Random regular digraphs: singularity and spectrum

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Probability Seminar, Stanford University
November 2, 2015

Talk outline

1. Universality for (global eigenvalue statistics of non-hermitian) random matrices
2. Random regular digraphs (adjacency matrices), and conjectured limiting spectral distributions
3. Two results:
   - Circular law for signed random regular digraphs
   - Bound on singularity probability for random regular digraphs

The circular law for i.i.d. matrices

Definition (i.i.d. matrix)

Let x be a $\mathbb{C}$-valued random variable with

$$\mathbb{E}\, x = 0, \qquad \mathbb{E}\, |x|^2 = 1.$$

For each n, let $X_n = (x_{ij})_{1 \le i,j \le n}$ have entries that are i.i.d. copies of x.

Theorem (Mehta, Girko, Edelman, Bai, Bai–Silverstein, Pan–Zhou, Götze–Tikhomirov, Tao–Vu '08)

Let $\{\lambda_k(X_n)\}_{k=1}^n$ be the eigenvalues of X_n.
Define the (rescaled) empirical spectral distribution (ESD) of X_n:

$$\mu_n := \frac{1}{n} \sum_{k=1}^{n} \delta_{\frac{1}{\sqrt{n}} \lambda_k(X_n)}.$$

Almost surely, $\mu_n \Rightarrow \frac{1}{\pi} \mathbb{1}_{B_{\mathbb{C}}(0,1)}\, dxdy$.

The circular law for i.i.d. matrices

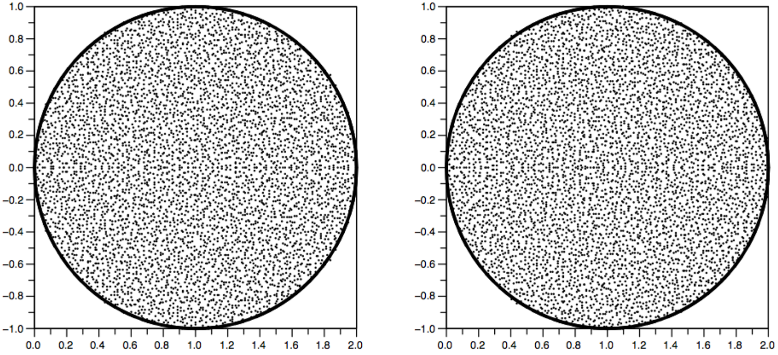
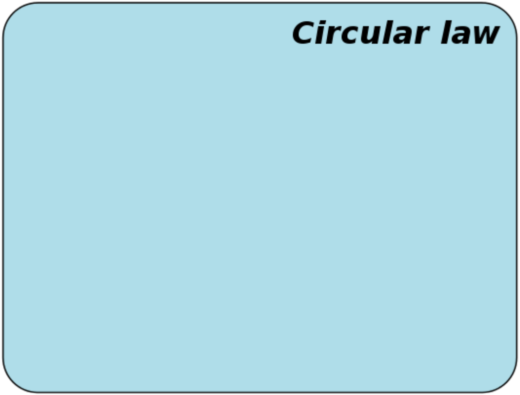
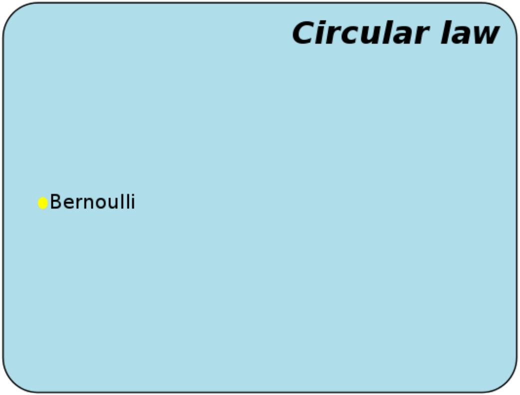


Figure: Circular law universality class: eigenvalue plots for randomly generated 5000×5000 matrices using Bernoulli random variables (left) and Gaussian random variables (right). Figure by Philip Matchett Wood.

The circular law universality class



The circular law universality class



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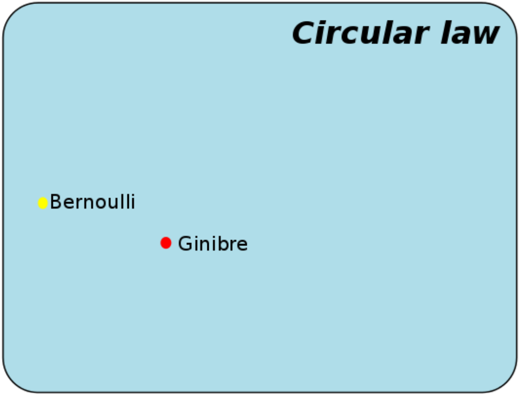

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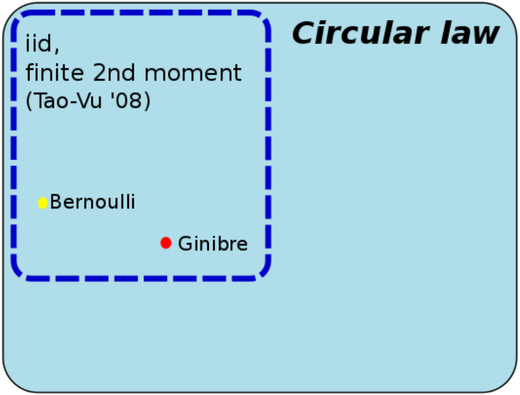

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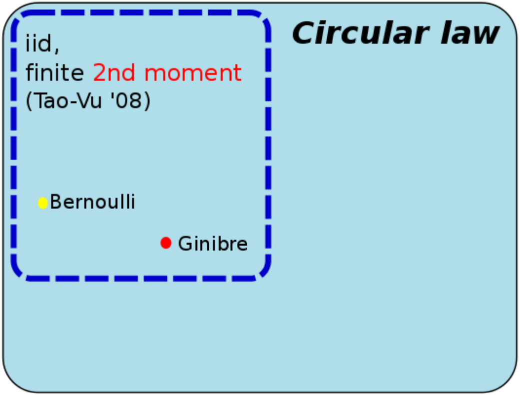

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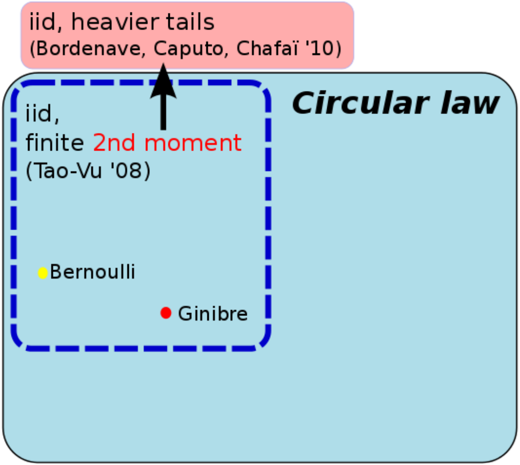

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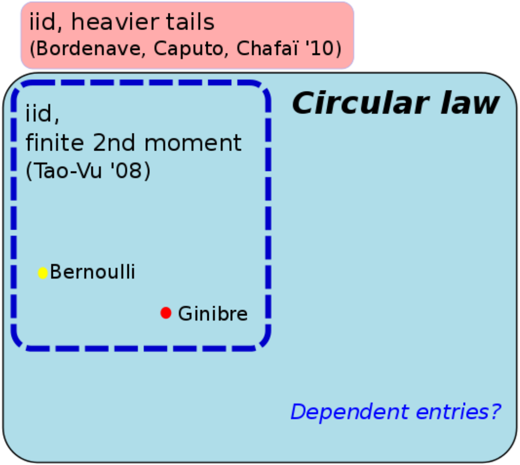

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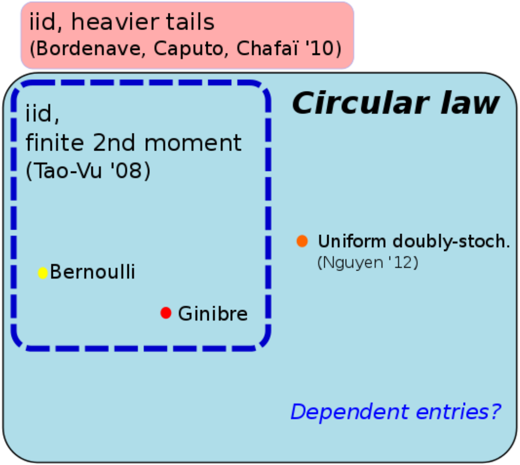

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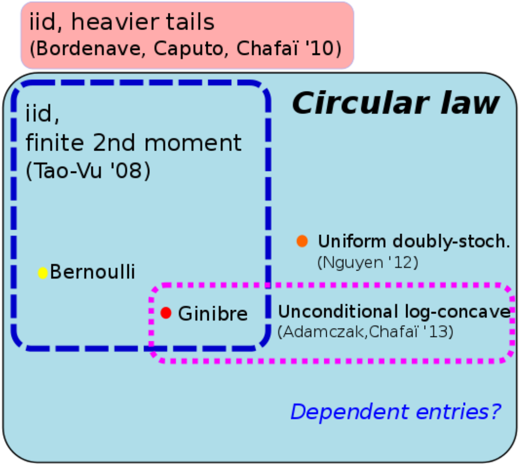

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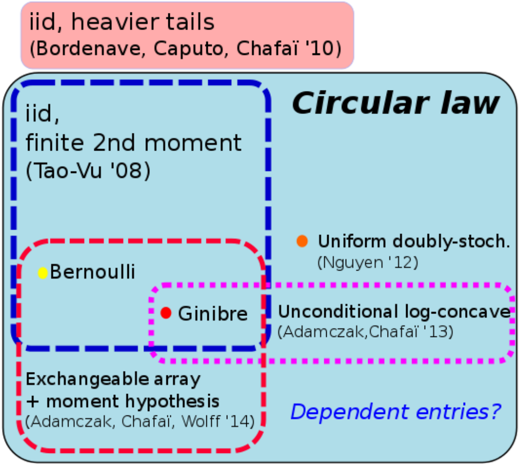


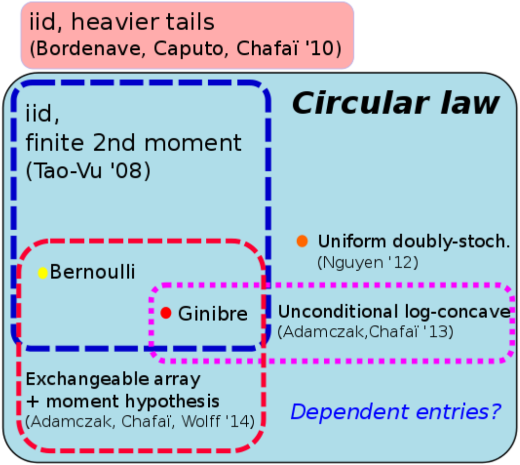
The r.r.d. matrix ensemble

- n large, $d \in [n]$
- $$\mathcal{M}_{n,d} := \Big\{ n \times n \text{ matrices}, \quad \text{entries} \in \{0,1\}, \\ \text{all row and column sums equal to } d \Big\}$$
$$= \big\{ \text{adjacency matrices of } d\text{-regular digraphs on } n \text{ vertices} \big\}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Let $A \in \mathcal{M}_{n,d}$ uniform random.
"Random regular digraph (r.r.d.) matrix"

The circular law universality class



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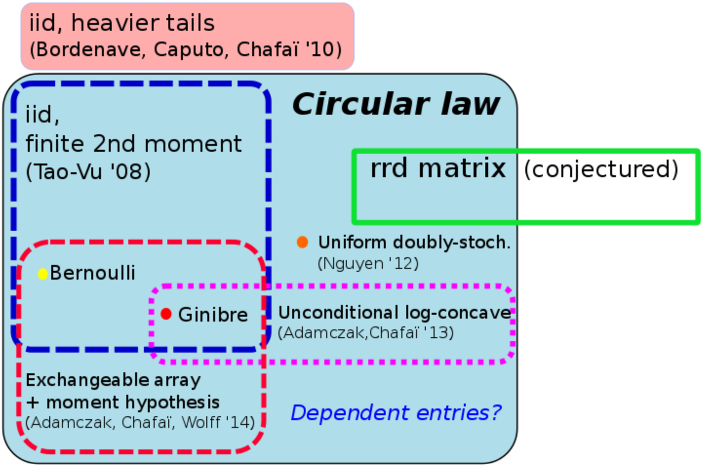

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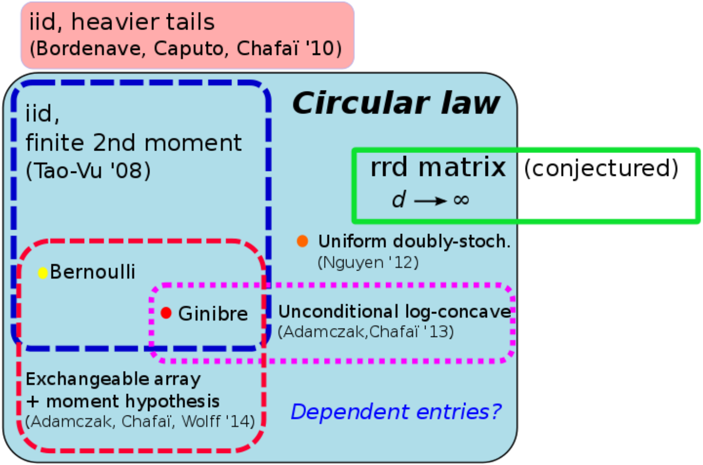

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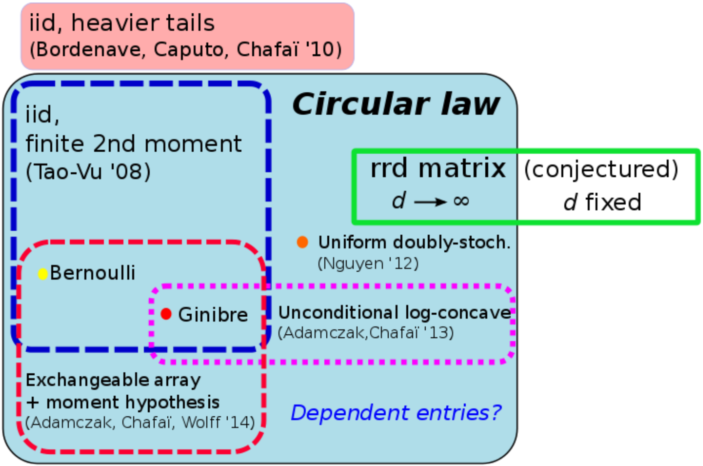

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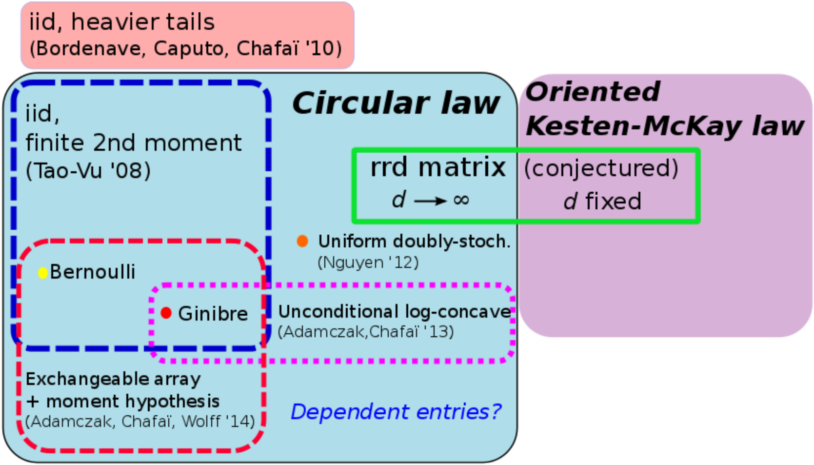

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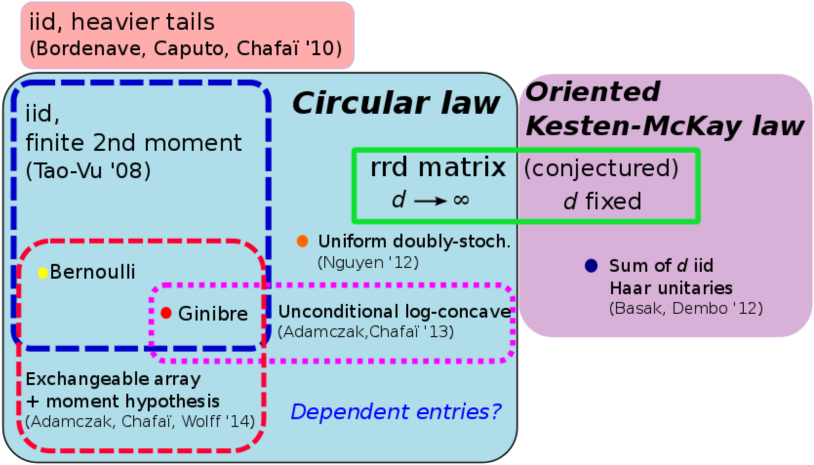

Simulations

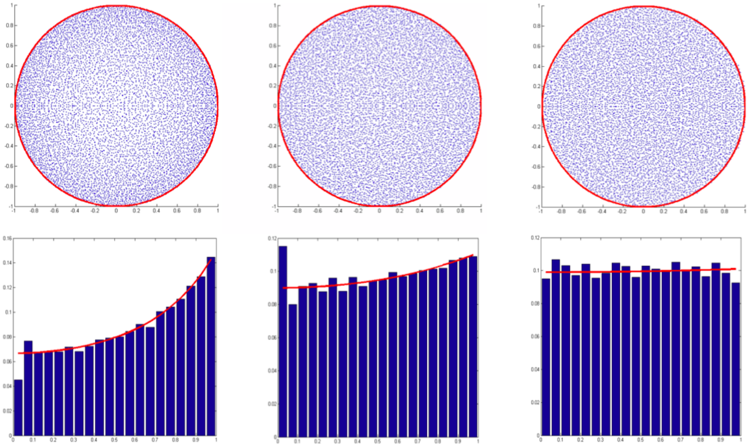

Figure: Empirical eigenvalue distributions for simulated 8000×8000 rescaled r.r.d. matrices $\frac{1}{\sqrt{d}}A$ for $d = 3$ (left), 10 (middle), and 100 (right). Predictions from the oriented Kesten–McKay law are plotted in red.

Circular law for signed r.r.d. matrices

We consider *signed r.r.d. matrices* $A \circ X = (a_{ij}x_{ij})$, where

- $A \in \mathcal{M}_{n,d}$ is an r.r.d. matrix,
- X is an i.i.d. matrix with ± 1 Bernoulli entries, independent of A.

Theorem (C. '15)

Fix $p \in (0,1)$ and put $d = \lfloor pn \rfloor$. Then as $n \to \infty$, the empirical spectral distribution of $\frac{1}{\sqrt{d}} A \circ X$ converges weakly in probability to the uniform measure on $B_{\mathbb{C}}(0,1)$.

- Stated for i.i.d. signs, but the proof only needs the entries of X to have $4 + \varepsilon$ finite moments.
- Work in progress: remove X, extend to sparse case $d = o(n)$ (more on this later).

How does one prove circular laws?

Girko's Hermitization approach

For a Borel probability measure μ, define the *log potential*:

$$U_\mu(z) := \int_{\mathbb{C}} \log|\lambda - z| d\mu(\lambda).$$

Two sides to why this is useful:

1) Borel measures on $\mathbb{C}$ are characterized by their log-potentials:

$$\mu = \frac{1}{2\pi}\Delta U_\mu.$$

2) Determinant identity:

$$\prod_{i=1}^n |\lambda_i(M)| = |\det(M)| = \prod_{i=1}^n s_i(M)$$

where $s_1(M) \geq \cdots \geq s_n(M)$ are the singular values.

How does one prove circular laws?

Putting these together:

- For a sequence of $n \times n$ matrices $(M_n)_{n\geq 1}$, to show $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{\lambda_i(M_n)}$ converges, suffices to show pointwise convergence of

$$U_{\mu_n}(z) = \int_{\mathbb{C}} \log|\lambda - z| d\mu_n(\lambda) = \frac{1}{n}\sum_{i=1}^n \log|\lambda_i(M_n - zI_n)|$$

$$= \frac{1}{n}\sum_{i=1}^n \log s_i(M_n - zI_n) = \int_{\mathbb{R}_+} \log(s) d\nu_{M_n - zI_n}(s).$$

- Gain: $\nu_{M_n - zI_n}$ are ESDs of *Hermitian* random matrices, which are (for our purposes) well understood.
- Loss: $s \mapsto \log(s) \notin BC(\mathbb{R}_+)$, has singularities at 0 and ∞.

Proof outline

For a signed r.r.d. matrix $A_n \circ X_n$, write $\nu_{n,z} = \nu_{\frac{1}{\sqrt{d}}A_n \circ X_n - zI_n}$.

Step 1: Show $\nu_{n,z}$ converges weakly in probability to a deterministic limit ν_z for all $z \in \mathbb{C}$.

i.e. $\forall f \in BC(\mathbb{R}_+)$, $\forall \varepsilon > 0$,

$$\mathbb{P}\left(\left| \int_{\mathbb{R}_+} f \, d\nu_{n,z} - \int_{\mathbb{R}_+} f \, d\nu_z \right| > \varepsilon \right) = o(1)$$

Step 2: Prove bounds on extreme singular values.

2a) Show $s_1(\frac{1}{\sqrt{d}}A_n \circ X_n - zI_n) = O(1)$ with high probability (w.h.p.)

2b) Show $s_n(\frac{1}{\sqrt{d}}A_n \circ X_n - zI_n) \geq n^{-C}$ w.h.p.

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Step 1: weak convergence of singular value distributions

Step 1: prove weak convergence of empirical singular value distributions

$$\nu_{n,z} = \nu_{\frac{1}{\sqrt{d}}A\circ X - zI} = \frac{1}{n}\sum_{i=1}^{n} \delta_{s_i(\frac{1}{\sqrt{d}}A\circ X - zI)}.$$

Idea (following Tran–Vu–Wang '10):

- Replace A with a 0/1 matrix

$$B = (b_{ij})_{1\le i,j\le n}, \quad b_{ij} \text{ i.i.d. Bernoulli}(d/n)$$

  independent of X. $B \circ X$ has i.i.d. entries.

- Note $A \stackrel{d}{=} B | \{B \in \mathcal{M}_{n,d}\}$.

  For a "bad event" $\mathcal{B}$ we can bound

$$\mathbb{P}(A \in \mathcal{B}) = \mathbb{P}(B \in \mathcal{B} | B \in \mathcal{M}_{n,d}) \le \frac{\mathbb{P}(B \in \mathcal{B})}{\mathbb{P}(B \in \mathcal{M}_{n,d})}.$$

Step 1: a comparison trick

For a "bad event" $\mathcal{B}$ we can bound

$$\mathbb{P}(A \in \mathcal{B}) = \mathbb{P}(B \in \mathcal{B} | \mathcal{E}_{n,d}) \leq \frac{\mathbb{P}(B \in \mathcal{B})}{\mathbb{P}(B \in \mathcal{M}_{n,d})}.$$

Lemma (Tran)

$\mathbb{P}\left(B \in \mathcal{M}_{n,d}\right) = \exp\left(- O(n\sqrt{d})\right).$

Want to show: for any $f \in BC(\mathbb{R}_+)$, $\varepsilon > 0$,

$$\mathbb{P}\left(\left| \int_{\mathbb{R}_+} f \, d\nu_{n,z} - \int_{\mathbb{R}_+} f \, d\nu_z \right| > \varepsilon \right) = o(1)$$

Denoting $\widetilde{\nu}_{n,z} = \nu_{\frac{1}{\sqrt{d}} B \circ X - zI}$, it suffices to show

$$\mathbb{P}\left(\left| \int_{\mathbb{R}_+} f \, d\widetilde{\nu}_{n,z} - \int_{\mathbb{R}_+} f \, d\nu_z \right| > \varepsilon \right) \ll e^{-Cn\sqrt{d}}.$$

Step 1: a comparison trick

Want to show: $\mathbb{P}\left(\left|\int_{\mathbb{R}_+} f\, d\widetilde{\nu}_{n,z} - \int_{\mathbb{R}_+} f\, d\nu_z\right| > \varepsilon\right) \ll e^{-Cn\sqrt{d}}$.

- Desired bound is too small to apply work of Bourgade–Yau–Yin '12 on the local law.
- Instead we go back to an argument of Guionnet–Zeitouni '00:
  - Lemma: if $f : \mathbb{R}_+ \to \mathbb{R}$ is convex and 1-Lipschitz, then

$$F = B \mapsto \int_{\mathbb{R}^+} f d\nu_{\frac{1}{\sqrt{d}}B\circ X - zI}$$

    is convex and 1-Lipschitz on $\mathcal{M}_n(\mathbb{C})$ (in Frobenius norm).
  - Applying Talagrand's isoperimetric inequality:

$$\mathbb{P}\left(|F(B) - \mathbb{E}\, F(B)| \geq \varepsilon\right) = O(e^{-c_\varepsilon nd}).$$

    Extend to general f by an approximation argument.
- This argument applies for A drawn uniformly from any set $\mathcal{S} \subset \mathcal{M}_n(\{0,1\})$ satisfying $\mathbb{P}(B \in \mathcal{S}) \geq \exp\left(-o(nd)\right)$.

Step 2: smallest singular value

- Consider a random $n \times n$ matrix of the form

$$M = A \circ X + B$$

with: X i.i.d., A fixed $0/1$ matrix, B fixed.

- We control the lower tail of $s_n(M)$ under a quasirandomness hypothesis on A ("super-regularity", c.f. Szemerédi's regularity lemma).

Theorem (C. '15)

Assume A satisfies [quasirandomness hypothesis], $\|B\| = O(\sqrt{n})$, and $|x_{ij}| = O(1)$ for all $i, j \in [n]$. Then for all $t > 0$,

$$\mathbb{P}\left(s_n(M) \le tn^{-1/2}\right) \lesssim t + n^{-1/2}.$$

- Similar result by Rudelson–Zeitouni for the case that x_{ij} are Gaussian, under a weaker expansion-type assumption on A.
- From (C. '14): the r.r.d. matrix A is super-regular w.h.p.

Extension to sparse, unsigned r.r.d. matrix?

- We can extend the argument for Step 1 (convergence of singular value distributions) to the r.r.d. matrix A with $d = n^{\varepsilon}$.
- The main difficulty is to obtain control of the least singular value.
- In this direction we have the following:

Theorem (C. '14)

There are absolute constants $C, c > 0$ such that the following holds. If $C \log^2 n \leq d \leq \frac{n}{2}$, then

$$\mathbb{P}(s_n(A) = 0) = O(d^{-c}).$$

(We can take $c = .05$.)

Conjecture

There are constants $C, c > 0$ such that for any $d \in [3, n-3]$,

$$\mathbb{P}(s_n(A) = 0) \leq Cn^{-c}.$$

Spectral concentration from classical concentration

- Proofs of upper bounds on $s_1(M) = ||M||_{op}$ reduce to an application of *concentration of measure*.
- Proofs of lower bounds on $s_n(M) = ||M^{-1}||_{op}^{-1}$ reduce to the application of *anti-concentration* or "small ball" estimates.

Theorem (Anti-concentration for random walks, Erdős '40s)

Let $\xi_1, \dots, \xi_n$ be i.i.d. uniform Bernoulli signs, and $x \in \mathbb{R}^n$. Then for any $a \in \mathbb{R}$,

$$\mathbb{P}\left(\sum_{j=1}^{n} \xi_j x_j = a\right) \lesssim \left|\{j : x_j \neq 0\}\right|^{-1/2}.$$

- More sophisticated bounds have been developed by Tao–Vu and Rudelson–Vershynin using *Inverse Littlewood-Offord theory*.
- This is our hammer – where is the nail?

Local symmetries: switchings (after McKay)

- In a regular digraph, we can change between

  at vertices i_1, i_2, j_1, j_2 and preserve d-regularity.
- In the adjacency matrix, this corresponds to switching between

$$\mathbf{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J}_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

  at the $(i_1, i_2) \times (j_1, j_2)$ minor.
- Idea: apply several independent switchings, encode outcomes with i.i.d. signs ξ_j.

Where is the nail?

$$
\begin{array}{c|ccccccccccccc|}
 & \multicolumn{3}{c}{\overbrace{\hphantom{1\cdots1}}^{Co(1,2)}} & \multicolumn{4}{c}{\overbrace{\hphantom{1\,1\cdots1}}^{Ex(1,2)}} & \multicolumn{4}{c}{\overbrace{\hphantom{0\,0\cdots0}}^{Ex(2,1)}} & & \\
 & & & & & j & & & & \pi(j) & & & & \\
1 & 1 & \cdots & 1 & 1 & {\color{red}1} & \cdots & 1 & 0 & {\color{red}0} & \cdots & 0 & 0 & \cdots \\
2 & 1 & \cdots & 1 & 0 & {\color{red}0} & \cdots & 0 & 1 & {\color{red}1} & \cdots & 1 & 0 & \cdots \\
 & & & & \vdots & & & & & & \vdots & & &
\end{array}
$$

Conditional on $R_3, \ldots, R_n$, the only randomness is in the choice of sets $Ex(1,2)$, $Ex(2,1)$.

Let $\pi : Ex(1,2) \to Ex(2,1)$ uniform random bijection.

Conditional on π, independently resample the 2×2 minors $M_{(1,2)\times(j,\pi(j))}$.

Where is the nail?

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccc}
\multicolumn{3}{c}{\overbrace{\qquad\qquad\qquad}^{Co(1,2)}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{Ex(1,2)}} & \multicolumn{4}{c}{\overbrace{\qquad\qquad\qquad\qquad}^{Ex(2,1)}} & & \\
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\end{array} \\
\begin{array}{c} 1 \\ 2 \\ \\ \end{array} & \left[\begin{array}{ccccccccccccc}
1 & \cdots & 1 & 1 & {\color{red}0} & \cdots & 1 & 0 & {\color{red}1} & \cdots & 0 & 0 & \cdots \\
1 & \cdots & 1 & 0 & {\color{red}1} & \cdots & 0 & 1 & {\color{red}0} & \cdots & 1 & 0 & \cdots \\
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\end{array}\right]
\end{array}
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 & & & & \multicolumn{5}{c}{\xrightarrow{\quad\quad\quad\quad\pi\quad\quad\quad\quad}} & & & \\
\multicolumn{3}{c}{\overbrace{\hphantom{1\quad\cdots\quad 1}}^{Co(1,2)}} & \multicolumn{4}{c}{\overbrace{\hphantom{1\quad 0\quad\cdots\quad 1}}^{Ex(1,2)}} & \multicolumn{4}{c}{\overbrace{\hphantom{0\quad 1\quad\cdots\quad 0}}^{Ex(2,1)}} & \\
 & & & & j & & & & \pi(j) & & &
\end{array} \\
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1 & \cdots & 1 & 1 & \color{red}{0} & \cdots & 1 & 0 & \color{red}{1} & \cdots & 0 & 0 & \cdots \\
1 & \cdots & 1 & 0 & \color{red}{1} & \cdots & 0 & 1 & \color{red}{0} & \cdots & 1 & 0 & \cdots \\
 & & & \vdots & & & & & & \vdots & & &
\end{matrix}\right]
\end{array}
$$

In the randomness of the resampling, $R_1 \cdot u$ is a random walk with steps $u_j - u_{\pi(j)}$. (Found the nail!)

Key technical proposition: normal vectors u have *small level sets*.

Combining this with the randomness of π guarantees most steps are non-zero.

What if $Ex(1,2)$ is small?

Concentration and expansion properties

- Problem: what if vertices 1, 2 have large codegree?
  - Solution: use the method of exchangeable pairs for concentration of measure (Chatterjee '06) with a "reflection" coupling to show codegrees concentrate around d^2/n.
- Also obtain control on edge densities:

  For $S, T \subset [n]$ and $\varepsilon \geq 0$,

$$\mathbb{P}\left(\left|\frac{e(S,T)}{\frac{d}{n}|S||T|} - 1\right| \geq \varepsilon\right) \leq 2\exp\left(-\frac{c\varepsilon^2}{1+\varepsilon}\frac{d}{n}|S||T|\right).$$

- In recent work with Larry Goldstein and Toby Johnson, we obtain exponential tail bounds for more general statistics using size biased couplings.
- Allowed us to extend a bound $\lambda_2(A) = O(\sqrt{d})$ on the second eigenvalue of a random regular (undirected) graph to allow $d = O(n^{2/3})$ (previous results were limited to $d = o(n^{1/2})$).

Summary of toy problem

- To show
$$\mathbb{P}\left(R_1 \in \operatorname{span}(R_3, \dots, R_n)\right) = o(1)$$
we defined a coupling $(M, \widetilde{M}, \pi, \xi)$ on an enlarged probability space, with $M \overset{d}{=} \widetilde{M}$, and sought to show
$$\mathbb{P}\left(\widetilde{R}_1 \in \operatorname{span}(R_3, \dots, R_n) \,\middle|\, M\right) = o(1).$$

1. *The randomness of M:* $Ex(1,2)$ is large with high probability.
2. *The randomness of π:* the random walk $\widetilde{R}_1 \cdot u$ takes many steps with high probability.
3. *The randomness of $\xi = (\xi_1, \dots, \xi_n)$* (encoding the resampling of 2×2 minors): used with Erdős' anti-concentration bound to finish the proof.