently of $L > 1$ (for example, see Corollary 5.5). In this case, Theorem 1.1 implies that $d_{TV}(\Gamma_{L,\beta}, W_{L,\beta}) = O(L^{-d/2})$ as $L \to \infty$, which is the optimal bound.

If $\beta = 0$ or if $\beta > 0$ is allowed to vanish as $L \to \infty$, estimating the moments $\mathbb{E}[\Phi_0^q]$ is a delicate issue. To estimate $\mathbb{E}[\Phi_0^q]$ in this situation one can use the arguments developed recently by Gloria and Otto in [13], which is the work most directly related to this article. In [13], the authors derive variance bounds for a discrete functional similar to $\Gamma_{L,\beta}$, involving an infinite network of random resistors on the bonds of the integer lattice $\mathbb{Z}^d$. The PDE (1.1) is replaced by a discrete difference equation on all of $\mathbb{Z}^d$, without the periodicity assumption. The stationary potential field $\phi(x)$ is defined at points $x \in \mathbb{Z}^d$; the gradient and divergence have interpretations as difference operators. For each edge e in the integer lattice, the conductivity $A = A(e)$ is a random variable which is stationary with respect to lattice translation, but it is not periodic. Consequently, ϕ depends nontrivially on the infinite set of conductances $A(e)$. Gloria and Otto then consider the random variable

$$\tilde{\Gamma}_{L,\beta} = \sum_{\mathbb{Z}^d} \left(A(e)|\nabla\phi(e) + e_1|^2 + \beta\phi(x)^2 \right) \eta_L(x)$$

where $\eta_L(x) \geq 0$ is a deterministic weight function that is supported on a cube of size L, and having total mass 1. In present setting, the periodization of the random field $a(x)$ over D_L serves a similar purpose to the weight function η_L. One of the main results of [13] is that there is a constant $C > 0$ such that

$$\mathrm{Var}(\tilde{\Gamma}_{L,\beta}) \leq \begin{cases} CL^{-d}, & \text{if } d \geq 3 \\ CL^{-d}|\log(\beta)|, & \text{if } d = 2 \end{cases}$$

holds for all $\beta > 0$ and $L > 1$. Another important result from [13], and a key step in the analysis of $\mathrm{Var}(\tilde{\Gamma}_{L,\beta})$, is the following bound on moments of the discrete corrector ϕ:

$$\mathbb{E}[\,|\phi(0)|^q\,] \leq \left\{ \begin{array}{c} C_q, \quad \text{if } \; d \geq 3 \\ C_q |\log(\beta)|^{\gamma_q}, \quad \text{if } \; d = 2. \end{array} \right. \tag{1.11}$$

The constants $C_q, \gamma_q > 0$ are independent of $L > 1$ and $\beta > 0$. Observe that in dimension $d = 2$, there is an extra factor that diverges as $\beta \to 0$. The extension of the analysis of [13] to the present setting (spatial continuum, with periodicity on D_L) can be carried out to estimate moments of both $\int_{Q_0} \phi(x)\,dx$ and Φ_0. In particular, the argument shows that for $d \geq 3$ all moments $\mathbb{E}[\Phi^q]$ are bounded independently of $L > 1$ and $\beta \geq 0$. Therefore, for $d \geq 3$, Theorem 1.1 implies that $d_{TV}(\Gamma_{L,\beta}, W_{L,\beta}) = O(L^{-d/2})$. In the case $d = 2$, however, the argument shows that $\mathbb{E}[\Phi^q]$ is bounded by $C|\log\beta|^{\gamma}$ for certain constants $C_q, \gamma_q > 0$ independent of $L > 1$ and $\beta > 0$. So, in the case $d = 2$, if $\beta = 0$ or if $|\log\beta|^{\gamma_q} \to \infty$ faster than $L^{d/2}$ as $L \to \infty$, we cannot conclude from this bound that $d_{TV}(\Gamma_{L,\beta}, W_{L,\beta}) \to 0$ as $L \to \infty$. In Section 5 we explain a few points about this method of bounding $\mathbb{E}[\Phi_0^q]$ and its relation to the present setting. However, the extension of the results of [13] to the periodic setting is being worked out in [12], so we do not pursue it further.

Other works related to Theorem 1.1 include those of Naddaf and Spencer [18], Conlon and Naddaf [8], and Boivin [4] in the discrete case and Yurinskii [26] in the continuum setting; they also derive upper bounds on the variance of quantities similar to $\tilde{\Gamma}_{L,\beta}$ and $\Gamma_{L,\beta}$. Komorowski and Ryzhik [15] have proved some related moment bounds on ϕ in the discrete case when $d = 1$. In the discrete setting the work of Wehr [24] contains a lower bound on the variance of a quantity analogous to $\Gamma_{L,0}$. However, none of the works we have mentioned address the issue of a central limit theorem: whether the distribution of $\Gamma_{L,\beta}$ is approximately normal for $L \gg 1$. If $\beta = 0$ and the dimension is $d = 1$, then equation (1.1) can be integrated, with the solution ϕ written in terms of integrals of $1/a(x)$. In that case it is known that the solution itself may satisfy a central limit theorem after suitable renormalization; see Borgeat and Piatnitski [5] Bal, Garnier, Motsch, Perrier [1] for precise statement of these results. In the multidimensional setting, however, those techniques do not apply.

The basis for our proof of Theorem 1.1 is the following general inequality developed recently by Chatterjee [7], based on Stein's method of normal approximation. From now on, we often use Γ for $\Gamma_{L,\beta}$, the dependence on L and β being understood (ϕ also depends on both L and β).

Theorem 1.2 ([7], Theorem 2.2) *Let $h \in C^2(\mathbb{R};\mathbb{R})$. Let $\{Z_k\}_{k\in\mathcal{I}}$ be a collection of independent, standard normal random variables, where $\mathcal{I}$ is a finite index set. For $k \in \mathcal{I}$, let $\zeta_k = h(Z_k)$, and let $\Gamma = \Gamma(\zeta) : \mathbb{R}^{|\mathcal{I}|} \to \mathbb{R}$ be a function of the random vector $\zeta = (\zeta_k)_{k\in\mathcal{I}}$. Define constants*

$$\kappa_0 = \left(\mathbb{E} \sum_{j\in\mathcal{I}} \left| \frac{\partial \Gamma}{\partial \zeta_j} \right|^4 \right)^{1/2},$$

and

$$\kappa_3 = \left(2\int_0^1 \left(\frac{1}{2} + \frac{1}{\sqrt{t}} \right) \mathbb{E}\left[\sum_{i\in\mathcal{I}} \left(\sum_{j\in\mathcal{I}} \frac{\partial^2 \Gamma}{\partial \zeta_i \partial \zeta_j}(\zeta) \frac{\partial \Gamma}{\partial \zeta_j}(\tilde{\zeta}(t)) h'(Z_i) h'(Z_j) h'(\tilde{Z}_j(t)) \right)^2 \right] dt \right)^{1/2}, \tag{1.12}$$

where $\tilde{\zeta}(t) = (\tilde{\zeta}_k(t))_{k\in\mathcal{I}}$ is the random vector defined by

$$\tilde{\zeta}_k(t) = h(\tilde{Z}_k(t)), \qquad \tilde{Z}_k(t) = \sqrt{t} Z_k + \sqrt{1-t} Z_k',$$

and $Z' = (Z'_k)_{k\in\mathcal{I}}$ *is an independent copy of the random vector* $Z = (Z_k)_{k\in\mathcal{I}}$*. If* W *is a normal random variable having the same mean and variance as* Γ*, then*

$$d_{TV}(\Gamma, W) \leq \frac{2\sqrt{5}c_1c_2\kappa_0 + 2\kappa_3}{\sigma^2}, \tag{1.13}$$

where $\sigma^2 = Var(\Gamma)$*,* $c_1 = \|h'\|_\infty$*, and* $c_2 = \|h''\|_\infty$*.*

We have stated this theorem differently from its statement in [7], yet the bound (1.13) follows directly from the anlaysis proving Theorem 2.2 of [7] (see p. 33-34 therein). One way to bound (1.12) uses the operator norm for the Hessian of Γ, as in [7]. With this approach, one obtains from (1.12) the estimate

$$\kappa_3 \leq c_1^3\sqrt{5}(\mathbb{E}\|\nabla_\zeta\Gamma\|^4)^{1/4}(\mathbb{E}\|\nabla_\zeta^2\Gamma\|^4)^{1/4}, \tag{1.14}$$

which implies that

$$d_{TV}(\Gamma, W) \leq \frac{2\sqrt{5}c_1c_2\kappa_0 + 2\sqrt{5}c_1^3\kappa_1\kappa_2}{\sigma^2}, \tag{1.15}$$

where

$$\kappa_1 = (\mathbb{E}\|\nabla_\zeta\Gamma\|^4)^{1/4}, \qquad \kappa_2 = (\mathbb{E}\|\nabla_\zeta^2\Gamma\|^4)^{1/4}.$$

Here $\nabla_\zeta\Gamma$ refers to the gradient with respect to the variables ζ_k for $k \in D_L$, and $\nabla_\zeta^2\Gamma$ is the Hessian, an $L^d \times L^d$ matrix. The norm $\|\nabla_\zeta^2\Gamma\|$ is the L^2 operator norm. The bound (1.15) is precisely the bound stated in Theorem 2.2 of [7]. Instead of using (1.14) and (1.15), however, we will use a different approach to bounding κ_3 that allows us to make better use of the structure of Γ.

For the moment, let us consider (1.15) instead of (1.13). What should we expect of the scaling of each of the terms in (1.15)? Consider a sum of random variables

$$S = \frac{1}{L^d}\sum_{j=1}^{L^d} g(Z_j) \tag{1.16}$$

where Z_j are independent standard normal random variables. Then $\partial_j S = L^{-d}g'(Z_j)$, so that $\kappa_0 = O(L^{-3d/2})$ if g' is bounded. Also, $\|\nabla S\|^4 = L^{-4d}(\sum_j g'(Z_j)^2)^2$, so that $\kappa_1 = O(L^{-d/2})$. For κ_2 notice that $\partial_j\partial_k S = L^{-d}\delta_{jk}g''(Z_j)$, so that $\kappa_2 = O(L^{-d})$ if g'' is bounded. Thus $\kappa_3 = O(L^{-3d/2})$. Finally, the variance is $\sigma^2 = O(L^{-d})$, so that the bound (1.15) is $O(L^{-d/2})$ for this simple sum of independent random variables. In general, if the dependence relations are sufficiently local, in the sense that $L^d\partial_j\partial_k S$ is typically small for $|j-k| >> 1$, we could still have $\kappa_3 = O(L^{-3d/2})$ and $d_{TV}(S, W) = O(L^{-d/2})$. Obviously $\Gamma_{L,\beta}$ can be written as the normalized sum

$$\Gamma_{L,\beta} = \frac{1}{L^d}\sum_{j\in D_L\cap\mathbb{Z}^d} \eta_j, \tag{1.17}$$

where the random variables η_j are

$$\eta_j = \int_{Q_j} a(x)|\nabla\phi(x) + e_1|^2 + \beta(\phi(x))^2\,dx.$$

Although the variables η_j in (1.17) are identically distributed, each η_j depends on the $O(L^d)$ variables ζ_k in a nonlinear way through solution of the PDE (1.1). Consequently, the terms in the sum (1.17) are mutually dependent, which makes the analysis of $\Gamma_{L,\beta}$ challenging.

Starting from (1.13), a proof of Theorem 1.1 will follow from a suitable upper bound on κ_0 and κ_3 as well as a lower bound on $\sigma^2 = \mathrm{Var}(\Gamma_{L,\beta})$, which appears in the denominator of (1.13). We will show that $\kappa_0 = O(L^{-3d/2})$ and $\kappa_3 = O(L^{-3d/2})$. We will also prove the following lower bound on σ^2, which is similar to a result in [24]:

Theorem 1.3 *Let $d \geq 1$. There is a constant $C > 0$ such that*

$$Var(\Gamma_{L,\beta}) \geq CL^{-d}\mathbb{E}[\Phi_0]^2 \geq CL^{-d}$$

holds for all $L \geq 1$ and $\beta \geq 0$.

Although the proof of Theorem 1.1 does not require an upper bound on the variance of $\Gamma_{L,\beta}$, the variance of $\Gamma_{L,\beta}$ can also be estimated from above in terms of Φ_0:

Proposition 1.4 *Let $d \geq 1$. There is a constant $C \geq 0$ such that*

$$Var(\Gamma_{L,\beta}) \leq CL^{-d}\mathbb{E}[\Phi_0^2]$$

holds for all $L \geq 1$ and $\beta \geq 0$.

As we have mentioned, if $d \geq 3$, or if $d = 2$ and $\beta > 0$ is fixed, then $\mathbb{E}[\Phi_0^2]$ is bounded as $L \to \infty$, and Proposition 1.4 implies that $\mathrm{Var}(\Gamma_{L,\beta}) = O(L^{-d})$.

Let us point out that the proof of Theorem 1.3 makes use of the structural assumptions on the coefficient $a(x)$. Specifically, the lower bound in (1.3) enables us to estimate $\frac{\partial \Gamma}{\partial \zeta_k}$ from below, a key step in proof of Theorem 1.3. This structural condition is not used anywhere else in the analysis. If (1.3) were replaced with

$$\left|\frac{\partial a}{\partial \zeta_k}(x)\right| \leq C_2 \mathbb{I}_{B_\tau(k)}(x), \quad a.e.\ x \in D_L, \quad \forall\, k \in \mathbb{Z}^d \cap D_L,$$

then (1.9) may be replaced with

$$d_{TV}(\Gamma_{L,\beta}, W_{L,\beta}) \leq CL^{-3d/2} \frac{\mathbb{E}[\Phi_0^q]^{2/q}}{\sigma^2}. \tag{1.18}$$

The rest of the paper is organized as follows: In Section 2 we prove Theorem 1.3, the lower bound on σ^2. Upper bounds on the constants κ_0 and κ_3 and the rest of the proof of Theorem 1.1 are developed in Sections 3 and 4. Section 3 contains some deterministic PDE estimates (Caccioppoli's inequality and a version of Meyers' estimate) which are useful in bounding κ_3 and do not rely on the statistical structure of the coefficients. Section 4 contains the main argument bounding κ_0 and κ_3. Finally, in Section 5, we prove Proposition 1.4 and Corollary 5.5, which is an estimate of $\mathbb{E}[\Phi_0^q]$ in the case that $\beta > 0$ is fixed. We also make some remarks about estimating $\mathbb{E}[\Phi_0^q]$ using the method of [13] to deal with the case that β vanishes as $L \to \infty$.

A few more comments about notation: throughout the article we will use the convention that summation over indices $j \in D_L$ means a summation over $j \in \mathbb{Z}^d \cap D_L$, with $j \in \mathbb{Z}^d$ being understood. For a given measureable set $A \subset \mathbb{R}^d$, we define the normalized integral

$$\fint_A u(x)\, dx = \frac{1}{|A|} \int_A u(x)\, dx.$$

We also use C to denote deterministic constants that may change from line to line, but do not depend on L or β. We will use Φ_j and Φ_j' to refer to the integrals

$$\Phi_j = \int_{Q_j} |\nabla \phi(x) + e_1|^2\, dx, \qquad \Phi_j' = \int_{B_\tau(j)} |\nabla \phi(x) + e_1|^2\, dx \tag{1.19}$$

which appear frequently in the analysis. Recall that $B_\tau(j) \supset Q_j$, so $\Phi'_j \geq \Phi_j$.

After this paper was submitted for publication, we learned of two other related works on discrete resistor network models. By making use of the martingale central limit theorem, Biskup, Salvi, and Wolff [2] have proved a central limit theorem for a discrete quantity similar to $\tilde{\Gamma}_{L,\beta}$ when ϕ satisfies linear Dirichet boundary conditions on a square box, in the regime of small ellipticity contrast (i.e. $|\frac{a^*}{a_*} - 1|$ is sufficiently small). Using different techniques, including generalized Walsh decomposition and concentration bounds, Rossignol [21] has proved a variance bound and a central limit theorem for effective resistence of a resistor network on the discrete torus. We refer to the recent review paper [3] for many other references on the random conductance model.

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2 A lower bound on the variance σ^2

In this section we prove Theorem 1.3, the lower bound on $\sigma^2 = \mathrm{Var}(\Gamma_{L,\beta})$ which appears in the denominator of (1.13). One approach to proving the lower bound is to use the argument of Wehr [24] who considered a functional similar to Γ for a discrete resistor network with random conductance (without a uniform ellipticity constraint). If we assume (1.5) with $M_k = Q_k$ and that the law of ζ_k is absolutely continuous with respect to Lebesgue measure on $[\zeta_{min}, \zeta_{max}]$, that argument can be adapted to the present setting, under the constraint

$$\int_{\zeta_{min}}^{\zeta_{max}} \frac{(\nu(s) + s\nu'(s))^2}{\nu(s)} \, ds < \infty, \tag{2.20}$$

where ν is the density for the law of ζ_k. Here we give a proof that allows for the more general structural condition (1.3) and allows for the law of ζ_k to be singular with respect to Lebesgue measure.

First, since the variables $\{\zeta_k\}_{k \in \mathbb{Z}^d}$ are independent, we have the lower bound

$$\mathrm{Var}(\Gamma) \geq \sum_{k \in D_L} \mathrm{Var}\left(\mathbb{E}[\, \Gamma \,|\, \zeta_k \,] \right), \tag{2.21}$$

where $\mathbb{E}[\, \Gamma \,|\, \zeta_k \,]$ is the conditional expectation of Γ, conditioned on the value of ζ_k. This inequality is proved in [25] (see Proposition 3.1, therein). Since the ζ_k are identically distributed, we have $\mathrm{Var}\left(\mathbb{E}[\, \Gamma \,|\, \zeta_k \,] \right) = \mathrm{Var}\left(\mathbb{E}[\, \Gamma \,|\, \zeta_j \,] \right)$ for all $j, k \in D_L$, so that

$$\mathrm{Var}(\Gamma) \geq L^d \mathrm{Var}\left(\mathbb{E}[\, \Gamma \,|\, \zeta_0 \,] \right).$$

Next, observe that

$$\mathrm{Var}\left(\mathbb{E}[\, \Gamma \,|\, \zeta_0 \,] \right) = \int_{\mathbb{R}} g(s)^2 \nu(ds) = \frac{1}{2} \int_{\mathbb{R}^2} (g(s) - g(s'))^2 \nu(ds)\nu(ds') \tag{2.22}$$

where

$$g(s) = \mathbb{E}[\, \Gamma \,|\, \zeta_0 = s \,] - \mathbb{E}[\Gamma] \tag{2.23}$$

and $\nu(ds)$ is the probability measure supported on $[\zeta_{min}, \zeta_{max}]$ which is the law of the random variable ζ_0.

Since $\zeta \mapsto a(x, \zeta)$ is nondecreasing with respect to each coordinate ζ_k, it follows from (1.8) that Γ is a nondecreasing function of each ζ_k, so we have $g'(s) \geq 0$. We will establish the following lower bound on the difference $g(s) - g(s')$:

Lemma 2.1 *Define $\rho(s) = \mathbb{E}\left[\Phi_0 \mid \zeta_0 = s\right] \geq 0$. There is a constant $\theta > 0$ such that*

$$|g(s') - g(s)| \geq L^{-d}\theta|s - s'| \, \max\left(\rho(s), \rho(s')\right)$$

holds for all $s', s \in [\zeta_{min}, \zeta_{max}]$, for all $L > 1$, $\beta \geq 0$.

Therefore, from (2.22) we have

$$\begin{aligned} \mathrm{Var}\left(\mathbb{E}[\,\Gamma\,|\,\zeta_0\,]\right) &= \frac{1}{2}\int_{\mathbb{R}^2}(g(s) - g(s'))^2\nu(ds)\nu(ds') \\ &\geq \frac{1}{2}\theta^2 L^{-2d}\int_{\mathbb{R}^2}(s - s')^2\rho^2(s)\nu(ds)\nu(ds') \\ &= \frac{1}{2}\theta^2 L^{-2d}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}(s - s')^2\nu(ds')\right)\rho^2(s)\nu(ds). \end{aligned} \tag{2.24}$$

Since $\int_{\mathbb{R}}(s - s')^2\nu(ds') = \mathbb{E}[(s - \zeta_0)^2] \geq \mathrm{Var}(\zeta_0)$, this implies

$$\begin{aligned} \mathrm{Var}\left(\mathbb{E}[\,\Gamma\,|\,\zeta_0\,]\right) &\geq \frac{1}{2}\theta^2 L^{-2d}\mathrm{Var}(\zeta_0)\int_{\mathbb{R}}\rho^2(s)\nu(ds) \\ &\geq \frac{1}{2}\theta^2 L^{-2d}\mathrm{Var}(\zeta_0)\left(\int_{\mathbb{R}}\rho(s)\nu(ds)\right)^2 \\ &= \frac{1}{2}\theta^2 L^{-2d}\mathrm{Var}(\zeta_0)\mathbb{E}[\Phi_0]^2. \end{aligned} \tag{2.25}$$

Hence, (2.21) implies that

$$\mathrm{Var}(\Gamma) \geq \frac{1}{2}L^{-d}\theta^2\mathrm{Var}(\zeta_0)\mathbb{E}[\Phi_0]^2.$$

Let us observe that $\mathbb{E}[\Phi_0] \geq 1$. Indeed, by stationarity of ϕ we have

$$\begin{aligned} \mathbb{E}[\Phi_0] &= \mathbb{E}[\int_{Q_0}|\nabla\phi + e_1|^2\,dx] \\ &= \mathbb{E}\left[\fint_{D_L}|\nabla\phi + e_1|^2\,dx\right] \\ &= \mathbb{E}\left[\fint_{D_L}(|\nabla\phi|^2 + 2\nabla\phi\cdot e_1 + 1)\,dx\right]. \end{aligned} \tag{2.26}$$

Since ϕ is periodic, $\int_{D_L}\nabla\phi\cdot e_1\,dx = 0$. Hence,

$$\mathbb{E}[\Phi_0] \geq 1 + \mathbb{E}[\fint_{D_L}|\nabla\phi|^2\,dx] \geq 1. \tag{2.27}$$

Except for a proof of Lemma 2.1, this establishes Theorem 1.3. To prove Lemma 2.1, we will make use of the following lemma:

Lemma 2.2 *Let $\beta \geq 0$. Suppose that $a(x)$ and $a'(x)$ are two measurable functions satisfying (1.2). Let $\phi, \phi' \in H^1_{per}(D_L)$ satisfy*

$$-\nabla \cdot (a(\nabla \phi + e_1)) + \beta \phi = 0, \qquad \text{and} \qquad - \nabla \cdot (a'(\nabla \phi' + e_1)) + \beta \phi' = 0,$$

respectively. If $(a - a')$ is supported on a measureable set $M \subset D_L$, then

$$\int_{D_L} |\nabla \phi(x) - \nabla \phi'(x)|^2 \, dx \leq \left(\frac{a^* - a_*}{a_*} \right)^2 \int_M |\nabla \phi(x) + e_1|^2 \, dx, \tag{2.28}$$

and

$$\int_M |\nabla \phi' + e_1|^2 \, dx \leq \left(1 + \frac{a^* - a_*}{a^*} \right)^2 \int_M |\nabla \phi + e_1|^2 \, dx. \tag{2.29}$$

In particular, if the vectors ζ and ζ' differ only in the j^{th} coordinate (i.e. $\zeta_k = \zeta'_k$ if $k \neq j$). Then $a(\cdot) = a(\cdot, \zeta)$ and $a'(\cdot) = a(\cdot, \zeta')$ differ only on the set $B_\tau(j)$ (by 1.3), so (2.28) and (2.29) hold with $M = B_\tau(j)$.

Proof of Lemma 2.2: The function $v(x) = \phi - \phi' \in H^1_{per}(D_L)$ is a weak solution to

$$-\nabla \cdot (a' \nabla v) + \beta v = -\nabla \cdot \left((a' - a)(\nabla \phi + e_1) \right).$$

Multiply by v and integrate by parts. The uniform ellipticity implies:

$$a_* \int_{D_L} |\nabla v(x)|^2 \, dx \leq \int_{D_L} \nabla v \cdot \left((a' - a)(\nabla \phi + e_1) \right) \, dx.$$

Since $a' - a = 0$ outside the set M, the Cauchy-Schwarz inequality leads to

$$a_* \int_{D_L} |\nabla v(x)|^2 \, dx \leq (a^* - a_*) \left(\int_{D_L} |\nabla v|^2 \, dx \right)^{1/2} \left(\int_M |\nabla \phi(x) + e_1|^2 \, dx \right)^{1/2},$$

which is (2.28). The bound (2.29) now follows by the triangle inequality in $(L^2(M))^d$. $\square$

Proof of Lemma 2.1: Recall that $g(s') - g(s) = \mathbb{E}[\Gamma \mid \zeta_0 = s'] - \mathbb{E}[\Gamma \mid \zeta_0 = s]$. Suppose the vectors ζ and ζ' differ only in the j^{th} coordinate (i.e. $\zeta_k = \zeta'_k$ if $k \neq j$) and that $s' = \zeta'_j > \zeta_j = s$. Define $a(\cdot) = a(\cdot, \zeta)$ and $a'(\cdot) = a(\cdot, \zeta')$. The difference $a' - a$ is supported in $B_\tau(j)$, but its support may not be confined to $\bar{Q}_j$. For this reason, we also define a function a'' according to

$$a''(x) = \begin{cases} a'(x) & x \in \bar{Q}_j \\ a(x), & x \notin \bar{Q}_j. \end{cases}$$

Since $a'(x) \geq a''(x) \geq a(x)$ almost everywhere, we must have $\Gamma(a') \geq \Gamma(a'') \geq \Gamma(a)$.

Let $\phi, \phi'' \in H^1_{per}(D_L)$ satisfy

$$-\nabla \cdot (a(\nabla \phi + e_1)) + \beta \phi = 0, \qquad \text{and} \qquad - \nabla \cdot (a''(\nabla \phi'' + e_1)) + \beta \phi'' = 0,$$

respectively. From the variational representation (1.8), we know that

$$\begin{aligned} L^d \Gamma(a) &= \min_{v \in H^1_{per}(D_L)} \int_{D_L} a |\nabla v + e_1|^2 + \beta v^2 \, dx \\ &\leq \int_{D_L} a |\nabla \phi'' + e_1|^2 \, dx + \beta (\phi'')^2 \, dx \\ &= \int_{D_L} a'' |\nabla \phi'' + e_1|^2 \, dx + \beta (\phi'')^2 \, dx + \int_{D_L} (a - a'') |\nabla \phi'' + e_1|^2 \, dx \\ &= L^d \Gamma(a'') + \int_{D_L} (a - a'') |\nabla \phi'' + e_1|^2 \, dx. \end{aligned} \tag{2.30}$$

Therefore,

$$\Gamma(a'') - \Gamma(a) \geq L^{-d} \int_{D_L} (a'' - a)|\nabla\phi'' + e_1|^2 \, dx. \tag{2.31}$$

Since $(a'' - a) \geq 0$ is supported on $\overline{Q_j}$, then by (2.31) and Lemma 2.2,

$$\begin{aligned} \Gamma(a'') - \Gamma(a) &\geq L^{-d} \left(\inf_{x \in Q_j} |a'' - a| \right) \frac{1}{2} \int_{Q_j} |\nabla\phi'' + e_1|^2 \, dx \\ &\geq L^{-d} \left(1 + \frac{a^* - a_*}{a_*} \right)^{-2} \left(\inf_{x \in Q_j} |a'' - a| \right) \frac{1}{2} \int_{Q_j} |\nabla\phi + e_1|^2 \, dx. \end{aligned} \tag{2.32}$$

Hence by using (1.3) and (2.32) we obtain

$$\begin{aligned} \Gamma(a') - \Gamma(a) \geq \Gamma(a'') - \Gamma(a) &\geq L^{-d} C \left(\inf_{x \in Q_j} |a'' - a| \right) \frac{1}{2} \int_{Q_j} |\nabla\phi + e_1|^2 \, dx \\ &= L^{-d} C \left(\inf_{x \in Q_j} |a' - a| \right) \frac{1}{2} \int_{Q_j} |\nabla\phi + e_1|^2 \, dx. \end{aligned} \tag{2.33}$$

By the lower bound in (1.3), this implies

$$\Gamma(a') - \Gamma(a) \geq L^{-d} C C_1 |\zeta_j' - \zeta_j| \int_{Q_j} |\nabla\phi + e_1|^2 \, dx. \tag{2.34}$$

On the other hand, arguing as at (2.31) and using (1.3) we also have

$$\Gamma(a') - \Gamma(a) \geq L^{-d} \int_{D_L} (a' - a)|\nabla\phi' + e_1|^2 \, dx \geq L^{-d} C_1 |\zeta_j' - \zeta_j| \int_{Q_j} |\nabla\phi' + e_1|^2 \, dx. \tag{2.35}$$

Lemma 2.2 now follows from (2.34) and (2.35) and the definition of $g(s)$. □

3 Deterministic estimates for solutions of the elliptic equation

Our next goal is to prove Theorem 1.1 by using Theorem 1.2. In this section, however, we first establish some regularity estimates that apply to solutions of elliptic equations. These estimates will be used in the process of bounding the constants κ_0 and κ_3 which appear in Theorem 1.2. These estimates rely only on the uniform ellipticity assumption, not on the statistical structure of the coefficient $a(x)$.

3.1 Caccioppoli's inequality

Recall that the Poincaré inequality tells us that for sufficiently regular sets D there is a constant $C_D > 0$ such that

$$\int_D (u(x) - \rho_D)^2 \, dx \leq C_D \int_D |\nabla u|^2 \, dx \tag{3.36}$$

holds for all $u \in H^1(D)$, where

$$\rho_D = \fint_D u(x) \, dx = \frac{1}{|D|} \int_D u(x) \, dx.$$

For solutions of elliptic equations Caccioppoli's inequality gives the reverse inequality, enabling us to control moments of $\nabla\phi$ by moments of ϕ itself. Here and at other points in the paper it will be convenient to use the notation $3Q_j$ and $5Q_j$ to refer to the cubes

$$3Q_j = j + [-1,2)^d \qquad \text{and} \qquad 5Q_j = j + [-2,3)^d,$$

which are concentric cubes of width 3 and 5, respectively, and containing $Q_j = j + [0,1)^d$ in their center. We also define the random variables

$$\rho_{3,j} = \fint_{3Q_j} \phi(x)\,dx \qquad \text{and} \qquad \rho_{5,j} = \fint_{5Q_j} \phi(x)\,dx. \tag{3.37}$$

Here is Caccioppoli's inequality, presented in two different forms for convenient reference later:

Lemma 3.1 *Let $d \geq 1$ and let $u \in H^1(3Q_j)$ be a weak solution to $-\nabla\cdot(a\nabla u) + \beta u = \nabla\cdot\xi$ for $x \in 3Q_j$, with $\xi \in (L^2(3Q_j))^d$. There is a constant K, depending only on a^* and a_* such that*

$$\int_{Q_j} |\nabla u|^2\,dx \quad \leq \quad K\left(\int_{3Q_j} |\xi|^2\,dx + \int_{3Q_j} (u(x)-b)^2\,dx + \beta b^2\right)$$

holds for any constant $b \in \mathbb{R}$. Similarly, there is a constant K such that if $R > 0$ and $u \in H^1(B_R(x_0))$ is a weak solution to $-\nabla\cdot(a\nabla u) + \beta u = \nabla\cdot\xi$ for $x \in B_R(x_0)$, with $\xi \in (L^2(B_R))^d$, then

$$\int_{B_{\frac{R}{2}}(x_0)} |\nabla u|^2\,dx \quad \leq \quad K\left(\int_{B_R(x_0)} |\xi|^2\,dx + \frac{1}{R^2}\int_{B_R(x_0)} (u(x)-b)^2\,dx + \beta b^2 R^d\right)$$

holds for any constant $b \in \mathbb{R}$.

Lemma 3.1 and variants are a consequence of the following:

Lemma 3.2 *Let $K_1 = 2/a_*$, $K_2 = (2/a_*) + 8(a^*/a_*)^2$, and $K_3 = (2/a_*) + 2/(a_*)^2$. Let Q be a bounded open subset of $\mathbb{R}^d$ with smooth boundary. If $\beta \geq 0$ and $u \in H^1(Q)$ is a weak solution to $-\nabla\cdot(a\nabla u) + \beta u = f + \nabla\cdot\xi$ for $x \in Q$, with $f \in L^2(Q)$ and $\xi \in (L^2(Q))^d$, then*

$$\begin{aligned}\int_Q \varphi^2 |\nabla u|^2\,dx \quad &\leq \quad K_1 \int_Q f(u-b)\varphi^2\,dx - K_1\beta\int_Q u(u-b)\varphi^2\,dx \\ &\quad + K_2\int_Q |\nabla\varphi|^2 (u-b)^2\,dx + K_3\int_Q |\xi|^2\varphi^2\,dx\end{aligned} \tag{3.38}$$

holds for any smooth function $\varphi \geq 0$ which vanishes on the boundary of Q, and any constant $b \in \mathbb{R}$.

Proof of Lemma 3.2: A proof of this sort can be found in various texts, for example Chapter III of [9]. Suppose that $u \in H^1(Q)$ solves $-\nabla\cdot(a\nabla u) + \beta u = \nabla\cdot\xi$ in the weak sense:

$$\int_Q a\nabla u\nabla v\,dx + \int_Q \beta uv\,dx = \int_Q fv\,dx - \int_Q \xi\cdot\nabla v\,dx \quad \forall v \in H_0^1(Q). \tag{3.39}$$

By choosing a test function $v = (u-b)\varphi^2 \in H_0^1(Q)$, using the uniform ellipticity, and the fact that $\varphi \geq 0$, we obtain from (3.39) the bound:

$$\begin{aligned}a_*\int_Q \varphi^2|\nabla u|^2\,dx \quad &\leq \quad -\int_Q 2\varphi(u-b)a\nabla u\nabla\varphi\,dx - \int_Q \beta u(u-b)\varphi^2\,dx + \int_Q f(u-b)\varphi^2\,dx \\ &\quad - \int_Q \xi\cdot(\nabla u)\varphi^2\,dx - 2\int_Q \xi\cdot(\nabla\varphi)\varphi(u-b)\,dx.\end{aligned} \tag{3.40}$$

By the Cauchy-Schwarz inequality, for any $\epsilon > 0$,

$$\int_Q 2\varphi(u-b)a\nabla u\nabla\varphi\,dx \le \epsilon\int_Q \varphi^2|\nabla u|^2\,dx + \frac{(a^*)^2}{\epsilon}\int_Q |\nabla\varphi|^2(u-b)^2\,dx$$

and

$$\int_Q \xi\cdot(\nabla u)\varphi^2\,dx \le \epsilon\int_Q \varphi^2|\nabla u|^2\,dx + \frac{\epsilon^{-1}}{4}\int_Q \varphi^2|\xi|^2\,dx \tag{3.41}$$

and

$$2\int_Q \xi\cdot(\nabla\varphi)\varphi(u-b)\,dx \le \int_Q |\xi|^2\varphi^2 + \int_Q (u-b)^2|\nabla\varphi|^2\,dx. \tag{3.42}$$

Now by choosing $\epsilon = a_*/4$, we infer from (3.40) the bound

$$\begin{aligned}\frac{a_*}{2}\int_Q \varphi^2|\nabla u|^2\,dx \;\;&\le\;\; -\int_Q \beta u(u-b)\varphi^2\,dx + \int_Q f(u-b)\varphi^2\,dx \\ &\quad + \left(1+\frac{4(a^*)^2}{a_*}\right)\int_Q |\nabla\varphi|^2(u-b)^2\,dx + \left(a_*^{-1}+1\right)\int_Q |\xi|^2\varphi^2\,dx. \end{aligned}\tag{3.43}$$

This completes the proof. □

Remark 3.3 *The conclusion of Lemma 3.2 also holds if we assume that $u \in H^1_{per}(D_L)$ and $\varphi \ge 0$ is periodic. In that case, $\varphi(x)$ need not vanish at any point D_L; the integration-by-parts is made possible by the periodicity.*

Proof of Lemma 3.1. Let $u(x)$ and b be as in Lemma 3.1. We apply Lemma 3.2 with $Q = 3Q_j$. We may choose $0 \le \varphi(x) \le 1$ to be a smooth function with support in $Q = 3Q_j \subset D_L$ and satisfying $\varphi \equiv 1$ in Q_j and $|\nabla\varphi| \le C$. In this case, observe that since $\beta \ge 0$ and $0 \le \varphi \le 1$,

$$-\beta\int_{3Q_j} u(u-b)\varphi^2\,dx \le \beta b^2|3Q_j|.$$

This proves the first part of Lemma 3.1. The second part also follows by a similar argument with $Q = B_R(x_0)$. In that case, we choose the test function $\varphi(x)$ to be a smooth function with support in $Q = B_R(x_0) \subset D_L$ and satisfying $\varphi \equiv 1$ in $B_R/2$ with $|\nabla\varphi| \le C/R$. □

3.2 Higher regularity – Meyers' estimate

If $u \in H^1_{per}(D_L)$ satisfies $-\nabla\cdot(a\nabla u) + \beta u = \nabla\cdot v$ with $v \in (L^2(D_L))^d$ and $\beta \ge 0$, then it is easy to see that

$$\int_{D_L} |\nabla u|^2\,dx \le \frac{1}{(a_*)^2}\int_{D_L} |v|^2\,dx$$

must hold. If $v \in (L^p(D_L))^d$ for some $p > 2$, a well-known result of Meyers [17] implies higher integrability of ∇u, as described by the following lemma.

Lemma 3.4 *For all $s > 2$, there is a constant $p^* > 2$ and $C > 0$ such that the following holds: If $L > 1$, $\beta \ge 0$, $v \in (L^s(D_L))^d$, and if $u \in H^1_{per}(D_L)$ satisfies $-\nabla\cdot(a\nabla u) + \beta u = \nabla\cdot v$, then*

$$⨍_{D_L} |\nabla u|^p\,dx < C⨍_{D_L} |v|^p\,dx \tag{3.44}$$

for all $p \in [2, p^*]$.

Proof of Lemma 3.4: For $\beta = 0$, (3.44) can be derived directly from Theorem 2 of Meyers [17] by using the periodicity of u. However, for $\beta > 0$ it is more convenient to give a proof based on a result of Giaquinta and Modica [10]. By the Caccioppoli inequality (Lemma 3.1) applied to $u(x)$ we have

$$\int_{K_R(x_0)} |\nabla_x u(x)|^2 \, dx \quad \leq \quad C\beta \bar{u}^2 R^d + \frac{C}{R^2} \int_{K_{2R}(x_0)} |u(x) - \bar{u}|^2 \, dx + C \int_{K_{2R}(x_0)} |v|^2 \, dx \quad (3.45)$$

where

$$\bar{u} = \frac{1}{|K_{2R}|} \int_{K_{2R}(x_0)} u(x) \, dx$$

and $K_R(x_0)$ is a cube of width $2R$ centered at a point x_0. By the Poincaré-Sobolev inequality, this implies

$$\begin{aligned} \fint_{K_R(x_0)} |\nabla u|^2 \, dx \quad &\leq \quad C\beta \fint_{K_{2R}(x_0)} u^2 \, dx + C \left(\fint_{K_{2R}(x_0)} |\nabla u|^{2d/(d+2)} \, dx \right)^{(d+2)/d} \\ &\quad + C \fint_{K_{2R}(x_0)} |v|^2 \, dx. \end{aligned} \quad (3.46)$$

Now we apply Proposition 5.1 of [10] (see also Theorem V.1.2 of [9]) with $g = |\nabla u|^{2d/(d+2)}$, $q = (d+2)/d$, and $f = (C\beta u^2 + C|v|^2)^{1/q}$. Since

$$\fint_{K_R(x_0)} g^q \, dx \leq C \left(\fint_{K_{2R}(x_0)} g \, dx \right)^q + \fint_{K_{2R}(x_0)} f^q \, dx$$

holds for all $x_0 \in D_L$ and $R > 0$, that proposition implies that for some $\epsilon > 0$ and $p = q(1+\epsilon)$, we must have

$$\fint_{K_R(x_0)} g^p \, dx \leq C \left(\fint_{K_{2R}(x_0)} g^q \, dx \right)^{1+\epsilon} + C \fint_{K_{2R}(x_0)} f^p \, dx$$

for any $R > 0$ and $x_0 \in D_L$. This means that

$$\begin{aligned} \fint_{K_{R/2}(x_0)} |\nabla u(x)|^r \, dx \quad &\leq \quad C \left(\fint_{K_R(x_0)} |\nabla u(x)|^2 \, dx \right)^{r/2} \\ &\quad + C\beta^{r/2} \fint_{K_R} |u|^r \, dx + C \fint_{K_R} |v|^r \, dx, \end{aligned} \quad (3.47)$$

where $r = 2(1+\epsilon) > 2$. In particular, we may choose $R = L$ so that $K_{R/2}(x_0) = D_L$. Since u and v are periodic over D_L, we conclude that

$$\begin{aligned} \left(\fint_{D_L} |\nabla u(x)|^r \, dx \right)^{1/r} \quad &\leq \quad C \left(\fint_{D_L} |\nabla u(x)|^2 \, dx \right)^{1/2} + C \left(\fint_{D_L} |v|^r \, dx \right)^{1/r} \\ &\quad + C\beta^{1/2} \left(\fint_{D_L} |u|^r \, dx \right)^{1/r}. \end{aligned} \quad (3.48)$$

If $\beta > 0$, the last term in (3.48) may be absorbed into the others, as follows. Since $r > 2$, we may use the test function $\eta = |u|^{r-1} \text{sign}(u) \in H^1_{per}(D_L)$ in the equality

$$\int_{D_L} a \nabla u \nabla \eta \, dx + \beta \int_{D_L} u \eta \, dx = - \int_{D_L} v \cdot \nabla \eta \, dx,$$

to obtain:

$$\begin{aligned} a_*(r-1)\int_{D_L}|\nabla u|^2|u|^{r-2}\,dx+\beta\int_{D_L}|u|^r\,dx &\le (r-1)\int_{D_L}|v||\nabla u||u|^{r-2}\,dx, \\ &\le \frac{(r-1)\epsilon^{-1}}{2}\int_{D_L}|v|^2|u|^{r-2}\,dx \\ &\quad +\frac{(r-1)\epsilon}{2}\int_{D_L}|\nabla u|^2|u|^{r-2}\,dx \end{aligned}$$

for any $\epsilon > 0$. Therefore, with $\epsilon = 2a_*$, we obtain

$$\beta\int_{D_L}|u|^r\,dx \le \frac{(r-1)}{4a_*}\int_{D_L}|v|^2|u|^{r-2}\,dx.$$

By Hölder's inequality, this implies

$$\beta^{r/2}\int_{D_L}|u|^r\,dx \le \left(\frac{(r-1)}{4a_*}\right)^{r/2}\int_{D_L}|v|^r\,dx.$$

Now we substitute this bound for the last term in (3.48), and we conclude that

$$\left(\fint_{D_L}|\nabla u(x)|^r\,dx\right)^{1/r} \le C\left(\fint_{D_L}|\nabla u(x)|^2\,dx\right)^{1/2}+C\left(\fint_{D_L}|v|^r\,dx\right)^{1/r} \tag{3.49}$$

holds for all $L > 1$.

Since u satisfies $-\nabla\cdot(a\nabla u)+\beta u = \nabla\cdot v$, we have

$$\int_{D_L}|\nabla u|^2\,dx \le \frac{1}{(a_*)^2}\int_{D_L}|v|^2\,dx.$$

Therefore, since $r > 2$, Jensen's inequality implies

$$\left(\fint_{D_L}|\nabla u(x)|^2\,dx\right)^{1/2} \le \frac{1}{a_*}\left(\fint_{D_L}|v(x)|^r\,dx\right)^{1/r}.$$

Combining this with (3.49) we conclude the proof. □

4 The Proof of Theorem 1.1

In this section we prove that the constants κ_0 and κ_3 appearing in Theorem 1.2 are bounded according to

$$\kappa_0 = L^{-3d/2}\mathbb{E}[\Phi_0^4]^{1/2} \quad \text{and} \quad \kappa_3 \le CL^{-3d/2}\mathbb{E}[\Phi_0^q]^{3/(2q)}+CL^{-3d/2}\mathbb{E}[\Phi_0^4]^{1/2} \tag{4.50}$$

for all $L > 1$ and $\beta \ge 0$, if $q > 4$ is sufficiently large. By combining this with Theorem 1.3 and Theorem 1.2, we obtain Theorem 1.1.

To obtain these bounds, we will need to compute derivatives of Γ and ϕ with respect to the variables ζ_k. First, we establish the differentiability of $\phi(x,\zeta)$ with respect to ζ_k.

Lemma 4.1 *For $\beta \ge 0$, the function*

$$w_k(x) = \frac{\partial}{\partial\zeta_k}\phi(x,\zeta)$$

is in $H^1_{per}(D_L)$ and it is a weak solution of the equation

$$-\nabla \cdot (a(x)\nabla w_k) + \beta w_k = \nabla \cdot \xi_k, \quad x \in D_L \tag{4.51}$$

where $\xi_k \in (L^2_{per}(D_L))^d$ is the vector field

$$\xi_k(x) = \frac{\partial a}{\partial \zeta_k}(x,\zeta)(\nabla \phi(x,\zeta) + e_1).$$

Proof of Lemma 4.1: For $\epsilon > 0$ small, let $\zeta'_j = \zeta_j$ for all $j \neq k$ and let $\zeta'_k = \zeta_k - \epsilon$. Let

$$v^\epsilon(x) = \epsilon^{-1}(\phi - \phi')$$

where $\phi' = \phi(x,\zeta')$. The function $\phi - \phi' \in H^1_{per}(D_L)$ satisfies

$$-\nabla \cdot (a\nabla(\phi - \phi')) + \beta(\phi - \phi') = \nabla \cdot \left((a(x,\zeta) - a(x,\zeta'))(\nabla \phi' + e_1)\right). \tag{4.52}$$

Using $\phi - \phi'$ as a test function against (4.52), we integrate by parts and apply the Cauchy-Schwarz inequality to obtain

$$\int_{D_L} |\nabla(\phi - \phi')|^2 \, dx \le \frac{1}{(a_*)^2} \|a(x,\zeta) - a(x,\zeta')\|^2_\infty \int_{D_L} |\nabla \phi' + e_1|^2 \, dx. \tag{4.53}$$

As $\epsilon \to 0$, $\|a(x,\zeta) - a(x,\zeta')\|^2_\infty \to 0$. So, combining (4.53) and the fact that $\int_{D_L} \phi \, dx = \int_{D_L} \phi' \, dx = 0$, we conclude that $\phi' \to \phi$ strongly in $H^1_{per}(D_L)$ as $\epsilon \to 0$.

Let w_k be the unique weak solution of (4.51) satisfying $\int_{D_L} w_k \, dx = 0$. The function $v^\epsilon \in H^1_{per}(D_L)$ satisfies

$$-\nabla \cdot (a\nabla v^\epsilon) + \beta v^\epsilon = \nabla \cdot \left(\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon}(\nabla \phi' + e_1)\right). \tag{4.54}$$

Therefore, $v^\epsilon - w_k$ satisfies

$$\begin{aligned} -\nabla \cdot (a\nabla(v^\epsilon - w_k)) + \beta(v^\epsilon - w_k) &= \nabla \cdot \left(\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon}(\nabla \phi' + e_1) - \xi_k\right) \\ &= \nabla \cdot \left(\left(\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon} - \frac{\partial a}{\partial \zeta_k}(x,\zeta)\right)(\nabla \phi + e_1)\right) \\ &\quad + \nabla \cdot \left(\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon}(\nabla \phi' - \nabla \phi)\right). \end{aligned} \tag{4.55}$$

Using $v^\epsilon - w_k$ as a test function against (4.55), we obtain

$$\begin{aligned} \int_{D_L} |\nabla v^\epsilon|^2 \, dx \quad &\le \quad \frac{2}{(a_*)^2} \left\| \frac{a(x,\zeta) - a(x,\zeta')}{\epsilon} - \frac{\partial a}{\partial \zeta_k}(x,\zeta) \right\|^2_\infty \int_{D_L} |\nabla \phi + e_1|^2 \, dx. \\ &\quad + \frac{2}{(a_*)^2} \left\| \frac{a(x,\zeta) - a(x,\zeta')}{\epsilon} \right\|^2_\infty \int_{D_L} |\nabla \phi' - \nabla \phi|^2 \, dx. \end{aligned} \tag{4.56}$$

Since $\zeta \mapsto a(\cdot,\zeta)$ is Fréchet differentiable, we know that $\|\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon} - \frac{\partial a}{\partial \zeta_k}(x,\zeta)\|_\infty \to 0$ as $\epsilon \to 0$. Furthermore, $\|\frac{a(x,\zeta) - a(x,\zeta')}{\epsilon}\|$ is bounded as $\epsilon \to 0$. Since $(\phi' - \phi) \to 0$ in H^1_{per} as $\epsilon \to 0$, the right side of (4.56) vanishes as $\epsilon \to 0$. This and the Poincaré inequality implies that $v^\epsilon \to w_k$ strongly in

H^1_{per}. Elliptic regularity implies that the limit holds pointwise in x and that $w_k(x)$ is a continuous function. $\square$

Using the dominated convergence theorem and the fact that $\zeta \mapsto a(\cdot,\zeta) \in L^\infty_{per}(D_L)$ is Fréchet differentiable, we find that

$$\frac{\partial \Gamma}{\partial \zeta_j} = L^{-d} \int_{D_L} \frac{\partial a}{\partial \zeta_j}(x) |\nabla \phi + e_1|^2 \, dx + 2L^{-d} \int_{D_L} \left(a(x)(\nabla \phi + e_1) \cdot \nabla w_j + 2\beta \phi w_j \right) dx, \tag{4.57}$$

where $w_j(x) = \frac{\partial}{\partial \zeta_j} \phi(x,\zeta) \in H^1_{per}(D_L)$ was established in Lemma 4.1. Then, using integration by parts and (1.1), we see that the last term vanishes so that

$$\partial_j \Gamma = \frac{\partial \Gamma}{\partial \zeta_j} = L^{-d} \int_{D_L} \frac{\partial a}{\partial \zeta_j} |\nabla \phi + e_1|^2 \, dx. \tag{4.58}$$

In particular, the structural assumption (1.3) implies

$$0 \le C_1 L^{-d} \Phi_j \le \frac{\partial \Gamma}{\partial \zeta_j} \le C_2 L^{-d} \Phi'_j. \tag{4.59}$$

Recall Φ'_j defined at (1.19). From this and the stationarity of ϕ we see immediately that

$$\kappa_0 = \left(\mathbb{E} \sum_{j \in D_L} |\partial_j \Gamma(Z)|^4 \right)^{1/2} \le C L^{-2d} \left(\sum_{j \in D_L} \mathbb{E}[(\Phi'_j)^4] \right)^{1/2} = C L^{-3d/2} \left(\mathbb{E}[(\Phi'_0)^4] \right)^{1/2}. \tag{4.60}$$

Moments of Φ'_0 are related to moments of Φ_0, as follows:

Lemma 4.2 *For any power $p \ge 1$, there is a constant C such that $\mathbb{E}[(\Phi'_0)^p] \le C\mathbb{E}[(\Phi_0)^p]$ for all $\beta \ge 0$, $L \ge 1$.*

Proof: By Minkowski's inequality and the stationarity of ϕ:

$$\begin{aligned} \mathbb{E}[(\Phi'_0)^p] &\le \mathbb{E}\left[\left(\sum_{\substack{j \in D_L \\ |B_\tau(0) \cap Q_j| > 0}} \int_{Q_j} |\nabla \phi + e_1|^2 \, dx \right)^p \right] \\ &\le \left(\sum_{\substack{j \in D_L \\ |B_\tau(0) \cap Q_j| > 0}} \mathbb{E}\left[\left(\int_{Q_j} |\nabla \phi + e_1|^2 \, dx \right)^p \right]^{1/p} \right)^p \\ &= \left(\sum_{\substack{j \in D_L \\ |B_\tau(0) \cap Q_j| > 0}} \mathbb{E}\left[(\Phi_j)^p \right]^{1/p} \right)^p \le C \tau^{pd} \mathbb{E}\left[(\Phi_0)^p \right]. \end{aligned} \tag{4.61}$$

$\square$

By combining (4.60) and Lemma 4.2, we now have

$$\kappa_0 \le C L^{-3d/2} \mathbb{E}[(\Phi_0)^4]^{1/2}, \tag{4.62}$$

which is the first estimate in (4.50).

Now we bound κ_3. The term κ_3 involves the Hessian $\nabla_\zeta^2 \Gamma$, and from (4.58) we compute

$$\frac{\partial^2 \Gamma}{\partial \zeta_j \partial \zeta_i} \quad = \quad L^{-d} 2 \int_{D_L} \frac{\partial a}{\partial \zeta_j} (\nabla \phi(x) + e_1) \cdot \nabla(\partial_i \phi) \, dx + L^{-d} \int_{D_L} \frac{\partial^2 a}{\partial \zeta_j \partial \zeta_i} |\nabla \phi + e_1|^2 \, dx, \quad (4.63)$$

where $\partial_i \phi$ denotes the function $\partial_i \phi = \frac{\partial}{\partial \zeta_i} \phi$. Recall that the function $x \mapsto \frac{\partial a}{\partial \zeta_j}$ is supported in $B_\tau(j)$. In particular, the bounds (1.3) and (1.4) imply

$$\left| \frac{\partial^2 \Gamma}{\partial \zeta_j \partial \zeta_i} \right| \quad \leq \quad C L^{-d} \left(\int_{B_\tau(j)} |(\nabla \phi(x) + e_1) \cdot \nabla(\partial_i \phi)| \, dx + \int_{B_\tau(i) \cap B_\tau(j)} |\nabla \phi + e_1|^2 \, dx \right). \quad (4.64)$$

We will make use of the following observations:

Lemma 4.3 *There is a constant $p^* > 2$ and $C > 0$ such that*

$$\mathbb{E}\left[\int_{Q_0} |\nabla \phi|^p \, dx \right] < C \quad (4.65)$$

holds for all $L > 1$, $\beta \geq 0$ and $p \in [2, p^*]$.

Proof of Lemma 4.3: This is a consequence of Lemma 3.4 and the stationarity of ϕ. Applying Lemma 3.4 to ϕ, with $v(x) = a(x) e_1 \in (L^\infty(D_L))^d$ we conclude that for some $p^* > 2$, there is $C > 0$ such that, almost surely,

$$\left(⨍_{D_L} |\nabla \phi(x)|^p \, dx \right)^{1/p} \quad \leq \quad C \left(⨍_{D_L} |a e_1|^p \, dx \right)^{1/p} \leq C \quad (4.66)$$

holds for all $p \in [2, p^*]$, $L > 1$, and $\beta \geq 0$. Now, by the stationarity of ϕ,

$$\mathbb{E}\left[\int_{Q_0} |\nabla \phi|^p \, dx \right] = \mathbb{E}\left[⨍_{D_L} |\nabla \phi|^p \, dx \right] \leq C^p.$$

□

Lemma 4.4 *Let $\tilde{\zeta}(t) = h(\tilde{Z}(t))$ be the random vector defined in Theorem 1.2, and let $\tilde{a}_t = a(x, \tilde{\zeta}(t))$ denote the associated conductivity. There are constants $q > 4$, $C > 0$ such that*

$$\mathbb{E}\left[\sum_i \left(\sum_j \frac{\partial^2 \Gamma}{\partial \zeta_i \partial \zeta_j}(a) \frac{\partial \Gamma}{\partial \zeta_j}(\tilde{a}_t) h'(Z_i) h'(Z_j) h'(\tilde{Z}_j(t)) \right)^2 \right] \leq C L^{-3d} \mathbb{E}[\Phi_0^q]^{3/q} + C L^{-3d} \mathbb{E}[\Phi_0^4] \quad (4.67)$$

holds for all $L > 1$, $\beta \geq 0$, $t \in [0, 1]$.

Proof of Lemma 4.4: For each index $i \in D_L$, let

$$H_i = \sum_j \frac{\partial^2 \Gamma}{\partial \zeta_i \partial \zeta_j}(a) \frac{\partial \Gamma}{\partial \zeta_j}(\tilde{a}_t) h'(Z_j) h'(\tilde{Z}_j(t)).$$

We claim that there is a constant C, independent of t, β, and L, such that

$$|H_i| \le CL^{-2d} \int_{B_\tau(i)} |\nabla\phi(x,a) + e_1||\nabla u(x)|\, dx + CL^{-2d} \sum_{j \in D_L \cap B_{2\tau}(i)} \Phi_j'(a)\Phi_j'(\tilde{a}_t) \tag{4.68}$$

where $u(x) \in H^1_{per}(D_L)$ satisfies

$$-\nabla \cdot (a\nabla u) + \beta u = -\nabla \cdot \left((\nabla\phi(x,a) + e_1) \sum_{j \in D_L} \left(s_j \mathbb{I}_{B_\tau(j)}(x) \frac{\partial a}{\partial \zeta_j}(x,\zeta) \right) \right), \quad x \in D_L \tag{4.69}$$

and the random variables $\{s_j\}_{j \in D_L}$ are defined by

$$s_j = L^d \frac{\partial \Gamma}{\partial \zeta_j}(\tilde{a}_t) h'(Z_j) h'(\tilde{Z}_j).$$

These variables are identically distributed and satisfy $|s_j| \le C\Phi_j'(\tilde{a}_t)$, by (4.58) and the fact that $|h'| \le c_1$. If $\beta = 0$, we may assume $\int_{D_L} u\, dx = 0$, so that u is uniquely defined. To see why (4.68) must be true, observe from (4.63) that

$$\begin{aligned} L^{2d} H_i &= 2 \sum_{j \in D_L} s_j \int_{B_\tau(j)} \frac{\partial a}{\partial \zeta_j}(x)(\nabla\phi(x,a) + e_1) \cdot (\nabla \partial_i \phi(x,a))\, dx \\ &\quad + \sum_{j \in D_L} s_j \int_{B_\tau(j) \cap B_\tau(i)} \frac{\partial^2 a}{\partial \zeta_j \partial \zeta_i}(x) |\nabla\phi(x,a) + e_1|^2\, dx. \end{aligned} \tag{4.70}$$

The second sum in (4.70) is bounded by

$$\left| \sum_{j \in D_L} s_j \int_{B_\tau(j) \cap B_\tau(i)} \frac{\partial^2 a}{\partial \zeta_j \partial \zeta_i}(x) |\nabla\phi(x,a) + e_1|^2\, dx \right| \le C \sum_{j \in B_{2\tau}(i)} \Phi_j'(\tilde{a}_t)\Phi_j'(a).$$

The first sum in (4.70) is exactly

$$\sum_{j \in D_L} s_j \int_{B_\tau(j)} \frac{\partial a}{\partial \zeta_j}(x)(\nabla\phi(x,a) + e_1) \cdot (\nabla \partial_i \phi(x,a))\, dx = \int_{D_L} v(x) \cdot \nabla \partial_i \phi(x,a)\, dx \tag{4.71}$$

where the vector field $v \in (L^2(D_L))^d$ is

$$v(x) = (\nabla\phi(x,a) + e_1) \sum_{j \in D_L} s_j \mathbb{I}_{B_\tau(j)}(x) \frac{\partial a}{\partial \zeta_j}(x,\zeta), \tag{4.72}$$

which depends on the random vectors Z and $\tilde{Z}(t)$. Therefore,

$$L^{2d}|H_i| \le C \left| \int_{D_L} v(x) \cdot \nabla \partial_i \phi(x,a)\, dx \right| + C \sum_{j \in D_L \cap B_{2\tau}(i)} \Phi_j'(a)\Phi_j'(\tilde{a}_t). \tag{4.73}$$

Using equation (4.69) for $u(x)$ and equation (4.51) for $w_i(x) = \partial_i \phi(x)$, we have

$$\begin{aligned} \int_{D_L} v(x) \cdot \nabla \partial_i \phi(x,a)\, dx &= \int_{D_L} a(x) \nabla u \nabla w_i + \beta u w_i\, dx \\ &= -\int_{B_\tau(i)} \frac{\partial a}{\partial \zeta_i} (\nabla\phi(x,a) + e_1) \cdot \nabla u(x)\, dx. \end{aligned}$$

Hence

$$\left|\int_{D_L} v(x)\cdot\nabla\partial_i\phi(x,a)\,dx\right| \le C\int_{B_\tau(i)} |\nabla\phi(x,a)+e_1||\nabla u|\,dx.$$

This combined with (4.73) establishes (4.68). Observe that the vector field v is stationary with respect to integer shifts in x, and it is independent of the index i. Consequently, u is also statistically stationary and independent of i.

To establish (4.67) we must bound $\sum_i \mathbb{E}\left[H_i^2\right]$, and we will use (4.68). First, observe that there is $p>2$ such that $v\in(L^p(D_L))^d$ almost surely. This is a consequence of Lemma 4.3. Therefore, since u satisfies (4.69), Lemma 3.4 immediately implies the following:

Corollary 4.5 *There is an exponent $p^*>2$ and a constant C such that, with probability one,*

$$\fint_{D_L} |\nabla u|^p\,dx \le C\fint_{D_L} |v|^p\,dx$$

holds for all $p\in[2,p^*]$ and $L>1$ and $\beta\ge 0$, where v is the vector field defined by (4.72).

Now we proceed with the proof of (4.67). From (4.68) we have

$$L^{4d}\sum_{i\in D_L}\mathbb{E}\left[H_i^2\right] \le C(S_1+S_2),$$

where

$$S_1=\sum_{i\in D_L}\mathbb{E}\left[\left(\int_{B_\tau(i)}|\nabla\phi+e_1|\,|\nabla u|\,dx\right)^2\right],\quad S_2=\sum_{i\in D_L}\mathbb{E}\left[\left(\sum_{j\in D_L\cap B_{2\tau}(i)}\Phi_j'(a)\Phi_j'(\tilde a_t)\right)^2\right].$$

First we bound S_1. Let $p\in(2,p^*)$ be as in Corollary 4.5, and let $r=p/2>1$ and $q=r/(r-1)$. By Hölder's inequality and the fact that τ is independent of L,

$$\begin{aligned} S_1 &\le C\sum_{i\in D_L}\mathbb{E}\left[\int_{B_\tau(i)}|\nabla\phi+e_1|^2\,dx\int_{B_\tau(i)}|\nabla u|^2\,dx\right] \\ &\le \sum_{i\in D_L}\mathbb{E}\left[\Phi_i'(a)^q\right]^{1/q}\mathbb{E}\left[\left(\int_{B_\tau(i)}|\nabla u|^2\,dx\right)^r\right]^{1/r} \\ &= CL^d\mathbb{E}\left[\Phi_0'(a)^q\right]^{1/q}\mathbb{E}\left[\left(\int_{B_\tau(0)}|\nabla u|^2\,dx\right)^r\right]^{1/r}. \end{aligned} \tag{4.74}$$

In this last step we have used the stationarity of both u and ϕ. Now, as in the proof of Lemma 4.2, the stationarity of u implies

$$\begin{aligned} \mathbb{E}\left[\left(\int_{B_\tau(0)}|\nabla u|^2\,dx\right)^r\right]^{1/r} &\le \mathbb{E}\left[\left(\sum_{\substack{\ell\in\mathbb{Z}^d\\ |Q_\ell\cap B_\tau(0)|>0}}\int_{Q_\ell}|\nabla u|^2\,dx\right)^r\right]^{1/r} \\ &\le \sum_{\substack{\ell\in\mathbb{Z}^d\\ |Q_\ell\cap B_\tau(0)|>0}}\mathbb{E}\left[\left(\int_{Q_\ell}|\nabla u|^2\,dx\right)^r\right]^{1/r} \le C\tau^d\mathbb{E}\left[\left(\int_{Q_0}|\nabla u|^2\,dx\right)^r\right]^{1/r}. \end{aligned}$$

So, by Jensen's inequality, this implies

$$\mathbb{E}\left[\left(\int_{B_\tau(0)} |\nabla u|^2\,dx\right)^r\right]^{1/r} \le C\mathbb{E}\left[\int_{Q_0} |\nabla u|^{2r}\,dx\right]^{1/r} = C\mathbb{E}\left[\fint_{D_L} |\nabla u|^{2r}\,dx\right]^{1/r}.$$

Combining this with Corollary 4.5 ($p = 2r$), we obtain

$$\mathbb{E}\left[\left(\int_{B_\tau(0)} |\nabla u|^2\,dx\right)^r\right]^{1/r} \le C\mathbb{E}\left[\fint_{D_L} |v|^p\,dx\right]^{1/r}.$$

By definition of v and the bound $|s_j| \le C\Phi_j'(\tilde{a}_t)$,

$$\begin{aligned}
\mathbb{E}\left[\fint_{D_L} |v|^p\,dx\right] &= \mathbb{E}\left[\int_{Q_0} |v|^p\,dx\right] \\
&\le C \sum_{\substack{\ell\in\mathbb{Z}^d \\ |B_\tau(\ell)\cap Q_0|>0}} \mathbb{E}\left[(\Phi_\ell'(\tilde{a}_t))^p \int_{Q_0} |\nabla\phi(x,a)+e_1|^p\,dx\right] \\
&\le C\mathbb{E}\left[(\Phi_0'(\tilde{a}_t))^{np}\right]^{1/n} \mathbb{E}\left[(\int_{Q_0} |\nabla\phi(x,a)+e_1|^p\,dx)^m\right]^{1/m} \\
&\le C\mathbb{E}\left[(\Phi_0(\tilde{a}_t))^{np}\right]^{1/n} \mathbb{E}\left[(\int_{Q_0} |\nabla\phi(x,a)+e_1|^p\,dx)^m\right]^{1/m} \qquad (4.75)
\end{aligned}$$

where $n > 2$ and $m = n/(n-1)$. In the last step we have used Lemma 4.2. Observe that $\Phi_0(\tilde{a}_t)$ has the same law as $\Phi_0(a)$, so $\mathbb{E}\left[(\Phi_0(\tilde{a}_t))^{np}\right]^{1/n} = \mathbb{E}\left[(\Phi_0(a))^{np}\right]^{1/n}$. Since $p < p^*$, we may choose n large so that $pm < p^*$. Then Jensen's inequality and Lemma 4.3 imply that

$$\mathbb{E}\left[(\int_{Q_0} |\nabla\phi(x,a)+e_1|^p\,dx)^m\right]^{1/m} \le \mathbb{E}\left[1+\int_{Q_0} |\nabla\phi(x,a)+e_1|^{p^*}\,dx\right]^{1/m} \le C,$$

so that

$$\mathbb{E}\left[\fint_{D_L} |v|^p\,dx\right] \le C\mathbb{E}\left[(\Phi_0)^{np}\right]^{1/n}.$$

Combining the above computations applying Lemma 4.2, we conclude that

$$S_1 \le CL^d\mathbb{E}\left[\Phi_0'(a)^q\right]^{1/q}\mathbb{E}\left[(\Phi_0(a))^{np}\right]^{1/rn} \le CL^d\mathbb{E}\left[\Phi_0(a)^q\right]^{1/q}\mathbb{E}\left[(\Phi_0(a))^{np}\right]^{1/rn}.$$

By choosing $p > 2$ smaller, if necessary, we may assume that $q \ge np$. Therefore, by Jensen's inequality, $S_1 \le CL^d\mathbb{E}\left[\Phi_0(a)^q\right]^{3/q}$.

Bounding S_2 involves similar arguments. By Minkowski's inequality and Hölder's inequality

$$\begin{aligned} S_2 &= \sum_{i\in D_L} \mathbb{E}\left[\left(\sum_{j\in D_L\cap B_{2\tau}(i)} \Phi_j'(a)\Phi_j'(\tilde{a}_t)\right)^2\right] \\ &\le \sum_{i\in D_L}\left(\sum_{j\in D_L\cap B_{2\tau}(i)} \mathbb{E}\left[\left(\Phi_j'(a)\Phi_j'(\tilde{a}_t)\right)^2\right]^{1/2}\right)^2 \\ &\le \sum_{i\in D_L}\left(\sum_{j\in D_L\cap B_{2\tau}(i)} \mathbb{E}\left[(\Phi_j'(a))^4\right]^{1/4}\mathbb{E}\left[(\Phi_j'(\tilde{a}_t))^4\right]^{1/4}\right)^2 \\ &= \sum_{i\in D_L}\left(\sum_{j\in D_L\cap B_{2\tau}(i)} \mathbb{E}\left[(\Phi_0'(a))^4\right]^{1/2}\right)^2 \le CL^d\mathbb{E}\left[(\Phi_0(a))^4\right]. \end{aligned} \tag{4.76}$$

Now we conclude that

$$\begin{aligned} &\mathbb{E}\left[\sum_{i\in D_L}\left(\sum_{j\in D_L}\frac{\partial^2\Gamma}{\partial\zeta_i\partial\zeta_j}(a)\frac{\partial\Gamma}{\partial\zeta_j}(\tilde{a}_t)h'(Z_i)h'(Z_j)h'(\tilde{Z}_j(t))\right)^2\right] = \mathbb{E}\left[\sum_{i\in D_L}|h'(Z_i)|^2H_i^2\right] \\ &\le C\mathbb{E}\left[\sum_{i\in D_L}H_i^2\right] \le CL^{-4d}(S_1+S_2) \le CL^{-3d}(\mathbb{E}\left[(\Phi_0)^q\right]^{3/q}+\mathbb{E}\left[(\Phi_0)^4\right]). \end{aligned} \tag{4.77}$$

□

Having proved (4.67), the bound $\kappa_3 \le CL^{-3d/2}\mathbb{E}[\Phi_0^q]^{3/(2q)} + CL^{-3d/2}\mathbb{E}\left[(\Phi_0)^4\right]^{1/2}$ now follows immediately from the definition of κ_3 in Theorem 1.2 and the fact that the right side of (4.67) is independent of $t\in[0,1]$. This completes the proof of Theorem 1.1.

5 Stochastic moment estimates

We close with some estimates on the moments of the random variable Φ_0 which appears in Theorem 1.1. First, we have an estimate which is Lemma 2.7 from [13]:

Lemma 5.1 *Let $d\ge 1$. Let $n\ge 0$ be an even integer. Then*

$$\mathbb{E}\left[\int_{Q_j}(\phi(x))^n|\nabla\phi(x)|^2\,dx\right] + \frac{2\beta}{(n+1)a_*}\mathbb{E}\left[\int_{Q_j}(\phi(x))^{n+2}\,dx\right] \le \left(\frac{a^*}{a_*}\right)^2\mathbb{E}[\int_{Q_j}(\phi(x))^n\,dx] \tag{5.78}$$

holds for any cube Q_j, $j\in D_L$.

This is proved by using the test function $v=\phi^{n+1}\in H^1_{per}(D_L)$ in the variational equality (1.6) satisfied by ϕ. By (1.6) and the Cauchy-Schwarz inequality one obtains

$$\begin{aligned} (n+1)\int_{D_L}a(x)|\nabla\phi|^2\phi^n\,dx + \beta\int_{D_L}\phi^{n+2}\,dx &= (n+1)\int_{D_L}\phi^n\nabla\phi\cdot a(x)e_1\,dx \\ &\le \frac{(n+1)}{2a_*}\int_{D_L}\phi^n|a(x)e_1|^2\,dx + \frac{(n+1)a_*}{2}\int_{D_L}\phi^n|\nabla\phi|^2\,dx. \end{aligned} \tag{5.79}$$

Therefore, since $a_* \le a(x) \le a^*$, we conclude that

$$\int_{D_L} |\nabla\phi|^2 \phi^n \, dx + \frac{2\beta}{(n+1)a_*} \int_{D_L} \phi^{n+2} \, dx \le \left(\frac{a^*}{a_*}\right)^2 \int_{D_L} \phi^n \, dx. \tag{5.80}$$

Then (5.78) follows by the stationarity of ϕ and $\nabla\phi$.

Corollary 5.2 *Let $d \ge 1$ and let m be a positive integer. Then*

$$\mathbb{E}\left[\int_{Q_0} (\phi(x))^{2m} \, dx\right] \le \left(\frac{a^*}{\sqrt{2a_*}}\right)^{2m} \beta^{-m} \prod_{k=1}^{m} (2m - 2k + 1) \tag{5.81}$$

holds for all $\beta > 0$ and $L > 1$.

Proof of Corollary 5.2: Observe that the final product over $k = 1, \ldots, m$ is bounded by $2^m(m!)$. By (5.78) with $n = 0$, we have

$$\mathbb{E}\left[\int_{Q_0} (\phi(x))^2 \, dx\right] \le \frac{(a^*)^2}{2a_*} \beta^{-1}.$$

So, (5.81) holds for $m = 1$. Now, arguing inductively, suppose that (5.81) holds for some integer $m \ge 1$. Then by (5.78) and the induction hypothesis

$$\begin{aligned} \mathbb{E}\left[\int_{Q_0} (\phi(x))^{2(m+1)} \, dx\right] &\le \frac{(a^*)^2}{2a_*} \beta^{-1} (2m+1) \mathbb{E}\left[\int_{Q_0} (\phi(x))^{2m} \, dx\right] \\ &\le \frac{(a^*)^2}{2a_*} \beta^{-1} (2m+1) \frac{(a^*)^{2m}}{(2a_*)^m} \beta^{-m} \prod_{k=1}^{m} (2m - 2k + 1) \\ &= \left(\frac{a^*}{\sqrt{2a_*}}\right)^{2(m+1)} \beta^{-(m+1)} \prod_{k=1}^{m+1} (2(m+1) - 2k + 1). \end{aligned} \tag{5.82}$$

So, (5.81) also holds for $m + 1$ and by induction on m it holds for all $m \ge 1$. □

Proposition 5.3 *Let $d \ge 1$. For each even integer $n \ge 0$, there is a constant C_n such that*

$$\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2 \, dx\right)^{n+1}\right] \le C_n + C_n \mathbb{E}\left[\left(\int_{Q_0} \phi \, dx\right)^{2n}\right] \tag{5.83}$$

holds for all $L > 1$ and $\beta \ge 0$.

Proof of Proposition 5.3: By Caccioppoli's inequality (Lemma 3.1) we know that

$$\int_{Q_0} |\nabla\phi|^2 \, dx \le C\left(1 + \beta b^2 + \int_{3Q_0} (\phi(x) - b)^2 \, dx\right)$$

holds with probability one, where b is the random constant

$$b = \left(\frac{1}{|3Q_0|} \int_{3Q_0} \phi^{n+1}(x)\, dx \right)^{\frac{1}{n+1}}.$$

Therefore, with probability one, we have

$$\left(\int_{Q_0} |\nabla \phi|^2 \, dx \right)^{n+1} \le C \left(1 + \beta^{n+1} b^{2(n+1)} + \int_{3Q_0} (\phi(x) - b)^{2(n+1)} \, dx \right). \tag{5.84}$$

Then, by (5.84) and Lemma 5.4 below, we have

$$\left(\int_{Q_0} |\nabla \phi|^2 \, dx \right)^{n+1} \le C \left(1 + \beta^{n+1} b^{2(n+1)} + \int_{3Q_0} (\phi^{n+1}(x) - b^{n+1})^2 \, dx \right).$$

Now by applying the Poincaré inequality in $3Q_0$ to the last integral, we conclude that

$$\begin{aligned} \left(\int_{Q_0} |\nabla \phi|^2 \, dx \right)^{n+1} &\le C \left(1 + \beta^{n+1} b^{2(n+1)} + \int_{3Q_0} |\nabla (\phi^{n+1})|^2 \, dx \right) \\ &= C \left(1 + \beta^{n+1} b^{2(n+1)} + (n+1)^2 \int_{3Q_0} |\nabla \phi|^2 \phi^{2n} \, dx \right). \end{aligned} \tag{5.85}$$

Consider the term $\beta^{n+1} b^{2(n+1)}$. By Jensen's inequality, the stationarity of ϕ, and Corollary 5.2 we know that

$$\mathbb{E}[\beta^{n+1} b^{2(n+1)}] \le \beta^{n+1} \mathbb{E}\left[\int_{Q_0} \phi^{2(n+1)} \, dx \right] \le (a^*/\sqrt{2a_*})^{n+1} \sqrt{(2n+2)!} \tag{5.86}$$

holds for all $L > 1$, $\beta \ge 0$. Also, by Lemma 5.1,

$$\mathbb{E}\left[\int_{3Q_0} |\nabla \phi|^2 \phi^{2n} \, dx \right] \le C \mathbb{E}\left[\int_{3Q_0} \phi^{2n} \, dx \right] = C 3^d \mathbb{E}\left[\int_{Q_0} \phi^{2n} \, dx \right].$$

So, returning to (5.85), we conclude that for a constant C_n independent of $L > 1$ and $\beta \ge 0$,

$$\mathbb{E}\left[\left(\int_{Q_0} |\nabla \phi|^2 \, dx \right)^{n+1} \right] \le C_n \left(1 + \mathbb{E}\left[\int_{Q_0} \phi^{2n} \, dx \right] \right). \tag{5.87}$$

By the De Giorgi-Nash-Moser theory (e.g. [11], Theorem 8.24), ϕ is Hölder continuous with

$$|\phi|_{C^\alpha(Q_0)} = \sup_{\substack{x,y \in Q_0 \\ x \neq y}} \frac{|\phi(x) - \phi(y)|}{|x-y|^\alpha} \le C(\|\phi(\cdot) - \rho_{3,0}\|_{L^2(3Q)} + 1)$$

for some deterministic constants $\alpha > 0$ and $C > 0$, which depend on a^* and a_* but not on L or $\beta \ge 0$. (Recall $\rho_{3,0}$ defined at (3.37).) There must be a point $x_0 \in Q_0$ such that $|\phi(x_0)| \le \|\phi(\cdot)\|_{L^2(Q_0)}$. Therefore, if $|\phi|_{C^\alpha(3Q)} = h$,

$$\begin{aligned} \int_{Q_0} \phi(x)^{2n} \, dx &\le \int_{Q_0} (|\phi(x_0)| + Ch)^{2n} \, dx \\ &\le |Q_0| \left(\|\phi\|_{L^2(Q_0)} + C(1 + \|\phi(\cdot) - \rho_{3,0}\|_{L^2(3Q)}) \right)^{2n} \\ &\le C \left(1 + \left(\int_{3Q_0} (\phi(x))^2 \, dx \right)^{1/2} \right)^{2n}. \end{aligned} \tag{5.88}$$

Now returning to (5.87), we conclude that

$$\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^{n+1}\right] \leq C_n\left(1+\mathbb{E}\left[\left(\int_{3Q_0} \phi^2\,dx\right)^{n}\right]\right) \leq C_n 3^{dn}\left(1+\mathbb{E}\left[\left(\int_{Q_0} \phi^2\,dx\right)^{n}\right]\right). \tag{5.89}$$

The last inequality follows from the stationarity of ϕ.

By the triangle inequality,

$$\int_{Q_0} \phi^2\,dx \leq 2\int_{Q_0} (\phi(x)-\rho)^2\,dx + 2\int_{Q_0} \rho^2\,dx$$

where $\rho = \int_{Q_0} \phi(x)\,dx$. Combining this with the Poincaré inequality in Q_0, we obtain

$$\left(\int_{Q_0} \phi^2\,dx\right)^n \leq C^n\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^n + C^n\rho^{2n}.$$

Therefore, by (5.89) we have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^{n+1}\right] &\leq C_n\left(1+\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^{n}\right] + \mathbb{E}[\rho^{2n}]\right). \\
&\leq C_n\left(1+\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^{n+1}\right]^{n/(n+1)} + \mathbb{E}[\rho^{2n}]\right).
\end{aligned}$$

The bound (5.83) now follows from Young's inequality. $\Box$

The following fact was used in the proof of Proposition 5.3:

Lemma 5.4 *Let $n \geq 2$ be an even integer. For all $z \in \mathbb{R}$ and $m \in \mathbb{R}$*

$$0 \leq (z-m)^{2(n+1)} \leq 3^{2(n+1)}(z^{n+1}-m^{n+1})^2. \tag{5.90}$$

Proof: If $m = 0$, the bound (5.90) obviously holds. If $m \neq 0$, then $m^{2(n+1)} > 0$ and we see that the bound is equivalent to

$$(\hat{z}-1)^{2(n+1)} \leq 3^{2(n+1)}(\hat{z}^{n+1}-1)^2 \tag{5.91}$$

where $\hat{z} = z/m$. Let $f(z) = (z^{n+1}-1)^2$ and $g(z) = (z-1)^{2(n+1)}$. Both of these polynomials are nonnegative for $z \in \mathbb{R}$ and $f(1) = g(1) = 0$. If $z \in [-2, 0]$, we observe that $g(z) \leq 3^{2(n+1)}$ and $f(z) \geq 1$, so $g(z) \leq 3^{2(n+1)}f(z)$ holds for $z \in [-2, 0]$. For other $z \in \mathbb{R}$, consider the factorization

$$g(z) = (z-1)^2\prod_{k=1}^{n}(z-1)^2, \qquad f(z) = (z-1)^2\prod_{k=1}^{n}(z-\omega_k)(z-\bar{\omega}_k),$$

where $\omega_k = e^{i2\pi k/(n+1)}$ is a $(n+1)^{th}$ root of unity. The products $(z-w_k)(z-\bar{w}_k)$ are real and positive for $z \in \mathbb{R}$, $k = 1, \ldots, n$. For $z \leq -2$, it is easy to see that $(z-1)^2 \leq 3^2(z-w_k)(z-\bar{w}_k)$. It follows that $g(z) \leq 3^{2n}f(z)$ for $z \leq -2$. For $z \geq 0$, we also have $(z-1)^2 \leq (z-w_k)(z-w_k)$ for each $k = 1, 2, \ldots, n$. Hence $g(z) \leq f(z)$ for $z \geq 0$. We have shown that $g(z) \leq 3^{2(n+1)}f(z)$ for all $z \in \mathbb{R}$. Hence (5.91) holds. $\Box$

From Corollary 5.2 and Proposition 5.3 we immediately obtain the following:

Corollary 5.5 *Let $d \geq 1$. For all positive odd integers m, there is a constant $C_m > 0$ such that*

$$\mathbb{E}\left[\left(\int_{Q_0} |\nabla\phi|^2\,dx\right)^m\right] \leq C_m(1+\beta^{1-m}) \tag{5.92}$$

holds for all $L > 1$, $\beta > 0$. Hence, $\mathbb{E}[\Phi_0^m] \leq C(1+\beta^{1-m})$.

Observe that the bound in Proposition 5.3 is better than what is immediately implied by the Caccioppoli inequality and the stationarity of ϕ, since the homogeneity of the integral term on the right side of (5.83) is less than that of the term on the left side. This fact plays an important role in the method of Gloria and Otto [13] to bound moments of Φ_0, independently of $\beta \geq 0$ and $L > 1$. Although [13] pertains to the discrete setting on all of $\mathbb{Z}^d$ (rather than continuum, periodic), that method can still be applied here. In view of Proposition 5.3, the moments of Φ_0 are bounded by $\mathbb{E}[\Phi_0^{n+1}] \leq C_n(1+\mathbb{E}[\rho^{2n}])$, where the random variable

$$\rho = \int_{Q_0} \phi(x)\,dx$$

has zero mean. In the discrete setting of [13], $\phi(0)$ is analogous to this ρ.

Let us briefly sketch the method of [13] to bound moments of ρ. For integers $m \geq 1$, define $E_m = \mathbb{E}[\rho^m]$ and $V_m = \mathrm{Var}[\rho^m]$. Therefore,

$$E_{2m} \quad = \quad V_m + (E_m)^2 \tag{5.93}$$

for all m. Because $E_1 = \mathbb{E}[\rho] = 0$ and $E_2 = V_1$, the equality (5.93) can be iterated to obtain

$$E_{2\cdot 2^\ell} \leq \sum_{q=0}^{\ell} C_q (V_{2^{\ell-q}})^{2^q} \tag{5.94}$$

for any integer $\ell > 2$, where $C_0 = 1$ and $C_q = 2^1 2^2 \cdots 2^{2^q}$ for $q > 0$. Of course, this bound is very general. However, in the discrete case on $\mathbb{Z}^d$, Gloria and Otto proved that for ℓ sufficiently large,

$$(V_{2^{\ell-q}})^{2^q} \leq K_\ell(1+(E_{2\cdot 2^\ell})^{r_\ell}) \tag{5.95}$$

holds for all $q = 0,\ldots,\ell$, for some power $r_\ell < 1$ and constant K_ℓ. For $d \geq 3$, the constant K_ℓ is independent of $L > 1$ and $\beta > 0$. Applying this fact and Young's inequality at (5.94), we conclude that $E_{2\cdot 2^\ell}$ must be bounded independently of $L > 1$ and $\beta \geq 0$, for $d \geq 3$; this is the first bound in (1.11). For $d = 2$, the constant K_ℓ is indepdendent of L, but it depends on $\log\beta$. So, in the $d = 2$ case, one obtains $E_{2\cdot 2^\ell} \leq C|\log\beta|^{\gamma_\ell}$ for some $\gamma_\ell > 0$. This is second bound in (1.11).

The bound (5.95) on the variances V_m is obtained from a spectral-gap estimate (e.g. the Efron-Stein inequality [22]). If $F(\zeta)$ is a function of the random vector $\zeta = (\zeta_j)$, $j \in D_L$, this inequality is

$$\mathrm{Var}[F(\zeta)] \quad \leq \quad \frac{1}{2}\mathbb{E}\left[\sum_{j\in D_L} |\Delta_j F(\zeta)|^2\right] \tag{5.96}$$

where

$$\Delta_j F(\zeta) = F(\zeta_1,\ldots,\zeta_{j-1},\zeta_j',\zeta_{j+1},\ldots,\zeta_N) - F(\zeta_1,\ldots,\zeta_{j-1},\zeta_j,\zeta_{j+1},\ldots,\zeta_N)$$

and ζ_j' is an independent copy of ζ_j. By the mean value theorem, this implies that

$$V_m \le C\mathbb{E}\left[\sum_{j\in D_L} \sup_{\zeta_j} |\partial_j(\rho^m)|^2\right]. \tag{5.97}$$

Therefore, since

$$\partial_j \rho^m \quad = \quad m\rho^{m-1}\int_{Q_0} \partial_j \phi(x)\, dx,$$

we have

$$|\partial_j \rho^m|^2 \le m^2 \rho^{2m-2} \Phi_j' \int_{Q_0} |\hat{w}_j(x)|^2\, dx \tag{5.98}$$

where

$$\hat{w}_j(x) = (\Phi_j')^{-1/2} \partial_j \phi(x), \qquad \Phi_j' = \int_{B_\tau(j)} |\nabla \phi(x,\zeta) + e_1|^2\, dx.$$

(Recall Lemma 4.1.) Hence,

$$V_m \le C \sum_{j\in D_L} \mathbb{E}\left[\sup_{\zeta_j} \rho^{2m-2} \Phi_j' \int_{Q_0} |\hat{w}_j(x)|^2\, dx\right]. \tag{5.99}$$

Now, suppose $1 \le m \le n$. By applying Hölder's inequality with $p = (n+1)$ and $p' = (n+1)/n$ to each term in (5.99) we have

$$\begin{aligned}\mathbb{E}\left[\sup_{\zeta_j} \rho^{2m-2}(\zeta) \int_{Q_0} |\partial_j \phi(x,\zeta)|^2\, dx\right] \quad &\le \quad \mathbb{E}\left[\sup_{\zeta_j} (\Phi_j')^{n+1}\right]^{\frac{1}{n+1}} \\ &\quad \times \mathbb{E}\left[\sup_{\zeta_j} \rho^{(2m-2)(n+1)/n} \left(\int_{Q_0} |\hat{w}_j(x)|^2\, dx\right)^{(n+1)/n}\right]^{n/(n+1)}.\end{aligned} \tag{5.100}$$

Consider the first term in the right side of (5.100). By Lemma 2.2, the stationarity of ϕ, Lemma 4.2, and then Proposition 5.3 we have

$$\begin{aligned}\mathbb{E}\left[\sup_{\zeta_j} \left(\Phi_j'\right)^{n+1}\right]^{\frac{1}{n+1}} \quad &\le \quad C\mathbb{E}\left[\left(\Phi_j'\right)^{n+1}\right]^{\frac{1}{n+1}} \\ &= \quad \mathbb{E}\left[\left(\Phi_0'\right)^{n+1}\right]^{\frac{1}{n+1}} \\ &\le \quad C\mathbb{E}\left[(\Phi_0)^{n+1}\right]^{\frac{1}{n+1}} \le C(1 + E_{2n}^{\frac{1}{n+1}}).\end{aligned} \tag{5.101}$$

Therefore,

$$V_m \le C(1 + E_{2n}^{\frac{1}{n+1}}) \sum_{j\in D_L} \mathbb{E}\left[\sup_{\zeta_j} \rho^{(2m-2)(n+1)/n} \left(\int_{Q_0} |\hat{w}_j(x)|^2\, dx\right)^{(n+1)/n}\right]^{n/(n+1)}. \tag{5.102}$$

To estimate the sum remaining in (5.102), one needs to control the random variables $\int_{Q_0} |\hat{w}_j(x)|^2\, dx$, which we expect to be small if $dist(B_\tau(j), Q_0)$ is large. In fact, it was observed in [13] that the function $\hat{w}_j$ is related to the gradient of the Green's function $G(x,y)$ for the operator $u \mapsto -\nabla \cdot a\nabla u + \beta u$

on D_L. As has been pointed out in [12], in the periodic setting it is important to choose the Green's function that respects the normalization $\int_D u(x)\,dx = 0$ in order to obtain the optimal estimates on ∇G, uniformly in $\beta \geq 0$. For each $y \in D_L$, this function $G(\cdot, y)$ is periodic over D_L, and for each $r > 0$, $G(\cdot, y) \in H^1_{loc}(D_L \setminus B_r(y))$. Also,

$$\int_{D_L} a(x)\nabla_x G(x,y)\nabla_x \varphi(x) + \beta G(x,y)\varphi(x)\,dx = \varphi(y) - \frac{1}{|D_L|}\int_{D_L} \varphi(y)\,dy$$

holds for all smooth, periodic functions φ. That is,

$$-\nabla_x \cdot (a(x)\nabla_x G(x,y)) + \beta G = \delta_y(x) - |D_L|^{-1}.$$

In the present setting, the connection between $\hat{w}_j(x)$ and $G(x,y)$ is as follows:

Lemma 5.6 *Let $d \geq 1$. Suppose $A \subset D_L$ is an open set for which $dist(A, B_\tau(j)) > 0$. Then we have*

$$\int_A (\hat{w}_j(y))^2\,dy \leq (C_2)^2 \int_{y\in A}\int_{x\in B_\tau(j)} |\nabla_x G(x,y)|^2\,dx\,dy. \tag{5.103}$$

In particular, if $dist(Q_0, B_\tau(j)) > 0$,

$$\left(\int_{Q_0} |\hat{w}_j(x)|^2\,dx\right)^{(n+1)/n} \leq (C_2)^{2(n+1)/n}\int_{y\in Q_0}\int_{x\in B_\tau(j)} |\nabla_x G(x,y)|^{2(n+1)/n}\,dx\,dy.$$

When n is large, the exponent $q = 2(n+1)/n$ is only slightly larger than 2, and Meyers' estimate implies that $|\nabla_x G| \in L^q_{loc}$ if $q - 2 > 0$ is small enough (away from the singularity at $x = y$). As shown in [13], this fact, the Caccioppoli inequality, and uniform decay estimates on $G(x,y)$ can be used to obtain optimal bounds on the decay of $|\nabla G|^q$ away from the singularity. This leads to the optimal estimate of (5.102). Extension of the Green's function estimates of [13] and of the moment estimates on Φ_0 to the periodic setting is being carried out in [12].

Proof of Lemma 5.6: Let $v \in H^1_{per}(D_L)$ satisfy

$$-\nabla\cdot(a\nabla v) + \beta v = \partial_j\phi \mathbb{I}_A(x) - \frac{1}{D_L}\int_A \partial_j\phi(x)\,dx.$$

By applying Lemma 4.1 to $\partial_j\phi = (\Phi_j')^{1/2}\hat{w}_j$ and using $\int_{D_L}\partial_j\phi(x)\,dx = 0$, we have

$$\begin{aligned}\int_A (\partial_j\phi(x))^2\,dx &= \int_{D_L} (\mathbb{I}_A(x)\partial_j\phi(x))\partial_j\phi(x)\,dx \\ &= \int_{D_L} a(x)\nabla v\nabla\partial_j\phi + \beta v\partial_j\phi\,dx \\ &= -\int_{D_L} \xi_j(x)\cdot\nabla v(x)\,dx \leq \left(\int_{B_\tau(j)}|\xi_j|^2\right)^{1/2}\left(\int_{B_\tau(j)}|\nabla v|^2\right)^{1/2}\end{aligned} \tag{5.104}$$

since ξ_j is supported in $B_\tau(j)$. On the other hand,

$$v(x) = \int_A G(x,y)\partial_j\phi(y)\,dy, \qquad \nabla v(x) = \int_A \nabla_x G(x,y)\partial_j\phi(y)\,dy$$

hold for almost every x outside A. Therefore, by Cauchy-Schwarz we have

$$|\nabla v(x)|^2 \leq \int_A |\nabla_x G(x,y)|^2\,dy \int_A (\partial_j \phi(y))^2\,dy$$

for almost every x in $B_\tau(j)$. Also, $\int_{B_\tau(j)} |\xi_j|^2\,dx \leq C_2^2 \Phi'_j$, by (1.3). Combining this with (5.104) we obtain (5.103). □

Proof of Proposition 1.4: This also follows from the inequality (5.96). Specifically, using (5.96) and (4.59), we obtain

$$\begin{aligned}
\mathrm{Var}(\Gamma) &\leq \frac{1}{2}\sum_{j\in D_L} \mathbb{E}[|\Delta_j \Gamma|^2] \\
&\leq \frac{(\zeta_{max}-\zeta_{min})^2}{2} \sum_{j\in D_L} \mathbb{E}\left[\sup_{\zeta_j}\left|\frac{\partial \Gamma}{\partial \zeta_j}\right|^2\right] \\
&\leq \frac{(\zeta_{max}-\zeta_{min})^2}{2} C_2^2 L^{-2d} \sum_{j\in D_L} \mathbb{E}[\sup_{\zeta_j}(\Phi'_j)^2].
\end{aligned}$$

By Lemma 2.2, $\mathbb{E}[\sup_{\zeta_j}(\Phi'_j)^2] \leq C\mathbb{E}[(\Phi'_j)^2]$. By stationarity of ϕ and Lemma 4.2, $\mathbb{E}[(\Phi'_j)^2] = \mathbb{E}[(\Phi'_0)^2] \leq C\mathbb{E}[(\Phi_0)^2]$. Hence

$$\mathrm{Var}(\Gamma) \leq CL^{-2d}\sum_{j\in D_L}\mathbb{E}[(\Phi_0)^2] = CL^{-d}\mathbb{E}[(\Phi_0)^2].$$

□

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