Pseudospectrum and spectral anti-concentration: beyond mean field models

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Partly based on joint work with Walid Hachem, Jamal Najim and David Renfrew

Outline of the talk

1. Background: Limiting ESDs and Controlling the Pseudospectrum
2. Spectral anti-concentration via the Schwinger–Dyson equations
3. Spectral anti-concentration: geometric approach

Background: Limiting ESDs and Controlling the Pseudospectrum

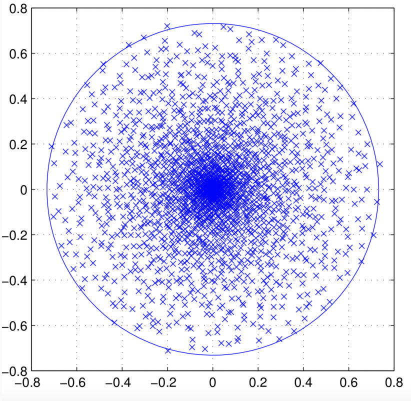
Structured non-Hermitian random matrices

- We study ESDs $\mu_M = \frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j(M)}$ for $n \times n$ matrices

$$M = \frac{1}{\sqrt{n}} A \odot X = (\frac{1}{\sqrt{n}} a_{ij}\xi_{ij})$$

  with ξ_{ij} iid, standardized, and deterministic weights $a_{ij} \in [0,1]$.
  For this talk take $a_{ij} \in \{0,1\}$.
- Ex: For $A \equiv \mathbf{1}\mathbf{1}^{\mathsf{T}}$, $\mu_M \to$ circular law (almost surely).
- Applications: stability analysis for large dynamical systems (ecology, neuroscience...). There is interest in allowing A to be structured and sparse.

Simulated ESDs for $M = (\frac{1}{\sqrt{n}} a_{ij} \xi_{ij})$

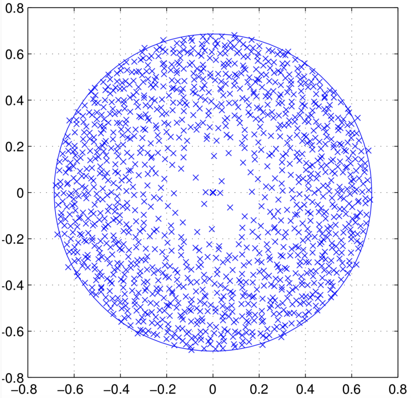


$n = 2000$

$\xi \in \{\pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}\}$ uniform.

$a_{ij} = \sigma(\frac{i}{n}, \frac{j}{n})$, with
$\sigma(x, y) = (x + y)^2 1(|x - y| \leq 0.1)$

Simulated ESDs for $M = (\frac{1}{\sqrt{n}} a_{ij} \xi_{ij})$



$n = 2001$

$\xi \in \{\pm \frac{1}{\sqrt{2}} \pm i \frac{1}{\sqrt{2}}\}$ uniform.

$$A_n = (a_{ij}) = \begin{pmatrix} 0 & \mathbf{1}_{n/3} & \mathbf{1}_{n/3} \\ \mathbf{1}_{n/3} & 0 & 0 \\ \mathbf{1}_{n/3} & 0 & 0 \end{pmatrix}.$$

Log-characteristic polynomials and Hermitization

To show ESDs of M converge weakly (a.s. or in probability) to μ, it's enough to show

$$\frac{1}{n}\log|\det(M-z)| = \frac{1}{n}\sum_{j=1}^{n}\log|\lambda_j(M)-z| \to \int_{\mathbb{C}}\log|\lambda-z|d\mu(\lambda) \quad a.e.\ z\in\mathbb{C}.$$

(Recover μ by taking Laplacian.)

Left hand side can be re-expressed as

$$\frac{1}{2n}\log\det[(M-z)^*(M-z)] = \int_0^\infty \log(s)d\mu_{|M-z|}(s).$$

(This is the template for the construction of Brown's spectral measure for non-normal elements of a W^* probability space.)

We need to be worried about very small singular values of $M-z$, i.e. large singular values of the resolvent $(M-z)^{-1}$.

(ε-)*pseudospectrum*: $\Lambda_\varepsilon(M) = \Lambda(M)\cup\left\{z\in\mathbb{C} : \|(M-z)^{-1}\|_{op}\geq 1/\varepsilon\right\}$

Structured random matrices have small pseudospectrum

Theorem (C. '16)
Let $M = \frac{1}{\sqrt{n}} A \odot X = (\frac{1}{\sqrt{n}} a_{ij}\xi_{ij})$ *with* $a_{ij} \in [0,1]$ *deterministic,* ξ_{ij} *iid, centered, unit variance and* $\mathbb{E}|\xi_{ij}|^{4+\varepsilon} < \infty$*. There are constants* $C(\varepsilon, r, R), c(\varepsilon) > 0$ *such that for any* $z \in \mathbb{C}$ *with* $|z| \in [r, R] \subset (0, \infty)$*,*

$$\mathbb{P}(\|(M-z)^{-1}\|_{op} \geq n^C) \lesssim_{\varepsilon,r,R} n^{-c}$$

- Earlier works on invertibility make strong pseudorandomness assumptions on A, don't use the shift (optimal condition found by Rudelson–Zeitouni '13, see also C. '16).
- *Question:* Proof gives tower dependence of C on $1/r$. Can we do better?

Spectral anti-concentration via the Schwinger–Dyson equations

Controlling other large singular values of $(M-z)^{-1}$

- To prove uniform integrability of $\log(s)$ by $\mu_{|M-z|}$, need more than just $\|(M-z)^{-1}\|_{op} = n^{O(1)}$. Would like a *spectral anti-concentration* estimate

$$\mu_{|M-z|}([0,\eta]) \lesssim \eta \qquad w.h.p.\ \forall \eta \geq n^{-c}.$$

- It's enough to bound

$$\frac{1}{\eta}\mu_{\boldsymbol{M}_z}([-\eta,\eta]) \lesssim \frac{1}{2n}\sum_{j=1}^{2n} \frac{\eta}{\lambda_j(\boldsymbol{M}_z)^2+\eta^2} = \frac{1}{2n}\mathrm{Im}\ \mathrm{Tr}(\boldsymbol{M}_z - i\eta)^{-1} \lesssim 1$$

for all $\eta \geq n^{-c}$, where $\boldsymbol{M}_z = \begin{pmatrix} 0 & M-z \\ M^* - \bar{z} & 0 \end{pmatrix}$.

Schwinger–Dyson equations

- In the mean field setting $A = \mathbf{1}\,\mathbf{1}^\mathsf{T}$, Stieltjes transform satisfies a (cubic) scalar polynomial equation.
- In non-mean field case, we have $\operatorname{Im}(\boldsymbol{M}_z - i\eta)^{-1} \approx \operatorname{diag}(\boldsymbol{u}, \boldsymbol{v})$, where $\boldsymbol{u}, \boldsymbol{v} : [n] \to \mathbb{R}_+$ solve the cubic polynomial system

$$
\begin{aligned}
-\frac{1}{\boldsymbol{u}} &= \eta + S\boldsymbol{v} - \frac{|z|^2}{\eta + S^\mathsf{T}\boldsymbol{u}} \\
-\frac{1}{\boldsymbol{v}} &= \eta + S\boldsymbol{u} - \frac{|z|^2}{\eta + S^\mathsf{T}\boldsymbol{v}}
\end{aligned}
$$

and $S(i,j) = \frac{1}{n} a_{ij}^2$.

Theorem (C., Hachem, Najim, Renfrew '16)

Assume $A = (a_{ij})$ is robustly irreducible. Then for any fixed $z \neq 0$,

$$
\|\boldsymbol{u}\|_\infty, \|\boldsymbol{v}\|_\infty \lesssim_{|z|} 1
$$

uniformly for $\eta > 0$.

Boundedness of solutions to the S–D equations

Idea: (Recall $a_{ij} \in \{0,1\}$.) View A as the adjacency matrix of a digraph. Up to constants, the S–D equations give the local constraints

$$\boldsymbol{u}(j) \asymp \min\left\{\langle \boldsymbol{u}\rangle_{\mathcal{N}_{in}(j)},\ \frac{1}{\langle \boldsymbol{v}\rangle_{\mathcal{N}_{out}(j)}}\right\}, \qquad \boldsymbol{v}(j) \quad \asymp \min\left\{\langle \boldsymbol{v}\rangle_{\mathcal{N}_{out}(j)}, \frac{1}{\langle \boldsymbol{u}\rangle_{\mathcal{N}_{in}(j)}}\right\}.$$

We also have a global trace identity

$$\sum_{j\in[n]} \boldsymbol{u}(j) = \sum_{j\in[n]} \boldsymbol{v}(j).$$

1. Assume $\boldsymbol{u}(j_0)$ is large for some $j_0 \in [n]$.
2. Use the robust irreducibility assumption and the local constraints to "propagate" this property to almost all other indices j.
3. On the other hand, we have $\boldsymbol{u}(j)\boldsymbol{v}(j) \lesssim 1$ for all j, so we conclude $\boldsymbol{v}(j)$ is small for almost all indices j. But this contradicts the trace identity.

Spectral anti-concentration: geometric approach

Bounding the density of states

Consider random symmetric matrices of the form $W = A \odot X = (a_{ij}\xi_{ij})$ where $\{\xi_{ij}\}_{1\le i\le j\le n}$ are independent, standardized, sub-Gaussian variables, $a_{ij} \in \{0, 1\}$ are deterministic weights.

For an interval $\mathcal{I} \subset \mathbb{R}$, when can we show

$$\mu_{\frac{1}{\sqrt{n}}W}(\mathcal{I}) \lesssim |\mathcal{I}| \qquad w.h.p.?$$

Theorem (Local semicircle law, Erdős–Schlein–Yau '08) Let $A = \mathbf{1}\mathbf{1}^{\mathsf{T}}$. Then with high probability, for any interval $\mathcal{I} \subset \mathbb{R}$ with $|\mathcal{I}| \ge n^{-1+\varepsilon}$,

$$|\mu_H(\mathcal{I}) - \mu_{sc}(\mathcal{I})| = o(|\mathcal{I}|).$$

Recent extensions to non-mean field models satisfying some technical hypotheses by Erdös et al., using careful analysis of associated Schwinger–Dyson equations (a.k.a. the quadratic vector equations).

Spectral anti-concentration under expanding support

Write $\mathcal{N}(i) = \{j \in [n] : a_{ij} = 1\}$, and define the δ-dense boundary of a set $J \subset [n]$ as

$$\mathcal{D}_\delta(J) = \{i \in J^c : |\mathcal{N}(i) \cap J| \geq \delta |J|\}.$$

Say that A is a (δ, ε)-*expander* if

$$|\mathcal{N}(i)| \geq \delta n \quad \forall i \in [n] \qquad \text{and} \qquad |\mathcal{D}_\delta(J)| \geq \min(\varepsilon |J|, |J^c|) \quad \forall J \subset [n].$$

Theorem (C. '18)

Let A be a (δ, ε)-expander for some $\delta, \varepsilon \in (0,1)$. Fix $\tau > 0$. With high probability, for any $\mathcal{I} \subset \mathbb{R} \setminus (-\tau, \tau)$ with $|\mathcal{I}| \geq Cn^{-1} \log n$, we have

$$\mu_{\frac{1}{\sqrt{n}} W}(\mathcal{I}) \lesssim |\mathcal{I}|.$$

*Note the assumptions on A permit an atom at 0.

*Could reach intervals of length $n^{-1}\sqrt{\log n}$ using recent ideas of Nguyen '17.

Spectral anti-concentration: ideas of the proof

Fix $\mathcal{I} \subset \mathbb{R}$ away from 0 of length η. We want to bound the spectral concentration event

$$Conc(W, n_0) = \{W \text{ has } \geq n_0 \text{ eigenvalues in } \sqrt{n}\mathcal{I}\}.$$

Idea: can use interlacing and weak delocalization of eigenvectors to reduce to bounding the probability that

$$s_{\min}(\Pi W_{J^c,J}) \leq C\eta n/\sqrt{m}.$$

where J is a random set of size $m \ll n_0$ and Π is a spectral projectionf or W_{J^c} of rank $\geq n_0 - m$.

Key: Π and $W_{J^c,J}$ are independent. We've reduced to invertibility of an $(n_0 - m) \times m$ (tall) projection $\Pi W_{J^c,J}$ of a (very tall) submatrix $W_{J^c,J}$.

Main technical step: show no-gaps delocalization holds for the eigenvectors of W_{J^c}.