Circular laws for random regular digraphs

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Non-Hermitian random matrices

- Wigner 1950s: used random matrix to model spectrum of Hamiltonian for large nucleus.
  - Hamiltonians are Hermitian.
- Are non-Hermitian random matrices relevant to physics?

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  - Hamiltonians are Hermitian.
- Are non-Hermitian random matrices relevant to physics?

  Yes:
  - Open quantum systems
  - Quantum chromodynamics
  - Stability analysis for large dynamical systems (food networks, neural networks)

- Fix a random variable $\xi \in \mathbb{C}$ with $\mathbb{E}\,\xi = 0$, $\mathbb{E}\,|\xi|^2 = 1$. For each n let X_n be $n \times n$ matrix of iid copies of ξ.
- **Theorem** (Tao–Vu '08): Almost surely, the rescaled ESDs

$$\mu_{\frac{1}{\sqrt{n}}X_n} = \frac{1}{n}\sum_{j=1}^{n} \delta_{\lambda_j(\frac{1}{\sqrt{n}}X_n)}$$

  converge weakly to $\frac{1}{\pi}1_{B(0,1)}dxdy$ as $n \to \infty$.
- Builds on ideas of (Girko '84) and (Bai '97), and previous advances of (Pan–Zhou '07) and (Götze–Tikhomirov '07). Complex Gaussian case goes back to work of Ginibre from 1960s.

The Circular Law

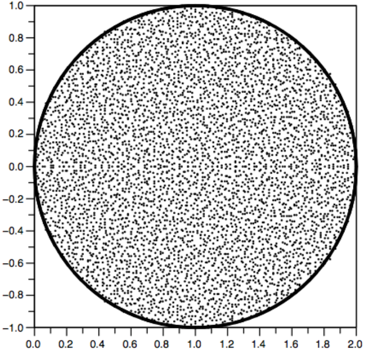

Figure: Circular law universality class: eigenvalue plots for randomly generated 5000×5000 matrices using Bernoulli random variables (left) and Gaussian random variables (right). Figure by P.M. Wood.

$$\dot{x}_i = -\alpha x_i + \sigma \sum_{j \neq i} S_{ij} x_j, \qquad 1 \leq i \leq n.$$

- n is large. Synaptic matrix S encodes (asymetric) interaction strength between nodes i and j.
- "Transition to chaos" when S has eigenvalues λ_i with $\Re\lambda_i > \alpha/\sigma$.
- S is difficult to specify in practice.
- (May '72) modeled S by a random matrix with iid entries. Used the Circular Law to argue food networks become unstable when the number of species is larger than the critical value α^2/σ^2.

Circular Law: Extensions and Universality

For better models of the real world, need random matrices that are sparse and inhomogeneous, with possibly dependent entries.

- Sparsity:
  - Götze–Tikhomirov '07, Tao–Vu '07, Wood '10, Basak–Rudelson '17
  - Also work of Bordenave–Caputo–Chafaï '10 on matrices with heavy tails
- Inhomogeneity:
  - C.–Hachem–Najim–Renfrew '16, Alt–Erdős–Krüger '16
- Weakly-dependent entries:
  - Elliptical law: Nguyen–O'Rourke '12
  - Stochastic matrices: Bordenave–Caputo–Chafaï '08, Nguyen '12
  - Log-concave distribution: Adamczak '12, Adamczak–Chafaï '13
  - Exchangeable entries: Adamczak–Chafaï–Wolff '14

Random regular graphs

- Consider an $n \times n$ random matrix $A = (a_{ij})$ with non-negative integer entries constrained to have all row and column sums equal to d.
  - Interpret A as the adjacency matrix for a d-regular directed multigraph.
  - Entries in $\{0, 1\}$: simple graph (allowing loops).
  - A symmetric: undirected graph.
- Random regular graphs have recently become a popular class of models in universality theory. Dependency structure calls for development of flexible arguments.
- Some recent advances:
  - Local semicircle/Kesten–McKay law: Dumitriu–Pal '09, Tran–Vu–Wang '10, Bauerschmidt–Knowles–Yau '15, Bauerschmidt–Huang–Yau '16
  - Sine kernel universality: Bauerschmidt–Huang–Knowles–Yau '15
  - Invertibility: C. '14, Litvak–Lytova–Tikhomirov–Tomczak-Jaegermann–Youssef '15, '17
  - Circular law: C. '17, Basak–C.–Zeitouni '17

For $d \geq 2$ define the *oriented Kesten–McKay law* as the measure on $\mathbb{C}$ with density

$$f_d(z) = \frac{1}{\pi} \frac{d^2(d-1)}{(d^2 - |z|^2)^2} \mathbf{1}_{B(0,\sqrt{d})}(z).$$

This is the *Brown measure* of the *free sum* of d *Haar unitary operators*.

Conjecture (Bordenave–Chafaï / folklore)

Fix $d \geq 2$ and let $P_n^1, \dots, P_n^d$ be iid uniform random $n \times n$ permutation matrices. As $n \to \infty$, the ESDs for $\frac{1}{\sqrt{d}}(P_n^1 + \cdots + P_n^d)$ converge in probability to to the oriented Kesten–McKay law.

The same should hold for the adjacency matrix of a uniform random d-regular digraph by contiguity.
Note that after rescaling by $\sqrt{d}$, as $d \to \infty$ the density f_d tends to the circular law.

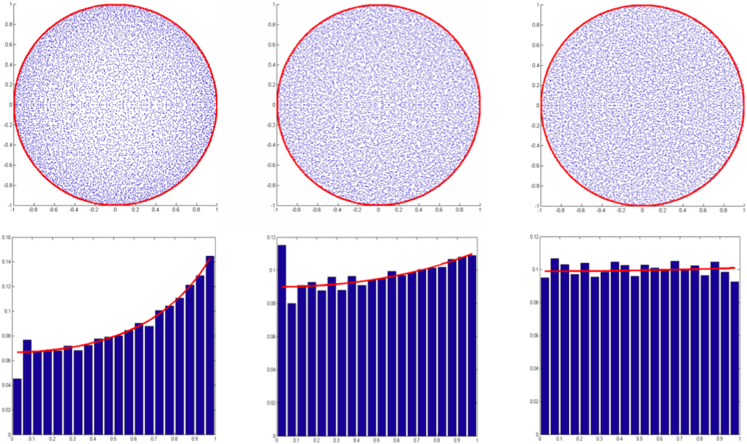

Figure: Empirical eigenvalue distributions for simulated 8000×8000 rescaled random regular digraph matrices $\frac{1}{\sqrt{d}}A$ for $d = 3$ (left), 10 (middle), and 100 (right). Predictions from the oriented Kesten–McKay law are plotted in red.

Uniform model:

Theorem (C. '17)

Let $\log^C n \le d \le n/2$ and let A_n be the adjacency matrix for a uniform random d-regular digraph on n vertices. Then the ESDs for $\frac{1}{\sqrt{d}}A_n$ converge in probability to the circular law.

Permutation model:

Theorem (Basak–C.–Zeitouni '17)

Let $\log^{16+\varepsilon} n \le d \lesssim n$, let $P_n^1, \dots, P_n^d$ be iid uniform random $n \times n$ permutation matrices, and put $S_n = P_n^1 + \cdots + P_n^d$. Then the ESDs for $\frac{1}{\sqrt{d}}S_n$ converge in probability to the circular law.

- Write M_n for $\frac{1}{\sqrt{d}}A_n$ or $\frac{1}{\sqrt{d}}S_n$. For any nice test function $f : \mathbb{C} \to \mathbb{R}$,

$$\int_{\mathbb{C}} f(z) d\mu_{M_n}(z) = \int_{\mathbb{C}} \Delta f(z)(\mu_{M_n} * \log)(z) dm(z)$$
$$= \int_{\mathbb{C}} \Delta f(z) \left(\frac{1}{n} \log |\det(M_n - z)| \right) dm(z).$$

- Hermitize: Letting $\boldsymbol{M}_n^z = \begin{pmatrix} 0 & M_n - z \\ M_n^* - \overline{z} & 0 \end{pmatrix}$, we have

$$\frac{1}{n} \log |\det(M_n - z)| = \frac{1}{2n} \log |\det \boldsymbol{M}_n^z| = \int_{\mathbb{R}} \log |x| d\mu_{\boldsymbol{M}_n^z}(x).$$

- Goal: show $\int_{\mathbb{R}} \log |x| d\mu_{\boldsymbol{M}_n^z}(x)$ converges in probability for a.e. $z \in \mathbb{C}$.

Bai's approach

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- Bai '97: for a.e. $z \in \mathbb{C}$, prove
  1. measures $\mu_{\boldsymbol{M}_n^z}$ converge in probability (to the right limit);
  2. (weak Wegner-type estimate) w.h.p., $\mu_{\boldsymbol{M}_n^z}(-t,t) = O(t)$ for all $t \geq n^{-c}$;
  3. $|\lambda_{\min}|(\boldsymbol{M}_n^z) = s_{\min}(M_n - z) \geq n^{-O(1)}$ w.h.p.
- Steps 1 and 2 both follow from quantitative convergence of Stieltjes transforms:

$$g_n^z(w) := \frac{1}{2n}\operatorname{Tr}(\boldsymbol{M}_n^z - w)^{-1} = \int_{\mathbb{R}} \frac{d\mu_{\boldsymbol{M}_n^z}(x)}{x-w}.$$

  Arguments for uniform and permutation models are completely different!
- Step 3 builds on earlier work on the invertibility problem (C. '14, LLTTY '15), using methods from Geometric Functional Analysis.

Comparison approach:

$$\begin{array}{ccccc} A_n & \longrightarrow & B_n & \longrightarrow & G_n \\ & & \text{iid Bernoulli}(d/n) & & \text{iid Gaussian} \end{array}$$

- $A_n \to B_n$: use an argument from Tran–Vu–Wang '10. Let $\mathcal{A}_{n,d}$ denote the set of d-regular digraph matrices. For a "bad" event $\mathcal{E}_n$,

$$\mathbb{P}(A_n \in \mathcal{E}_n) = \mathbb{P}(B_n \in \mathcal{E}_n | B_n \in \mathcal{A}_{n,d}) \leq \frac{\mathbb{P}(B_n \in \mathcal{E}_n)}{\mathbb{P}(B_n \in \mathcal{A}_{n,d})}.$$

  - Upper bound on $\mathbb{P}(B_n \in \mathcal{E}_n)$: Use Talagrand's concentration inequality applied to linear statistics of Bernoulli random matrices (as in Guionnet–Zeitouni '01).
  - Lower bound on $\mathbb{P}(B_n \in \mathcal{A}_{n,d})$ (i.e. lower bound on $|\mathcal{A}_{n,d}|$), following an argument of Shamir and Upfal.
- $B_n \to G_n$ Lindeberg replacement argument (following Chatterjee '06).

- Idea: use the group structure.
- Let $\boldsymbol{R} = \boldsymbol{R}^{z,w} = (\boldsymbol{M}^z - w)^{-1}$. Let T be $n \times n$ matrix for transposition of ℓ and m, and let $\widetilde{\boldsymbol{M}}^z$, $\widetilde{\boldsymbol{R}}$ be obtained by replacing P^k with $P^k T$. Then $\widetilde{\boldsymbol{M}}^z = \boldsymbol{M}^z + \frac{1}{\sqrt{d}}\Delta$, where

$$\Delta = \begin{pmatrix} 0 & P^k(T-I) \\ (T-I)^{\mathsf{T}}(P^k)^{\mathsf{T}} & 0 \end{pmatrix}.$$

- From the resolvent identity,

$$\boldsymbol{R} - \widetilde{\boldsymbol{R}} = \frac{1}{\sqrt{d}}\boldsymbol{R}\Delta\widetilde{\boldsymbol{R}} = \frac{1}{\sqrt{d}}\boldsymbol{R}\Delta\boldsymbol{R} - \frac{1}{d}\boldsymbol{R}\Delta\boldsymbol{R}\Delta\widetilde{\boldsymbol{R}} + \frac{1}{d^{3/2}}\boldsymbol{R}\Delta\boldsymbol{R}\Delta\boldsymbol{R}\Delta\widetilde{\boldsymbol{R}}.$$

- Taking expectations, the left hand side becomes zero. Then specialize to the (ℓ, m) entry, and average over k, ℓ, m. Yields approximate self-consistent relations between traces over the four $n \times n$ block sub-matrices of $\boldsymbol{R}$.