

# Universality of the minimum modulus for random trigonometric polynomials

(work with Hoi Nguyen)

## General questions:

- ① For a given random analytic function and curve  $y \subset \mathbb{C}$ ,  
what is the size & distribution of  $\min_{z \in y} |F(z)|$  ?

- ② How are near-minimizers distributed?

We consider  $y = S^1$  and the Kac polynomial:

$$F_n(z) = \sum_{j=0}^n \xi_j z^j. \quad (\text{among others})$$

with  $\xi_j$  iid,  $E\xi_j = 0$ ,  $E|\xi_j|^2 = 1$ , sub-Gaussian. e.g. Gaussian (over  $\mathbb{R}$  or  $\mathbb{C}$ ) or Rademacher  $\pm 1$

Much work has focused on the # of real zeros (for  $\xi_j \in \mathbb{R}$ )

Up to 1980:

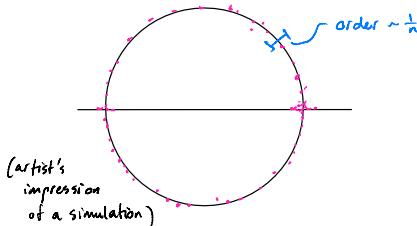
Bloch-Polya '32, Littlewood-Offord '40s, **(Rademacher case)**

- motivation: modeling Dirichlet series for  $\begin{cases} \text{Legendre symbols}, \\ \text{Liouville function}, \\ \text{M\"obius function}. \end{cases}$

Kac '50s:  $\sim \frac{2}{\pi} \log n$  for Gaussian case.

Erdős-Offord '56, Ibragimov-Maslava, Maslava (70s)

## Complex zeros:



Shepp & Vanderbei '95:  
intensity for zeros point process  
 $\sum_{z \in Z(n)} \delta_z$

Michalek - Sahasrabudhe '20: **(Gaussian case)**

$$\sum_{z \in Z(n)} \delta_{z^n (|z|-1)} \longrightarrow \text{PPP}(\frac{1}{12}) \quad \text{on } \mathbb{R}$$

## The minimum modulus on the circle

Rescale + parametrize:

$$f_n(t) = \frac{1}{\sqrt{n+1}} \sum_{j=0}^n \xi_j e^{i jt}, \quad t \in [-\pi, \pi] \quad (e(i\theta) = e^{i\theta})$$

$$m_n := \min_{t \in [-\pi, \pi]} |f_n(t)|.$$

For  $\xi_j$  Rademacher ( $\pm 1$ ):

- Littlewood conj.:  $m_n = o(1)$  whp. ✓
- Kashin '87:  $P(m_n > (\log n)^{-1/3}) = o(1)$ .
- Konyagin '94:  $P(m_n > n^{-1+\epsilon}) = o(1) \quad \forall \epsilon > 0$
- Konyagin-Schlag '99:  $P(m_n < \epsilon/n) \approx \epsilon + o(1)$

Theorem (Yakir, Zeitouni '20)

For  $\xi_j$  real or complex Gaussian,  $\forall x > 0$ ,

$$P(m_n > x/n) \longrightarrow e^{-\lambda x} \quad \lambda = 2\sqrt{\pi/3}.$$

Theorem (C., Nguyen '20)

For  $\xi_j$  sub-Gaussian,  $\forall x > 0$ ,

$$\left| P(m_n > \frac{x}{n}) - P_{N(0,1)}^{\text{R or C}}(m_n > x/n) \right| \longrightarrow 0.$$

In fact the proofs uncover a Poisson point process...

\* sub-Gaussian assumption mainly for convenience

- a sufficiently high moment would suffice

\* Rademacher case captures all of the challenges

- needs the whole proof.

(just as it does for invertibility of random matrices).

(Digression) At the other extreme...

The maximum modulus for Kac polynomials:

Salem-Zygmund '54  
Halász '73  
Kahane '84

Characteristic polynomials:

$$X_n(t) = \det(U_n - tI_n), \quad t \in [-\pi, \pi] \quad U_n \sim \text{Haar measure}$$

Model for:

$$\downarrow \begin{matrix} n \\ \log T \end{matrix}$$

$$S\left(\frac{1}{2} + i(\tau + t)\right), \quad t \in [-1, 1] \quad \tau \in [T, 2T] \text{ uniform}$$

For  $X_n$ : Arguin-Belius-Bourgade, Paquette-Zeitouni,

Chhaibi-Madaule-Najnudel '16

For  $\xi$ : AB + Rademacher (analogous upper bound)

(following ABBR-Soundararajan, Harper)

All: follow Brascamp paradigm for log-correlated fields

Max modulus for  $X_n$

LDPs

1st + 2nd moment  
(truncated)

Haar / permutation  
(C-Zeitouni)

Stationary + smooth

Min modulus for Kac

CLTs

all moments  
(also truncated)

Gaussian / Rademacher

Stationary + smooth  
(for C)  
near roots of unity

Some proof ideas ( $YZ + CN$ )

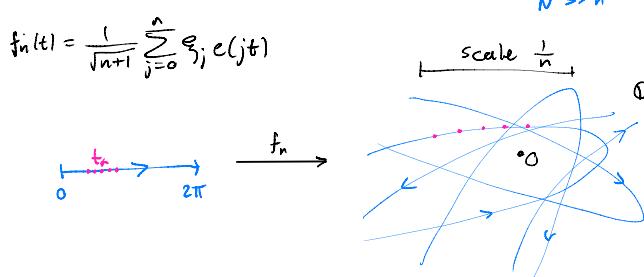
I. Find a point process of  $x$  local minima at scale  $1/n$ .

Discretize: Let  $t_\alpha = \frac{2\pi\alpha}{N}$ ,  $1 \leq \alpha \leq N = \lfloor n^{2-\varepsilon} \rfloor$ .

$$I_\alpha = [t_\alpha - \frac{\pi}{N}, t_\alpha + \frac{\pi}{N}]$$

Komagai + Schlag: careful union bound

For distribution, union bound is too wasteful:



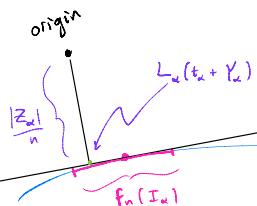
How to capture local minima?

Let linearization of  $f_n$  around  $t_\alpha$

Select  $t_\alpha$  if  $|Y_\alpha| < \frac{\pi}{N}$ .

Point process on  $\mathbb{R}$ :  $\mu_n = \sum_{\alpha=1}^N \delta_{X_\alpha}$

$$X_\alpha = Z_\alpha \mathbf{1}(|Y_\alpha| < \frac{\pi}{N}) + \infty \mathbf{1}(|Y_\alpha| > \frac{\pi}{N})$$



( $\vee$  II. Control on characteristic function)

### III. Comparison in phase space

Task has been reduced to showing

$$\mu_n = \sum_{\alpha=1}^N \delta_{X_\alpha} \Rightarrow \text{PPP}(\sqrt{\frac{\pi}{3}}) \text{ on } \mathbb{R}$$

Do this by computing moments of  $\mu_n(J)$  for intervals  $J \subset \mathbb{R}$ .

$$\mathbb{E} \mu_n(J)(\mu_n(J) - 1) \cdots (\mu_n(J) - m+1) = \sum_{\substack{\alpha_1, \dots, \alpha_m \\ \text{distinct in INT}}} P(X_{\alpha_1}, \dots, X_{\alpha_m} \in J)$$

$$\text{Claim} \longrightarrow \left( \sqrt{\frac{\pi}{3}} |J| \right)^m$$

Gaussian computation:

$$P(X_\alpha \in J) = P_{N(0,1)} \left( n \cdot \frac{\text{Im}(f_n(t_\alpha) \overline{f_n'(t_\alpha)})}{|f_n'(t_\alpha)|} \in J, \left| \frac{\text{Re}(f_n(t_\alpha) \overline{f_n'(t_\alpha)})}{|f_n'(t_\alpha)|^2} \right| \leq \frac{\pi}{N} \right)$$

$$= P_{N(0,1)} \left( (f(t_\alpha), f'(t_\alpha)) \in D_{n,J} \right) = (1+o(1)) \sqrt{\frac{\pi}{3}} \frac{|J|}{N}$$

cpt domain, piecewise smooth boundary

Prop Fix observables  $\phi_1, \dots, \phi_m : \mathbb{R}^d \rightarrow \mathbb{R}$  with  $\text{spts} \subset B(0, n^{O(1)})$ ,  $\|\nabla \phi_i\| = n^{O(1)}$   
+ times  $t_1, \dots, t_m \in [0, 2\pi]$  that are smooth + spread (TBD).

$$\text{then } \left| \left( \mathbb{E} - \mathbb{E}_{N(0,1)} \right) \prod_{i=1}^m \phi_i(f(t_i), f'(t_i)) \right| \leq O\left(\frac{1}{n}\right) \prod_{i=1}^m \int \phi_i d\mu_{\text{obs}}$$

Lemma: If  $t_1, \dots, t_m$  are well separated, then can take  $\mathbb{E}_{N(0,1)}$  under  $\prod_{i=1}^m$   
by  $> n^{1+\varepsilon}$ .

Prop: whp, times  $t_\alpha$  where  $X_\alpha \in J$  are well-separated

## II. Control on characteristic functions.

For prop, want to show distribution of the random vector

$$(f_n(t_1), \dots, f_n(t_m), \frac{f'_n(t_1)}{n}, \dots, \frac{f'_n(t_m)}{n}) \approx N_{\mathbb{R}^m}(0, \Sigma)$$

at fine scales  $n^{-C}$ .

But this is not true for arbitrary  $(t_1, \dots, t_m)!$

Consider position  $f_n(y) = \frac{1}{\sqrt{n+1}} \sum_{k=0}^n \xi_k e(kt)$  at a single time:

$$\begin{aligned} \text{at } t=0 : f_n(0) &= \frac{1}{\sqrt{n+1}} \sum_{k=0}^n \xi_k \xrightarrow{\text{at the slowest possible rate!}} N_{\mathbb{R}}(0, 1) \\ \text{at } t=\frac{\pi}{2} : f_n\left(\frac{\pi}{2}\right) &\xrightarrow{\text{slowly}} N_{\mathbb{R}}(0, 1) \quad \text{only looks Gaussian above scale } \sqrt{n}. \end{aligned}$$

To reach scales  $n^{-C}$  need

$$\left\| \frac{q_i t_i}{\pi} \right\|_{\mathbb{R}/\mathbb{Z}} > n^{5/4} \quad \forall q_i \in \mathbb{Z} \setminus \{0\}, |q_i| \leq n^5 \quad (\text{smooth})$$

$$\text{and } \left\| \frac{t_i \pm t_j}{\pi} \right\|_{\mathbb{R}/\mathbb{Z}} > n^{-1/8} \quad \forall 1 \leq i \neq j \leq m \quad (\text{spread})$$

$$\begin{aligned} \text{Then } W_n(t_1, \dots, t_m) &:= (f_n(t_1), \dots, f_n(t_m), \frac{f'_n(t_1)}{n}, \dots, \frac{f'_n(t_m)}{n}) \\ &= \frac{1}{\sqrt{n+1}} \sum_{j=0}^n \xi_j (e(jt_1), \dots, e(jt_m), \frac{1}{n} e(jt_1), \dots, \frac{1}{n} e(jt_m)) \\ &=: \frac{1}{\sqrt{n+1}} \sum_{j=0}^n \xi_j v_j \quad \approx \text{Gaussian at scales } n^{-C}, C > 0 \text{ arb.} \end{aligned}$$

Follows from

$$\otimes |\mathbb{E} e(\langle W_n(t_1, \dots, t_m), q \rangle)| = O_{A, B}(n^{-A}) \quad \forall n^c \leq \|q\| \leq n^B$$

(Inverse) Littlewood-Offord theory ( $\subset$  additive combinatorics)

For a collection of "steps"  $a_1, \dots, a_n$  in an additive group,

$$W = \sum_{j=1}^n \xi_j a_j \quad \text{concentrates} \iff$$

$\{a_1, \dots, a_n\}$  has additive structure

For singularity probability of random matrices,

$(a_1, \dots, a_n) \in \mathbb{R}^n$  is a normal vector to a random hyperplane.  
unlikely to have structure (Tao-Vu, Rudelson-Vershynin, ...)

For  $\otimes$ , need to rule out additive structure for steps

$$V = (v_1, \dots, v_n), \quad v_j = (e(jt_1), \dots, e(jt_m), \frac{1}{n} e(jt_1), \dots, \frac{1}{n} e(jt_m)).$$

(This is the main technical step of the proof.)

Show  $q^T V = (\langle q, v_1 \rangle, \dots, \langle q, v_n \rangle)$  avoids lattice  $\mathbb{Z}^n$   
for  $n^c \leq \|q\| \leq n^B$

Even showing it avoids  $\mathcal{Q}$  (i.e.  $\sigma_{\min}(V) \gtrsim n$ ) is nontrivial when  
 $t_1, \dots, t_m$  only slightly spread!  
( $\Rightarrow$  anti-concentration for the Gaussian case).