A beginner's guide to A_∞ algebras and Fukaya categories

Note Title 2/13/2017

Vague motivation: Homological Mirror Symmetry conjecture (Kontsevich '94)

Kähler mfd: (M, J, ω) M = even-dim mfd, J = integrable complex structure, ω = symplectic form

Certain Kähler manifolds come in "mirror pairs" $(M,J,\omega) \leftrightarrow (M^\vee, J^\vee, \omega^\vee)$.

Physics: Consider "branes" on Calabi-Yau mfds: come in two flavors.

A-model: branes = Lagrangian submfds wrt ω

B-model: branes = cx analytic submfds wrt J

Kontsevich HMS: equivalence of triangulated categories.

$$D^b\,\mathrm{Fuk}(M,\omega) \simeq D^b\,\mathrm{Coh}(M^\vee, J^\vee)$$

$$D^b\,\mathrm{Coh}(M,J) \simeq D^b\,\mathrm{Fuk}(M^\vee, \omega^\vee)$$

Coherent sheaf category: objects = coherent sheaves (think: hol. v.b.)
morphisms = Hom or Ext of sheaves

Fukaya category: objects = Lagrangian submfds
morphisms = intersection between these

important additional structure given by (Lagrangian intersection) Floer homology.
$\Rightarrow$ gives Fuk the structure of an A_∞ category.

Goal: define the Fukaya category and examine some properties.

this is the key.

Floer homology = "∞ dimensional version of Morse homology"

Outline:
- Morse homology (overview)
- Symplectic geometry intro: symplectic mfds, Lagrangian submfds, Hamiltonian diffeos, Arnold conjecture
- Lagrangian intersection Floer homology
- A_∞ algebras, A_∞ categories, twisted complexes
- Fukaya categories.

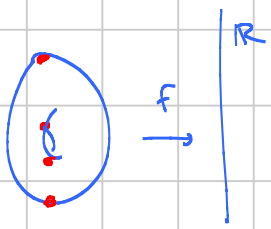
Morse homology review. (see: my minicourse notes)

M = smooth n-dim mfd.

$f: M \to \mathbb{R}$ is Morse if all critical points are nondegenerate.

$p \in M$, $(df)_p = 0$ ↓ Hessian $(Hf)_p$ is nondegenerate: $\det \frac{\partial^2 f}{\partial x_i \partial x_j} \neq 0$

ex: height function on T^2.



If $p \in \text{Crit } f = \{\text{critical pts of } f\}$ then the Morse index

$\text{ind}_f(p) = \#$ negative eigenvalues of $(Hf)_p$. 2, 1, 1, 0

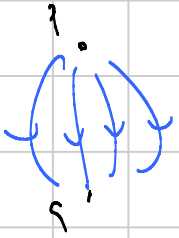
Given a metric $\langle , \rangle$ on M, define ∇f (vector field) as usual: $\langle \nabla f, v \rangle = df(v)$.

For $x \in M$, the (negative) gradient flow line $\gamma_x(t)$ is the flow of $-\nabla f$:

$$\gamma_x(0) = x$$
$$\frac{d}{dt}\gamma_x(t) = -\nabla f(\gamma_x(t)).$$

Note: M cpt $\Rightarrow$ every gradient flow line begins and ends at a crit pt:

$\exists\, p, q \in \text{Crit } f$ s.t. $\lim_{t \to -\infty} \gamma_x(t) = p$, $\lim_{t \to +\infty} \gamma_x(t) = q$.



$p, q \in \mathrm{Crit} f \rightsquigarrow$ define $W(p,q) = \{x \in M \mid \lim_{t\to-\infty} \gamma_x(t) = p, \lim_{t\to+\infty} \gamma_x(t) = q\}$.

We can write this as

$W(p,q) = W^u(p) \cap W^s(q)$

$W^u(p) = \{x \mid \lim_{t\to-\infty} \gamma_x(t) = p\}$, $W^s(q) = \{x \mid \lim_{t\to+\infty} \gamma_x(t) = q\}$.

disk of $\dim = \mathrm{ind}\, p$; disk of $\dim = n - \mathrm{ind}\, q$

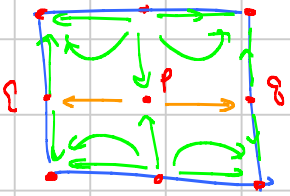
Def $(f, \langle,\rangle)$ is Morse-Smale if $W^u(p) \pitchfork W^s(q)$ $\forall p,q \in \mathrm{Crit} f$:

$\forall x \in W^u(p) \cap W^s(q)$, $T_x W^u(p) + T_x W^s(q) = T_x M$.

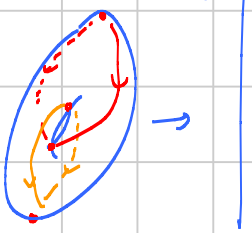
(set of Morse-Smale pairs is dense)

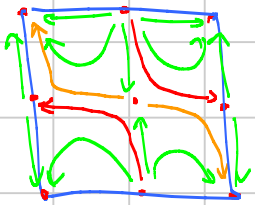
If $(f, \langle,\rangle)$ is Morse-Smale, then $\dim W(p,q) = \mathrm{ind}\, p - \mathrm{ind}\, q$.

not Morse-Smale ; $\dim W(p,q) = 1$



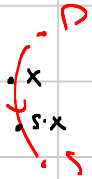
perturb





Note: $\mathbb{R}$ acts on $W(p,q)$ by time translation:

$s \cdot x = \gamma_x(s)$. This action is free (if $p \neq q$).



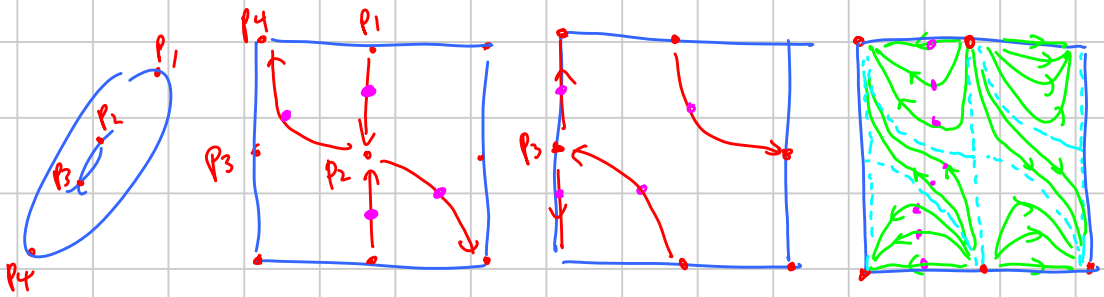
Def The moduli space of flow lines from p to q is

$M(p,q) := W(p,q)/\mathbb{R}$.

This is a smooth mfd of $\dim = \mathrm{ind}\, p - \mathrm{ind}\, q - 1$.

(rmk: given orientations on $W^u(p)$ $\forall p \in \mathrm{Crit} f$, can orient $M(p,q)$ $\forall p,q$)

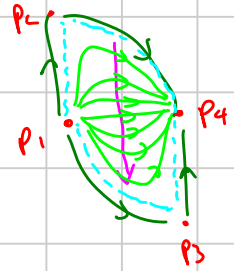
Ex.



$\mathcal{M}(p_1,p_2)=\{2\text{ pts}\}$ $\mathcal{M}(p_1,p_3)=\{2\text{ pts}\}$ $\mathcal{M}(p_1,p_4)=4\times\longleftrightarrow$

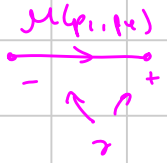
$\mathcal{M}(p_2,p_4)=\{2\text{ pts}\}$ $\mathcal{M}(p_3,p_4)=\{2\text{ pts}\}$

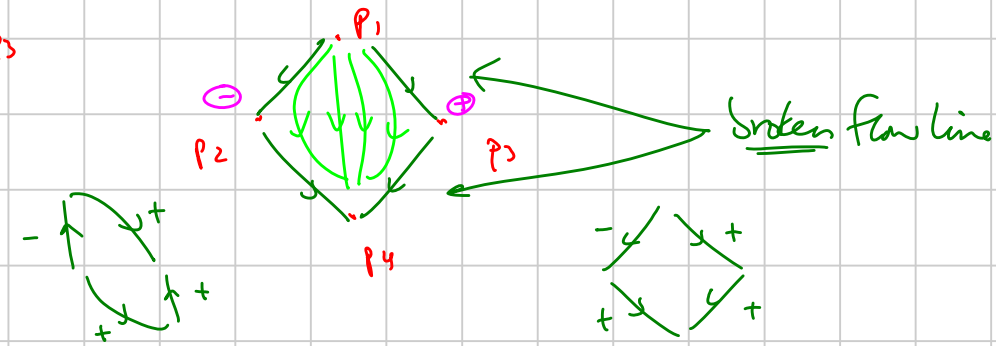
The key to Morse homology: compactification by broken flow lines.



$\mathcal{M}(p_1,p_4)$ = $\cup$ 4 open intervals; can compactify:

$\overline{\mathcal{M}}(p_1,p_4)$ = $\cup$ 4 closed intervals.





Thm 1. If $\operatorname{ind} p = \operatorname{ind} q + 1$ then

$\mathcal{M}(p,q)$ is compact (= finite union of points).

2. If $\operatorname{ind} p = \operatorname{ind} q + 2$ then $\mathcal{M}(p,q)$ has a compactification $\overline{\mathcal{M}}(p,q)$ = cpt 1-mfd with ∂ (= finite union of closed intervals and S^1's)

and

$$\partial\overline{\mathcal{M}}(p,q) = \bigcup_{\substack{\operatorname{ind} r = \operatorname{ind} q + 1 \\ = \operatorname{ind} p - 1}} \mathcal{M}(p,r)\times\mathcal{M}(r,q).$$

(even true with orientations!)

(In fact, in general $\overline{\mathcal{M}}(p,q)$ has a compactification to a mfd with corners whose strata are broken flow lines.)

2/14

Note: components of proof.

① transversality: moduli space have the expected dimension (built into the def)

② Compactness:

②a any sequence of flowlines in $\mathcal{M}(p,q)$ has a subsequence converging to a broken flowline

②b any broken flowline can be perturbed to an honest flowline.
("gluing theorem")

$M, (f,g)$ Morse-Smale $\Rightarrow (CM_*(M;f,g), \partial)$.

Def $CM_k = \mathbb{Z}_2 \mathrm{Crit}_k(f)$ (can take $\mathbb{Z}$, given orientations) $\quad \mathrm{Crit}_k(f) = \{\text{crit pts of index } k\}$

$\partial : CM_k \to CM_{k-1}$ defined by

$$\partial(p) = \sum_{q \in \mathrm{Crit}_{k-1}(f)} \underbrace{(\#\mathcal{M}(p,q))}_{\in \mathbb{Z}_2:\ \#\text{ of pts in } \mathcal{M}(p,q), \text{ counted with sign}} q$$

This is the Morse complex of M associated to (f,g).

Prop $\partial^2 = 0$.

Pf. $p \in \mathrm{Crit}_k f$, $q \in \mathrm{Crit}_{k-2}(f)$ $\Rightarrow$

$$\langle \partial p, q \rangle = \sum_{r \in \mathrm{Crit}_{k-1}(f)} \langle \partial p, r \rangle \langle \partial r, q \rangle$$

($\langle \partial p, q \rangle$ = coeff of q in ∂p)

$$= \#\bigcup_{r \in \mathrm{Crit}_{k-1}(f)} \mathcal{M}(p,r) \times \mathcal{M}(r,q)$$

$$= \# \partial \bar{\mathcal{M}}(p,q)$$

$$= 0. \quad \square$$

Def The Morse homology $HM_*(M;f,g) = H_*(CM_*(M;f,g), \partial)$.
(note depends on (f,g)).

Ex. $M = T^2$. $CM_* = \mathbb{Z}\langle p_1, p_2, p_3, p_4 \rangle$ (indices: p_1: 2, p_2: 1, p_3: 1, p_4: 0)

$\partial(p_1) = p_2 - p_2 + p_3 - p_3$

$\partial(p_2) = p_4 - p_4$

$\partial(p_3) = p_4 - p_4$

$\partial(p_4) = 0$

note $\partial^2(p_1)$ has 8 cancelling terms corresponding to 4 components of $M(p_1, p_4)$

$$HM_* = \begin{cases} \mathbb{Z} & * = 2 \\ \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 0 \end{cases}.$$

Thm M = cpt smooth mfd, (f, g) Morse-Smale. Then

$$HM_*(M; f, g) \cong H_*^{\text{sing}}(M).$$

Corollary:

b_k = k-th Betti number of M = $\dim_{\mathbb{Q}}(H_k(M) \otimes \mathbb{Q})$.

Thm f = Morse function on M. Then

(# critical points of index k) $\geq b_k$

(total # critical points of f) $\geq \sum_{k=0}^{n} b_k$.

Pf Find a metric g s.t. (f, g) is Morse-Smale (needs proof). Then

$$\begin{aligned} (\#\text{ critical points of index } k) &= \operatorname{rank} CM_k(f, g) \\ &\geq \operatorname{rank} HM_k(f, g) \\ &= b_k. \end{aligned}$$

□

Pf that $HM_* \cong H_*^{\text{sing}}$: several approaches, but "standard" one involves sublevel sets and seeing how homology changes as we pass thru crit pt.

A priori invariance

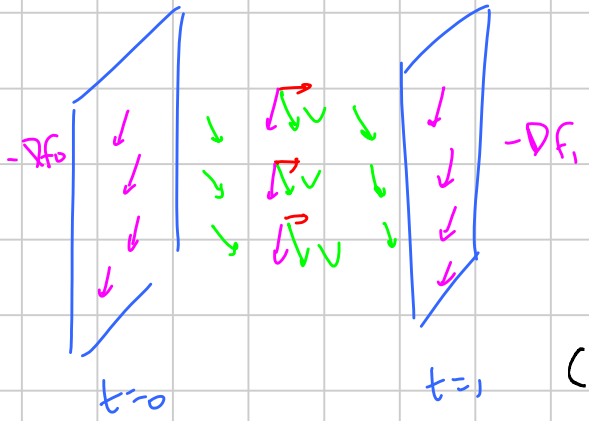
Suppose we didn't know $HM_* \cong H_*^{\text{sing}}$. We'd still deduce some interesting invt of M, provided $HM_*(M; f, g)$ indep of (f, g).

This can be proven directly by continuation (flows).

Suppose (f_0, g_0), (f_1, g_1) are Morse-Smale, and let (f_t, g_t) be a path between them.
On $[0,1] \times M$, let $M_t = \{t\} \times M$; then g_t is a metric on M_t.
Write $V_t = -\nabla_{g_t} f_t$ on M_t. Then define a vector field V on $[0,1] \times M$:

$$V = h(t)\frac{\partial}{\partial t} + V_t$$

$h(t) \geq 0$, $h(0) = h(1) = 0$.
$h(t) > 0$ for $0 < t < 1$



V is actually the negative gradient flow for $F = f_t - \int_0^t h(u)\,du$ with respect to the metric $g_t \oplus (dt \otimes dt)$ on $[0,1] \times M$.

(Technicality: need to perturb so that this is Morse-Smale).

$$\text{Crit}(F) = \underbrace{(\{0\} \times \text{Crit}(f_0))}_{M_0} \cup \underbrace{(\{1\} \times \text{Crit}(f_1))}_{M_1}$$

index $(k+1)$ ← index k
index k ← index k

As usual, $p, q \in \text{Crit}(F) \rightsquigarrow M(p,q) = \{\text{flowlines from } p \text{ to } q\}$.

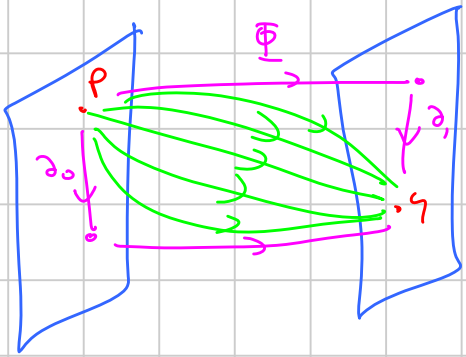
$p, q \in \mathrm{Crit} V \rightsquigarrow \mathcal{M}(p,q) = \{\text{flowlines from } p \text{ to } q\}$.

For $p \in \mathrm{Crit}(f_0)$ with $\mathrm{ind}(p) = k$, define

$$\Phi(p) = \sum_{\substack{q \in \mathrm{Crit}(f_1) \\ \mathrm{ind}\, q = k}} (\#\mathcal{M}(p,q))\, q .$$

Extend Φ linearly to get a map $\Phi : CM_k(f_0, g_0) \to CM_k(f_1, g_1)$.

Prop Φ is a chain map: $\Phi \circ \partial_0 = \partial_1 \circ \Phi$.



Pf Similar to proof that $\partial^2 = 0$.

For $p \in \mathrm{Crit}(f_0)$, $q \in \mathrm{Crit}(f_1)$.

$\mathrm{ind}\, q = \mathrm{ind}\, p - 1$, $\mathcal{M}(p,q)$ has boundary

$$\partial \mathcal{M}(p,q) = \bigcup_{\substack{r \in \mathrm{crit}(f_0) \subset M_0 \\ \mathrm{ind}\, r = \mathrm{ind}\, p - 1}} \mathcal{M}(p,r) \times \mathcal{M}(r,q) \;\cup\; \bigcup_{\substack{r \in \mathrm{Crit}(f_1) \subset M_1 \\ \mathrm{ind}\, r = \mathrm{ind}\, q}} \mathcal{M}(p,r) \times \mathcal{M}(r,q) .$$

□

Note: this says that the Morse complex $CM_*([0,1] \times M, F, g)$ is a mapping cone, $\mathrm{Cone}\left(CM_*(M, f_0, g_0), CM_*(M, f_1, g_1)\right)$.

Next: let Γ, Γ' be two generic paths $\rightsquigarrow \Phi_\Gamma, \Phi_{\Gamma'} : CM_*(f_0, g_0) \to CM_*(f_1, g_1)$.

Prop If Γ, Γ' are homotopic then $\Phi_\Gamma, \Phi_{\Gamma'}$ are chain homotopic:

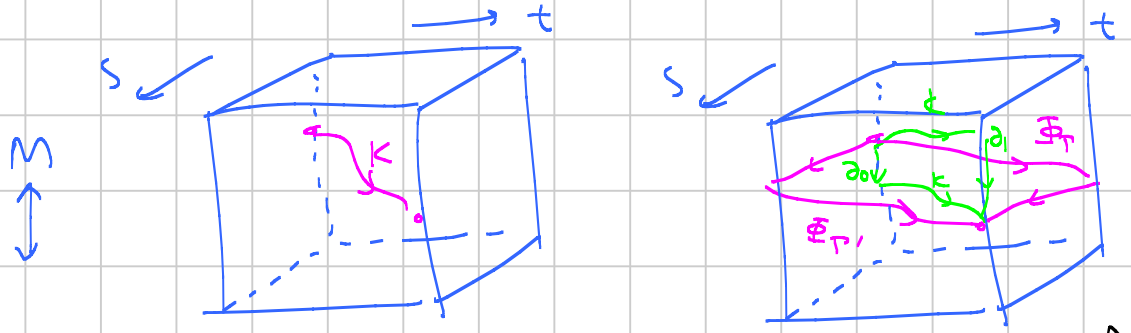
$\exists\ K : CM_*(f_0, g_0) \to CM_*(f_1, g_1)$ such that

$$\partial_1 K + K \partial_0 = \Phi_\Gamma - \Phi_{\Gamma'} .$$

Idea: instead of using $[0,1]\times M$, use $[0,1]^2\times M$. (f_0,g_0) | homotopy | (f_1,g_1)

On $[0,1]^2\times M$, consider gradient flow for $f_t - j(t)-j(s)$ with respect to $g_t + dt\otimes dt + ds\otimes ds$

Homotopy counts flows from $(s,t)=(0,0)$ to $(s,t)=(1,1)$.



boundaries of 1D moduli spaces

Cor $HM_*(f,g)$ indep of (f,g).

Pf. $(f_0,g_0) \xrightarrow{\Gamma} (f_1,g_1) \xrightarrow{-\Gamma} (f_0,g_0)$

$\Gamma\cup(-\Gamma)$ homotopic to constant path.

$\rightsquigarrow$ $HM_*(f_0,g_0) \xrightarrow{\Phi_\Gamma} HM_*(f_1,g_1) \xrightarrow{\Phi_{-\Gamma}} HM_*(f_0,g_0)$, with $\Phi_{-\Gamma}\circ\Phi_\Gamma = \Phi_{\text{const}} = id$ ☐

2/16

Symplectic Geometry background.

Def M^{2n} smooth. A symplectic form is $\omega\in\Omega^2(M)$ with

- $d\omega = 0$
- ω nondegenerate: $\omega^n \in \Omega^{2n}(M)$ is nowhere 0.

(M,ω) is a symplectic manifold.

Important examples:
- Σ = Riemann surface, ω = any area form
- $\mathbb{R}^{2n}$, $\omega = dq_1\wedge dp_1 + \cdots + dq_n\wedge dp_n$

· T^*Q, Q = any smooth mfd, $\omega = -d\lambda$ defined as follows.

$q_1, \dots, q_n$ coords on Q, $p_1, \dots, p_n$ dual coords on T^*Q ($\sum p_i dq_i \in T^*_x Q$)

The Liouville 1-form $\lambda \in \Omega^1(T^*Q)$ is given in coordinates by

$\lambda = \sum p_i dq_i$. $\quad (\Rightarrow \omega = \sum dq_i \wedge dp_i)$

Fact: λ is indep of coordinates.

Coord-free def of λ:

$$\begin{array}{ccc} & & T(T^*Q) \\ & {\scriptstyle d\pi = \pi_*} \swarrow & \quad \searrow {\scriptstyle \pi} \\ TQ & \longleftrightarrow & T^*Q \end{array}$$

$\lambda(v) = \langle \pi_*(v), \pi(v) \rangle$.

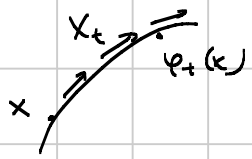
Def A symplectomorphism between (M,ω) and (M',ω') is a diffeo $\varphi : M \to M'$ with $\varphi^*(\omega') = \omega$.

Special case: $\mathrm{Symp}(M,\omega)$ = group of symplectomorphisms $(M,\omega) \circlearrowright$

Infinitesimal symplectomorphisms

X_t = time-dependent vector field on M $\quad (t \in \mathbb{R})$

generates a family of diffeomorphisms $\varphi_t : M \to M$:



$\varphi_0 = id$, $\frac{d}{dt}\varphi_t = X_t \circ \varphi_t$.

Conversely $\{\varphi_t\}$ determines X_t by the same equation.

$(i_X \omega)(v) = \omega(X, v)$

Def A vector field X on M is symplectic if $i_X \omega$ is closed.

Prop (M,ω) symplectic. $\{\varphi_t\}$ = family of diffeos gen'd by $\{X_t\}$ time dep v.f.

Then $\varphi_t \in \mathrm{Symp}(M)\ \forall t \iff X_t$ is symplectic $\forall t$.

Pf Cartan's magic formula.

$\varphi_t \in \mathrm{Symp}(M)\ \forall t \iff \mathcal{L}_{X_t}\omega = 0\ \forall t \iff i_{X_t}\omega$ closed. $\square$

$\mathcal{L}_{X_t}\omega = d i_{X_t}\omega + i_{X_t} d\omega$

Special case: $i_{X_t}\omega$ is exact.

Def $H \in C^\infty(M)$. The Hamiltonian vector field $X_H \in \mathrm{Vect}\, M$ is determined by

$$dH = i_{X_H}\omega.$$

→ this is symplectic

Rmk: By nondegeneracy of ω, we have an isom $TM \cong T^*M$, $X \longmapsto i_X\omega = \omega(X, \cdot)$, and $X_H \longleftrightarrow dH$ under this isom.

If instead we used a metric g to get $TM \cong T^*M$, $X \mapsto g(X, \cdot)$, then $dH \leftrightarrow \nabla H$.

- If we have an almost Kähler triple (ω, J, g), $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$, then $X_H = -J\nabla H$.

If we have $H: M \times [0,1] \to \mathbb{R}$, $H_t = H(\cdot, t)$, then get X_{H_t}, $t \in [0,1]$, and we get a corresponding $\varphi_t \in \mathrm{Symp}\, M$.

Def $\varphi: M \to M$ is a Hamiltonian diffeomorphism if $\varphi = \varphi_1$ for some (time-dependent) Hamiltonian $H: M \times [0,1] \to \mathbb{R}$.

Rmks 1. If $H^1(M, \mathbb{R}) = 0$ then all symplectomorphisms are Hamiltonian diffeomorphisms.

2. $M = T^*Q$, H = any function on M.

$\omega(X_H, \frac{\partial}{\partial q_i}) = dH(\frac{\partial}{\partial q_i}) = \frac{\partial H}{\partial q_i}$ $\rightsquigarrow X_H = \sum \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}$

$\omega(X_H, \frac{\partial}{\partial p_i}) = \frac{\partial H}{\partial p_i}$

So if we flow along X_H, get $\boxed{\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}}$

Hamilton's equation of motion!

(Side consequence: Conservation of energy:

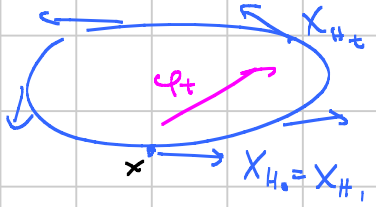
$dH(X_H) = (i_{X_H}\omega)(X_H) = \omega(X_H, X_H) = 0$ so Hamiltonian flow is tangent to level sets of H).

Arnold conjecture

Setting: 1-periodic Hamiltonian H_t on (M,ω):

$H_t : M \to \mathbb{R}$, $H_{t+1} = H_t$. $\rightsquigarrow \varphi_t$ = time t-flow of X_{H_t}

$\rightsquigarrow \varphi_1$ is a Hamiltonian diffeo and fixed pts of φ_1 are periodic orbits of Hamiltonian flow:



Say that a fixed point x of φ_1 is nondegenerate if $\det(1 - d\varphi_1(x)) \neq 0$.

Arnold conjecture (M^{2n}, ω) compact symplectic, H_t 1-periodic. Suppose all fixed pts of φ_1 are nondegenerate. Then

$$\#\mathrm{Fix}(\varphi_1) \geq \sum_{k=0}^{2n} b_k(M).$$

Special case: $H \in \mathcal{H}$ time-independent. Then $x \in \mathrm{Crit}\, H \Rightarrow dH(x) = 0$
$\to X_H(x) = 0 \to x \in \mathrm{Fix}(\varphi_1)$. So

$\#\mathrm{Fix}(\varphi_1) \geq \#\mathrm{Crit}\, H \geq \sum b_k(M)$

↑ Morse inequality.

Arnold conjecture now proven:

1979 Eliashberg Riemann surfaces

1983 Conley-Zehnder for T^{2n}

1988 Floer for $\pi_2(M) = 0$ ←

1989 Floer for monotone symplectic $[\omega] = \lambda c_1(M)$ $\lambda > 0$

≥90s Fukaya-Ono, Liu-Tian, Hofer-Salamon in general.

Lagrangian submanifolds

(M, ω) symplectic $\rightsquigarrow \forall x \in M$, (T_xM, ω) is a symplectic vector space: $\mathbb{R}^{2n}$ equipped with a skew-symm, nondeg bilinear form.

Given (V^{2n}, ω) symplectic, a subspace $W \subset V$ is isotropic if $\omega|_W \equiv 0$.
Linear alg: $\dim W \leq n$. W is Lagrangian if $\omega|_W \equiv 0$ and $\dim W = n$.

Def $L \subset M$ is Lagrangian if $T_xL \subset T_xM$ is Lagrangian $\forall x \in L$:
2/21 ↑ $\omega|_L \equiv 0$, $\dim L = n$.

Ex $M = T^*Q$. Q = zero section in M is Lagrangian.

Lagrangian neighborhood thm If $L \subset (M,\omega)$ is Lagrangian then there is a nbd $N(L)$ s.t. $(N(L),\omega)$ is symplectomorphic to $N(\text{zero section}) \subset T^*L$ where $L \to$ zero section.

More general than 0 sections:

$\alpha \in \Omega^1 Q$ induces a section S_α of T^*Q

T^*Q over Q via π

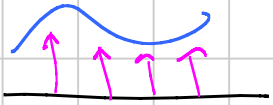
Claim: S_α is Lagrangian $\Leftrightarrow$ α is closed.

Pf: recall Liouville form $\lambda \in \Omega^1(T^*Q)$. Then $\lambda|_{S_\alpha} = \pi^*\alpha$:

at $(x,\alpha(x)) \in T^*Q$, $\lambda(v) = (\alpha(x))(\pi_* v) = (\pi^*\alpha)(v)$.

So $\omega|_{S_\alpha} = 0 \Leftrightarrow d\pi^*\alpha|_{S_\alpha} = 0 \Leftrightarrow \pi^* d\alpha|_{S_\alpha} = 0$ and π^* is an isom. from T_xQ to $T_{(x,\alpha(x))} S_\alpha$ so $\Leftrightarrow d\alpha = 0$. ☐

In particular, the graph Γ_{df} of any function $f \in C^\infty(M)$ is Lagrangian.



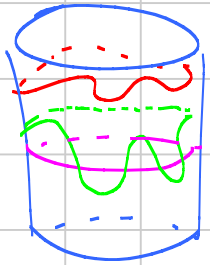
Note for next page: get Γ_{df} from 0 section by flowing along the vector field given in fibers by df itself: this is the Hamiltonian vector field for $H = f \circ \pi : T^*Q \to \mathbb{R}$.

Note: plenty of other Lagrangians.

ex: • cotangent fiber $T^*_x Q$

• $Q = S^1$, $T^*Q = S^1 \times \mathbb{R} = \mathbb{R}^2_{xy}/((x,y) \sim (x+1,y))$, $\lambda = y\,dx$, $\omega = dx \wedge dy$.

Any 1-dim'l submfd is Lagrangian.



Arnold conjecture for Lagrangian intersections

Let $\varphi = \varphi_1$ be a Hamiltonian diffeomorphism $(H_t : M \to \mathbb{R},\ 0 \le t \le 1)$.

Suppose • L is compact Lagrangian
• $L \pitchfork \varphi(L)$
• some additional assumption e.g. $\pi_2(M, L) = 0$.

Then

$$\#(L \cap \varphi(L)) \geq \sum b_k(L)$$ ← mod $\mathbb{Z}_2$-Betti numbers.

Ex $M = T^*Q$, L = 0 section, $\varphi(L) = \Gamma_{df}$; $\Gamma_{df} \cap (0\text{section}) = \mathrm{Crit}(f)$. This is the Morse inequality.
↳ see previous page

Thm (Floer 1988)

(M, ω) cpt spl, L cpt Lagr. If

(1) φ = Hamilt. difféo. ($\Rightarrow$ $\varphi(L)$ Lagrangian)

(2) $L \pitchfork \varphi(L)$

(3) $\forall u : (D^2, \partial D^2) \to (M, L)$, $\int_{D^2} u^*\omega = 0$

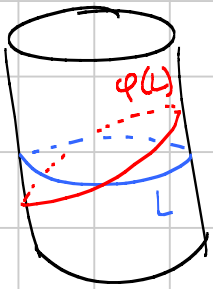
Then $\#(L \cap \varphi(L)) \geq \sum \dim H^k(L, \mathbb{Z}_2)$.

Note: by Stokes, since $\omega|_L = 0$,

$\int_{D^2} u^*\omega = \int_{u(\partial D^2)} \omega$ depends only on the class of u in $\pi_2(M, L)$;

(3) is satisfied if $\pi_2(M, L) = 0$.

Ex $S^1 \times \mathbb{R}$.

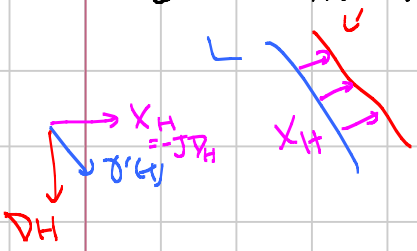


$L = S^1$; Floer says $\#(L \cap \varphi(L)) \geq 2$.

Claim: the signed area bounded by L and $\varphi(L)$ is 0.

Then $\#(L \cap \varphi(L)) \geq 2$ follows.

Claim is true because it's true infinitesimally: $L = \gamma(t)$,



infinitesimal change in area is

$$\int (\gamma'(t) \times X_H(\gamma(t))) \cdot \frac{\partial}{\partial z}\, dt = \int \underbrace{(\gamma'(t) \cdot \nabla H)}_{\frac{d}{dt} H(\gamma(t))}\, dt = 0.$$

Note: not true if:

- φ isn't Hamiltonian: vertical translation

- L is homotopically trivial: bounds a disk of positive area

Special case: apply Floer to $(\tilde{M} = M \times M, \omega \times (-\omega))$.

The diagonal $\Delta = \{(x,x) \mid x \in M\} \subset \tilde{M}$ is Lagrangian.

More generally, if $\varphi: M \to M$ is a symplectomorphism then

$\Gamma_\varphi = \{(x, \varphi(x))\} \subset \tilde{M}$ is Lagrangian.

If $H_t: M \to \mathbb{R}$ then $\tilde{H}_t: M \times M \to \mathbb{R}$

$\tilde{H}_t(x,y) = -H_t(y)$

Hamiltonian flow for $\tilde{H}_t$ is constant in 1st factor, follows Ham. flow in 2nd $\rightsquigarrow$ $\tilde{\varphi}_1: \tilde{M} \to \tilde{M}$

$\tilde{\varphi}_1(\Delta) = \Gamma_{\varphi_1}$

$\Delta \cap \Gamma_{\varphi_1} = \{(x,x) \mid x = \varphi_1(x)\}$.

Cor (Floer 1988) (M,ω) cpt symplectic, $\pi_2(M) = 0$, φ = Hamiltonian diffeo with all fixed pts nondegenerate. Then

$$\#\,\mathrm{Fix}(\varphi) \geq \sum (\mathbb{Z}/2\text{-Betti numbers of } M).$$

2/23

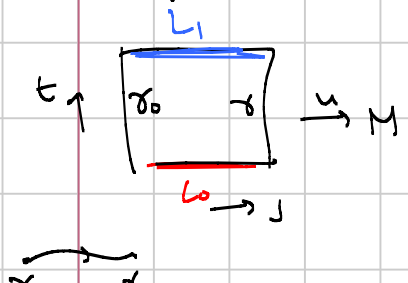
Setting up Lagrangian Floer theory: Motivation.

(M,ω) symplectic, L_0, L_1 Lagrangians, $L_0 \pitchfork L_1$.
(of special interest: $L_1 = \varphi(L_0)$ for $\varphi \in \mathrm{Ham}(M)$)

Write $\mathcal{P}(L_0,L_1)$ = path space from L_0 to L_1
$= \{\gamma : [0,1] \to M \mid \gamma(0) \in L_0,\ \gamma(1) \in L_1\}$

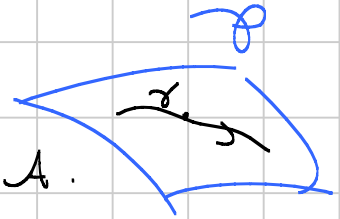
For $\gamma_0 \in \mathcal{P}(L_0,L_1)$, let $\mathcal{P}_{\gamma_0}(L_0,L_1)$ = connected component of $\mathcal{P}(L_0,L_1)$ containing γ_0.
$= \{\gamma \mid \gamma(0) \in L_0, \gamma(1) \in L_1, \gamma$ homotopic (rel endpts on L_0, L_1) to $\gamma_0\}$.

Define $\widetilde{\mathcal{P}}_{\gamma_0}(L_0,L_1)$ = universal cover of $\mathcal{P}_{\gamma_0}(L_0,L_1)$
$= \{u : [0,1]\times[0,1] \to M \mid u(\cdot,0) \in L_0,\ u(\cdot,1) \in L_1,\ u(0,\cdot) = \gamma_0\}$ / homotopy.
Write $\widetilde{\mathcal{P}}$ for short.



Now define an action functional
$\mathcal{A} : \widetilde{\mathcal{P}} \to \mathbb{R}$ by

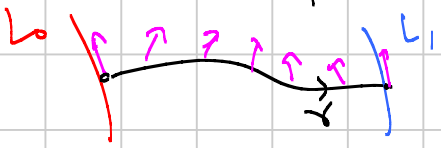
$$\mathcal{A}(u) = -\int u^*\omega = \int_0^1 \int_0^1 \omega\left(\frac{\partial u}{\partial t}, \frac{\partial u}{\partial s}\right) dt\, ds.$$



Goal: perform Morse theory on $\mathcal{P}$ with Morse function $\mathcal{A}$.

Simplifying assumption to make notation nicer: $\mathcal{A}$ descends to $\mathcal{P}(L_0,L_1)$.
(This happens under some topological assumption; else it's well-defined locally up to a constant.)

$$T_\gamma \mathcal{P} = \{\xi : [0,1] \to TM \mid \xi(t) \in T_{\gamma(t)} M,\ \xi(0) \in T_{\gamma(0)} L_0,\ \xi(1) \in T_{\gamma(1)} L_1\}.$$

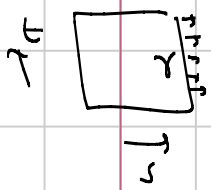


To define gradient flow on $\mathcal{P}$, need a metric.

Def J is an almost complex structure on M compatible with ω if
- $J : TM \to TM$ with $J^2 = -1$
- $\omega(Ju, Jv) = \omega(u,v)$ and $g_J(u,v) := \omega(u, Jv)$ is a (symmetric) pos def bilinear form.

Prop $\exists$ J compatible with ω, and the space $\mathcal{J}(M,\omega)$ of almost cx str compatible with ω is contractible.

Given J compatible with ω, get a metric on $\mathcal{P}$ defined by
$$\langle \xi_1, \xi_2 \rangle = \int_0^1 g_J(\xi_1(t), \xi_2(t))\, dt.$$
Calculate gradient of $\mathcal{A}$:



$$d\mathcal{A}_\gamma(\xi) = \int_{t=0}^1 \omega\left(\frac{d\gamma}{dt}, \xi\right) dt$$
$$= \int_0^1 g_J\left(J\frac{d\gamma}{dt}, \xi\right) dt$$
$$= \left\langle J\frac{d\gamma}{dt}, \xi \right\rangle$$
so $\boxed{\nabla_\gamma \mathcal{A} = J\frac{d\gamma}{dt}}$. (note: Strictly speaking $J\frac{d\gamma}{dt} \notin T_\gamma \mathcal{P}$ because of boundary conditions).

- Critical points of $\mathcal{A}$ satisfy $\frac{d\gamma}{dt} \equiv 0$ so $\gamma = $ const: those are intersection points $L_0 \cap L_1$.
- Gradient flows:

A gradient flow $u: \mathbb{R}_s \to \mathcal{P}$ with $\lim_{s\to-\infty} u = \gamma_p$, $\lim_{s\to+\infty} u = \gamma_q$ can be written as

$u: \mathbb{R}_s \times [0,1]_t \to M$ with:

- $\lim_{s\to-\infty} u(s,[0,1]) = p$
- $\lim_{s\to+\infty} u(s,[0,1]) = q$
- $\frac{\partial u}{\partial s} = -J\frac{\partial u}{\partial t}$.

Now: define complex structure j on $\mathbb{R}\times[0,1]$ as usual: $j\frac{\partial}{\partial s} = \frac{\partial}{\partial t}$, $j\frac{\partial}{\partial t} = -\frac{\partial}{\partial s}$.

Then $du\left(j\frac{\partial}{\partial s}\right) = du\left(\frac{\partial}{\partial t}\right) = J\frac{\partial u}{\partial s} = J\,du\left(\frac{\partial}{\partial s}\right)$

and $du\left(j\frac{\partial}{\partial t}\right) = -\frac{\partial u}{\partial s} = J\,du\left(\frac{\partial}{\partial t}\right)$

so $du\circ j = J\circ du$: u is a holomorphic curve $(\mathbb{R}\times[0,1], j) \to (M, J)$.

Floer: use these hol. curves to define a complex $(CF_*(L_0,L_1), \partial)$.

$CF_*(L_0,L_1) = \mathbb{Z}_2\langle$ intersections of L_0 and $L_1\rangle$, ∂ counts hol. curves:

$\langle \partial p, q\rangle = \#\mathcal{M}(p,q)$ ← hol. strips from p to q

Morse theory	Lagrangian intersection Floer theory
M	$\mathcal{P}(L_0,L_1)$ (or $\widetilde{\mathcal{P}}_{\gamma_0}(L_0,L_1)$)
f	$\mathcal{A}$
f Morse	$L_0 \pitchfork L_1$
$\mathrm{Crit} f$	$L_0 \cap L_1$
Morse index	Maslov index
gradient flows	hol. strips
$HM_*(M; f, g)$	$HF_*(L_0, L_1;$ choices $)$

Note: "Morse index" in ∞-dim setting is usually ∞ since $\dim W^u(p) = \infty$.
But we only need relative indices of crit pts: any map
$\text{ind}: \text{Crit } f \to \mathbb{Z}$ will do as long as it satisfies $\dim \mathcal{M}(p,q) = \text{ind } p - \text{ind } q - 1$.
For Lagrangian case, "renormalization" of index is given by Maslov index.

Steps to proving Floer's result.

1. $HF_*(L_0, L_1)$ is well-defined
   - set up grading, check moduli spaces are manifolds of the expected dimension
     $\rightsquigarrow \partial: CF_* \to CF_{*-1}$ TRANSVERSALITY
   - $\partial^2 = 0$: compactification of moduli spaces COMPACTNESS
     $\rightsquigarrow HF_*(L_0, L_1) = H_*(CF_*, \partial)$
   - check indep of all choices made along the way, e.g. J INVARIANCE
2. behavior under Hamiltonian diffeomorphism
   - if L_1, L_1' are Hamiltonian isotopic ($\exists \varphi \in \text{Ham}(M)$ with $\varphi(L_1) = L_1'$)
     then $HF_*(L_0, L_1) \cong HF_*(L_0, L_1')$ also INVARIANCE
   - if L_0, L_1 are Hamiltonian isotopic
     then $HF_*(L_0, L_1) \cong H_*(L_0)$.

Then: $L = \text{Lag}$, $\varphi \in \text{Ham}(M) \Rightarrow$
$$\#(L \cap \varphi(L)) = \text{rk } CF_*(L, \varphi(L)) \geq \text{rk } HF_*(L, \varphi(L)) = \text{rk } H_*(L).$$

Remarks
- Don't use ∞-dim'l Morse homology directly: lots of analytic difficulties. (eg. long-time existence of gradient flow; transversality, compactness)
- Use cohomology instead of homology (i.e. reverse the arrows, $d: CF^* \to CF^{*+1}$) because of product operations (to come).

Lagrangian Intersection Floer Cohomology

Setup as before: • (M,ω) symplectic
• $L_0, L_1 \subset M$ cpt lagrangian, $L_0 \pitchfork L_1$ ($\Rightarrow \#(L_0 \cap L_1) < \infty$)
• J almost cx str compatible with ω.
(note: the space of such is nonempty + contractible).

Simplest setting: Floer complex $CF^*(L_0,L_1) = \bigoplus_{p \in L_0 \cap L_1} \mathbb{Z}_2 \cdot p$.
(we'll talk more about coefficients later). or some field k.

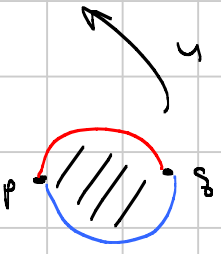
Def $p, q \in L_0 \cap L_1$. $W(p,q)$ is the set of finite energy holomorphic strips between p and q:

- holomorphic strip: $u: \mathbb{R}_s \times [0,1]_t \to M$ satisfying:
  - Cauchy-Riemann equation $\bar\partial_J u = \frac{\partial u}{\partial s} + J \frac{\partial u}{\partial t} = 0$ (i.e. $j\,du = du\,J$)
  - boundary conditions $u(\cdot,0) \in L_0$, $u(\cdot,1) \in L_1$
    $\lim_{s\to\infty} u(s,t) = p$, $\lim_{s\to-\infty} u(s,t) = q$
- finite energy:
$$E(u) := \int_{\mathbb{R}\times[0,1]} u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 ds\,dt < \infty.$$

(*)

2/28

Rmk • $\mathbb{R}\times[0,1]$ is biholomorphic to $\overline{D}^2 - \{\pm 1\}$ and u extends to $\overline{D}^2$ with $+1 \to q$, $-1 \to p$.



• $\mathbb{R}$ acts on $W(p,q)$ by translation in s direction.
$a \in \mathbb{R}$: $u \longmapsto u(s-a,t)$

Def $\mathcal{M}(p,q) = W(p,q)/\mathbb{R}$

For $[u] \in \pi_2(M, L_0 \cup L_1)$, $W(p,q;[u]) = \{u \in W(p,q) \text{ representing } [u]\}$

$\mathcal{M}(p,q;[u]) = W(p,q;[u])/\mathbb{R}$.

The conditions (*) determining a finite energy holomorphic strip determine a Fredholm problem: if u is such a strip, then the linearization $D\bar{\partial}_{J,u}$ of $\bar{\partial}_J$ at u is Fredholm on a suitable space of sections of u^*TM with boundary conditions: eg.

$$W^{1,p}\left(\mathbb{R}\times[0,1], \mathbb{R}\times\{0,1\}; u^*TM, u^*|_{t=0}TL_0 \cup u^*|_{t=1}TL_1\right).$$

Then assuming that all solutions to (*) are regular ($D\bar{\partial}_{J,u}$ is surjective), ↳ TRANSVERSALITY then

$$W(p,q;[u]) = \{u \text{ in class } [u] \text{ satisfying (*)}\}$$

is a smooth manifold of dimension = Fredholm index:

$$\text{ind}([u]) = \dim\ker D\bar{\partial}_{J,u} - \dim \text{ind}\, D\bar{\partial}_{J,u}.$$

Hope: we can define degrees on $L_0 \cap L_1$ so that

$$\text{ind}([u]) = \deg q - \deg p.$$

We'd then like to define

$$d: CF^*(L_0, L_1) \longrightarrow CF^{*+1}(L_0, L_1)$$

by

$$d(p) = \sum_{\substack{q \in L_0\cap L_1 \\ [u]:\, \text{ind}[u]=1}} \#\mathcal{M}(p,q;[u])\, q.$$

Problem: in general this sum is infinite; but it's not too infinite (Gromov compactness).

Def k = field. The Novikov ring over k is

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in k,\ \lambda_i \geq 0,\ \lambda_i \to \infty \right\}$$

The Novikov field over k is the field of fractions of Λ_0:

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} \;\middle|\; a_i \in k,\ \lambda_i \in \mathbb{R},\ \lambda_i \to \infty \right\}.$$

Then define $CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda \cdot p$.

Def The Floer differential $d: CF(L_0, L_1) \to CF(L_0, L_1)$ is given by

$$d(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ [u]:\ \mu([u])=1}} \left(\#\mathcal{M}(p, q; [u]) \right) T^{E([u])} \{q\}$$

where $E([u])$ = energy $= \int u^* \omega \geq 0$.

* In fact: for transversality, need to consider solutions to perturbed CR eqn.

Gromov compactness: given any E_0, there are finitely many $[u]$ with $\mathcal{M}(p, q; [u]) \neq \emptyset$ and $E([u]) \leq E_0$. So d is well-defined when we work with Novikov coefficients.

→ Special case: if ω is exact, $\omega = d\theta$ for $\theta \in \Omega^1(M)$, and L_0, L_1 are exact, $\theta|_{L_i} = df_i$ for $f_i: L_i \to \mathbb{R}$, then

$$E([u]) = \int u^* \omega = \underset{\text{Stokes}}{\left(f_1(q) - f_0(q)\right) - \left(f_1(p) - f_0(p)\right)} \quad \text{indep of } [u].$$

So can drop Λ coefficients and work with k coeffs by rescaling $p \mapsto T^{f_1(p) - f_0(p)} p$.

Thm (Floer) $k = \mathbb{Z}_2$. Assume $[\omega] \cdot \pi_2(M, L_i) = 0$ $i = 0, 1$.

Then d is well-defined and satisfies $d^2 = 0$ and $\deg d = +1$ (for a choice of grading of $CF(L_0, L_1)$), and $HF^*(L_0, L_1)$ is indep of choice of J and invt under Hamiltonian isotopies of L_0, L_1.

Grading Let $[u]$ be a homotopy class for a strip between p and q.

There is a Maslov index $\text{ind}([u]) \in \mathbb{Z}$

such that when transversality holds,

$$\dim \mathcal{M}(p,q;[u]) = \text{ind}([u]) - 1.$$

In nice circumstances, can assign degrees to p, q so that

$$\text{ind}([u]) = \deg q - \deg p \qquad \forall\, u$$

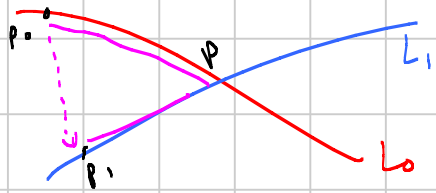
so that $\dim \mathcal{M}(p,q;[u]) = \deg q - \deg p - 1$ and d increases degree by 1.

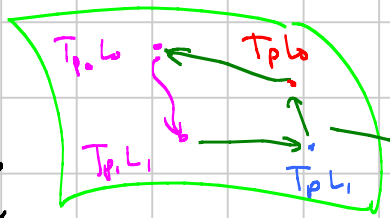
To define degree to $p \in L_0 \cap L_1$:

Use the fact that $\pi_1(LGr(n)) = \mathbb{Z}$ where

$LGr(n) = \{\text{Lagrangian } n\text{-planes in } (\mathbb{R}^{2n}, \omega_0)\}$.

Pick a base pt on L_0, L_1. Connect $T_{p_0}L_0$ to $T_{p_1}L_1$ by a fixed path in $LGr(n)$.





This path gives an elt of $\pi_1 = \mathbb{Z}$: grading of 1.

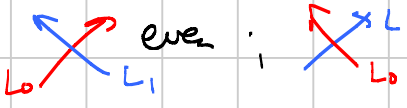
What's needed for this: • $2c_1(TM) = 0$ (to trivialize $LGr(TM)$)

• Maslov class of $L_i \in \text{Hom}(\pi_1(L_i), \mathbb{Z})$ vanishes (so answer is indep of paths in L_0, L_1)

Then: $\dim \mathcal{M}(p,q) = \deg(p) - \deg(q) - 1$. (calculation of Fredholm index)

(In general: grading lies in $\mathbb{Z}/2k\mathbb{Z}$ for some k; have to define index for $[u]$. If Lagrangians are oriented then grading is always at least $\mathbb{Z}/2$.

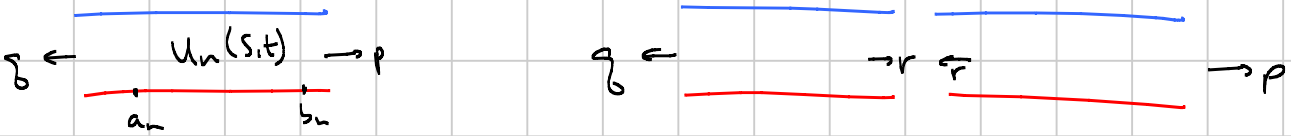
Ex: L_0, L_1 oriented curves in a surface.

3/2 ↑ 

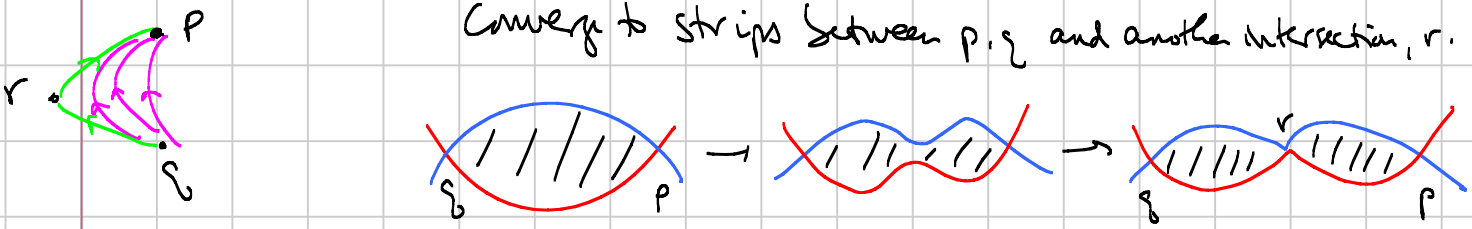
Compactness ($\partial^2=0$).

Gromov compactness: any sequence of hol. strips in $\mathcal{M}(p,q)$ with bounded energy has a subsequence converging to:

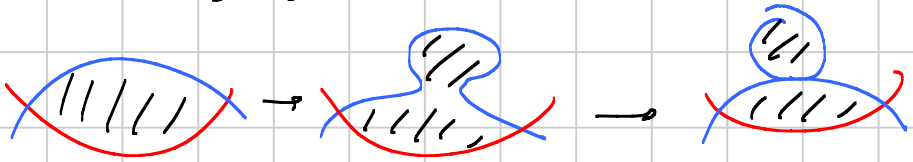
0. an honest holomorphic strip
1. a broken strip



$\exists\ a_n \to -\infty, b_n \to \infty$ s.t. translated strips $u_n(s-a_n,t)$, $u_n(s-b_n,t)$ converge to strips between p,q and another intersection, r.



2. a strip with a disk bubble: energy concentrates at a point in ∂(strip), and rescaling gives a hol. disk with ∂ on L_0 or L_1.



3. a strip with a sphere bubble: energy concentrates at an interior point, and rescaling gives a hol. sphere.

The assumption $[\omega]\cdot\pi_2(M,L_i)=0$ rules out disk and sphere bubbles:
if $u:(\mathbb{D}^2,\partial\mathbb{D}^2)\to(M,L_i)$ is a disk bubble then

$$[\omega]\cdot[u]=\int_{\mathbb{D}^2} u^*\omega = \iint \left|\frac{\partial u}{\partial s}\right|^2 ds\,dt > 0 \quad \text{(unless } u=\text{const)}.$$

Thm $p,q \in L_0 \cap L_1$, $[\omega] \cdot \pi_2(M, L_i) = 0$.

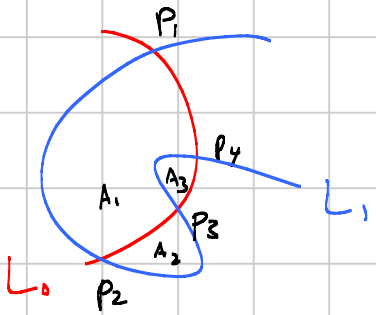
1. If $\operatorname{ind}[u] = 1$ then $\mathcal{M}(p,q;[u])$ is a compact 0-mfd.
2. If $\operatorname{ind}[u] = 2$ then $\mathcal{M}(p,q;[u])$ can be compactified to a compact 1-mfd $\overline{\mathcal{M}}(p,q;[u])$ with

$$\partial \overline{\mathcal{M}}(p,q;[u]) = \bigcup_{\substack{r \in L_0 \cap L_1 \\ [u'] + [u''] = [u] \\ \operatorname{ind}[u'] = \operatorname{ind}[u''] = 1}} \mathcal{M}(p,r;[u']) \times \mathcal{M}(r,q;[u'']).$$

compactness + gluing

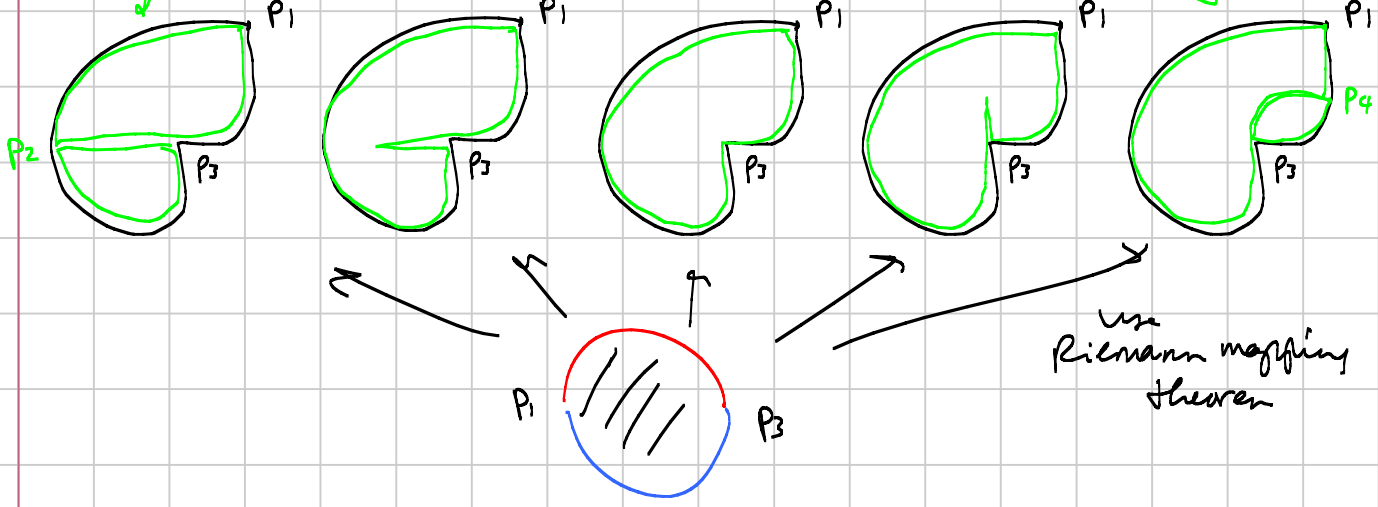
This allows us to define $d : CF^*(L_0, L_1) \to CF^{*+1}(L_0, L_1)$ and proves $d^2 = 0$.

Ex M = surface;
hol. maps are just usual conformal maps.



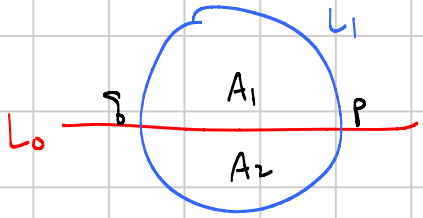
$$d p_1 = T^{A_1+A_3} p_2 + T^{A_1+A_2} p_4, \quad d p_2 = T^{A_2} p_3, \quad d p_4 = T^{A_3} p_3 \Rightarrow d^2 p_1 = 0.$$

$\mathcal{M}(p_1,p_2) \times \mathcal{M}(p_2,p_3)$ $\quad$ $\mathcal{M}(p_1,p_3)$ $\quad$ $\mathcal{M}(p_1,p_4) \times \mathcal{M}(p_4,p_3)$



What if $[\omega] \cdot \pi_2(M, L_i) \neq 0$? Then it's quite possible that $d^2 \neq 0$.

Ex. M = surface, but now L_1 bounds a disk.

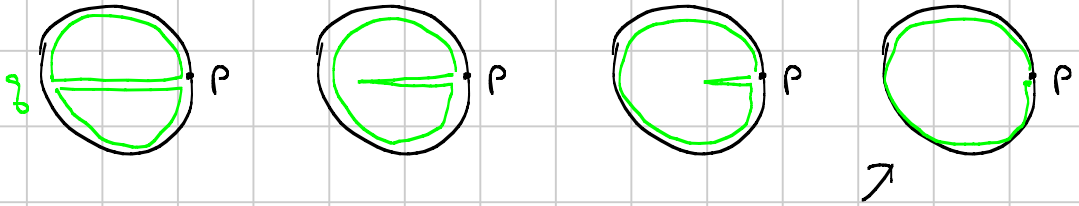


$d(p) = T^{A_1} q$, $d(q) = T^{A_2} p$.

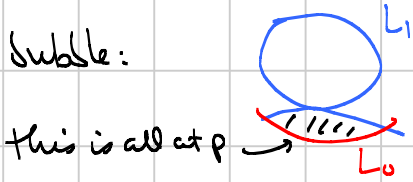
$\Rightarrow d^2(p) = T^{A_1 + A_2} p \neq 0$!

(note: in this case degrees are only defined mod 2)

$\mathcal{M}(p,q) \times \mathcal{M}(q,p)$ $\qquad$ $\mathcal{M}(p,p)$



This is a trivial strip plus a disk bubble:



Rmk: Sometimes possible to rule out disk bubbling in other circumstances besides $\pi_2 = 0$, e.g. when it's disallowed for degree reasons.

Also $\exists$ approaches where disk bubbling happens — e.g. Cornea-Lalonde cluster approach.

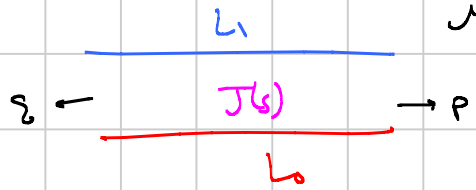
Invariance

Claim: $HF^*(L_0, L_1; J) \cong HF^*(L_0, L_1; J')$.

This uses a continuation argument: construct family of almost cx strs compatible with ω, $J(s)$, s.t. $J(s) = \begin{cases} J', & s \ll 0 \\ J, & s \gg 0. \end{cases}$

Then look at 0-d moduli spaces of hol. strips $u: \mathbb{R}\times[0,1]\to M$ (note no $\mathbb{R}$ translation)

$\mathcal{M}_{J,J'}(p,q)$

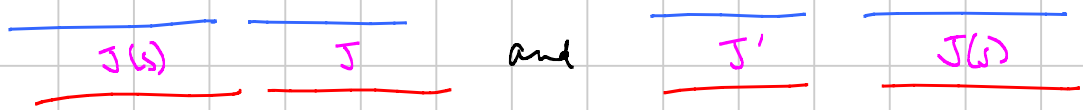


$$\frac{\partial u}{\partial s} + J(s)\frac{\partial u}{\partial t} = 0.$$

We get a map $\Phi: CF^*(L_0,L_1;J)\to CF^*(L_0,L_1;J')$ by

$$\Phi(p) = \sum_{\dim \mathcal{M}_{J,J'}(p,q)=0} (\#\mathcal{M}_{J,J'}(p,q))\cdot q.$$

This is a chain map: look at 1-dim moduli spaces. Boundaries look like



$$\Phi \quad d \quad = \quad d \quad \Phi.$$

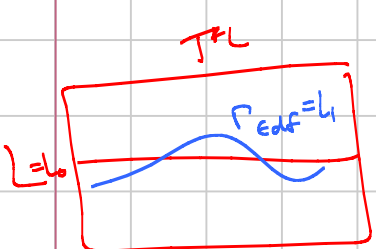
Similar argument to show invariance under Hamiltonian isotopy.

$HF^*(L,L)\cong H^*(L)$: perturb L by a small Hamiltonian isotopy ϕ to $\phi(L)$: then $HF^*(L,L)$ means $HF^*(L,\phi(L))$ and this is indep. of choice of ϕ.

Energy estimate $\Rightarrow$ strips contributing to d on $CF^*(L,\phi(L))$ lie close to L. So we can replace (M,L) by (T^*L, L) using Lagr. nbd thm.
(L ← 0 section)

Choose a Morse function $f: L\to\mathbb{R}$ and let $L_0 = L$,

$$L_1 = \phi(L) = \Gamma_{\epsilon df} \subset T^*L.$$



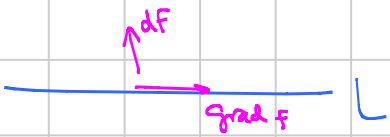
Note $L_0\cap L_1 = \mathrm{Crit}\, f$.

Prop (Floer) For a suitable choice of almost cx str J on T^*L, we have a chain isomorphism

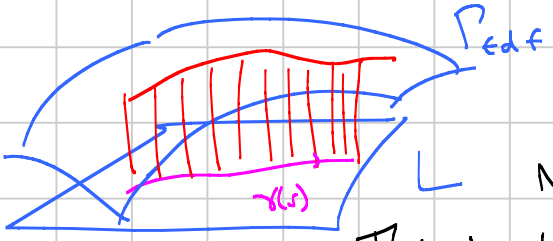
$$(CF^*(L_0,L_1), d) \cong (CM^*(f), d).$$
$$(\Rightarrow HF^*(L,\phi(L)) \cong HM^*(L,f)).$$

Idea of pf: Choose metric g on L. This induces a natural almost cx str J on T^*L such that at $(q,0) \in T^*L$, J maps

$$\text{grad } f \in T_{(q,0)}L \subset T_{(q,0)}T^*L$$
$$\downarrow$$
$$df \in T^*_{(q,0)}L \subset T_{(q,0)}T^*L.$$



Now let $\gamma(s)$ be a gradient flow for ϵf in L: $\dot\gamma(s) = \epsilon \text{ grad } f$.



Define

$$u(s,t) = (\gamma(s), \epsilon t\, df(s)).$$

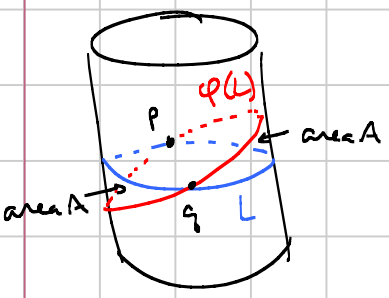
Note this has boundary on 0 section and $\Gamma_{\epsilon df}$.

This is almost holomorphic:

$$\frac{\partial u}{\partial s} \approx (\dot\gamma(s), 0) \xrightarrow{J} (0, \epsilon\, df(s)) = \frac{\partial u}{\partial t}.$$

Floer: there is a 1-1 corr. between gradient flows for f and honest holomorphic strips. □

EX L = zero section in $S^1 \times \mathbb{R}$.



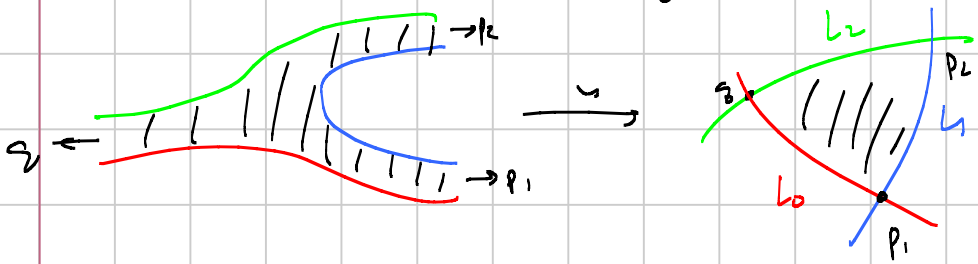
$$d(p) = T^A q - T^A q_0 = 0$$
$$d(q) = 0$$
$$HF^*(L,\varphi(L)) = \langle p, q\rangle \cong H^*(S^1).$$

Product structure.

– in everything that follows, may need to perturb.

Suppose now we have three Lagrangians L_0, L_1, L_2, $L_i \pitchfork L_j$, and suppose bubbling doesn't occur. (In particular, no hol. disks with ∂ on L_i).

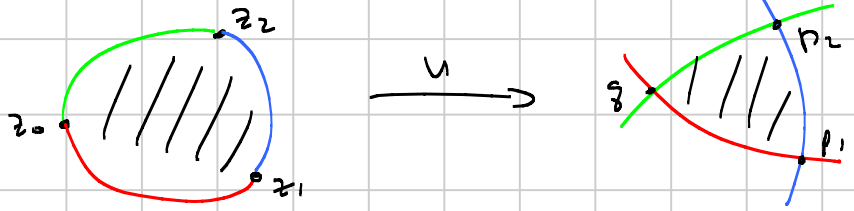
For $p_1 \in L_0 \cap L_1$, $p_2 \in L_1 \cap L_2$, $q \in L_0 \cap L_2$, consider holomorphic maps



The domain is conformally equivalent to $\overline{D}^2 - \{3 \text{ pts on boundary}\}$.

More precisely, given a homotopy class $[u] \in \pi_2(M, L_0 \cup L_1 \cup L_2)$, define

$\mathcal{M}(p_1, p_2, q; [u])$ = space of maps $u: D^2 \to M$



such that:

- u extends continuously to ∂D^2, mapping $z_0 \to q$, $z_1 \to p_1$, $z_2 \to p_2$, and arcs to L_0, L_1, L_2 as drawn
- u is holomorphic: $\overline{\partial}_J u = 0$
- u has finite energy: $E([u]) < \infty$.

Assuming transversality, $\mathcal{M}(p_1, p_2, q; [u])$ is a smooth manifold with dimension $= \text{ind}([u]) = \deg(q) - \deg(p_1) - \deg(p_2)$

(as before, assuming $2c_1(TM) = 0$ and Maslov classes of $L_0, L_1, L_2 = 0$).

Def The product is the map

$$CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_2)$$

defined by

$$p_2 \cdot p_1 = \sum_{\substack{q \in L_0 \cap L_2 \\ \text{ind}[u]=0}} \left(\#\mathcal{M}(p_1, p_2, q; [u])\right) T^{E([u])} q .$$

Thm If $[\omega] \cdot \pi_2(M, L_i) = 0 \ \forall i$, then the product satisfies Leibniz rule:

$$d(p_2 \cdot p_1) = (dp_2) \cdot p_1 + p_2 \cdot (dp_1).$$

(if we work out signs then this is $(-1)^{\deg p_2}$)

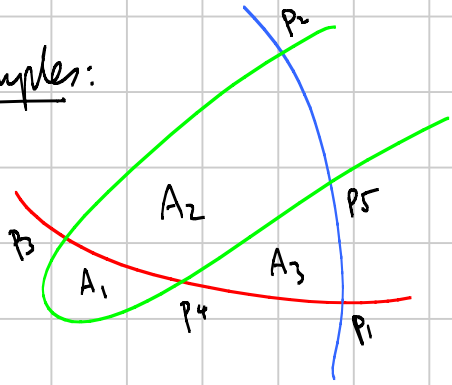
Thus this induces a product

$$HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \to HF^*(L_0, L_2).$$

Furthermore, this product on cohomology is associative and independent of choice of almost complex structure.

Associativity: later.

Examples:



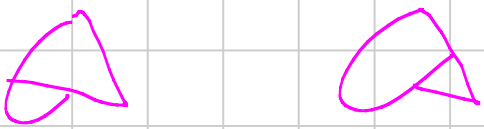
$p_2 \cdot p_1 = T^{A_2 + A_3} p_3$.

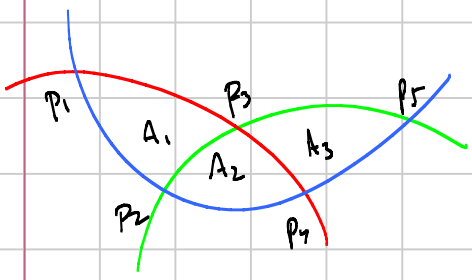
$dp_3 = T^{A_1} p_4$

$dp_2 = T^{A_1 + A_2} p_5$

$p_5 \cdot p_1 = T^{A_3} p_4$

$d(p_2 \cdot p_1) = T^{A_1 + A_2 + A_3} p_4 = (dp_2) \cdot p_1 + p_2 \cdot (dp_1)$



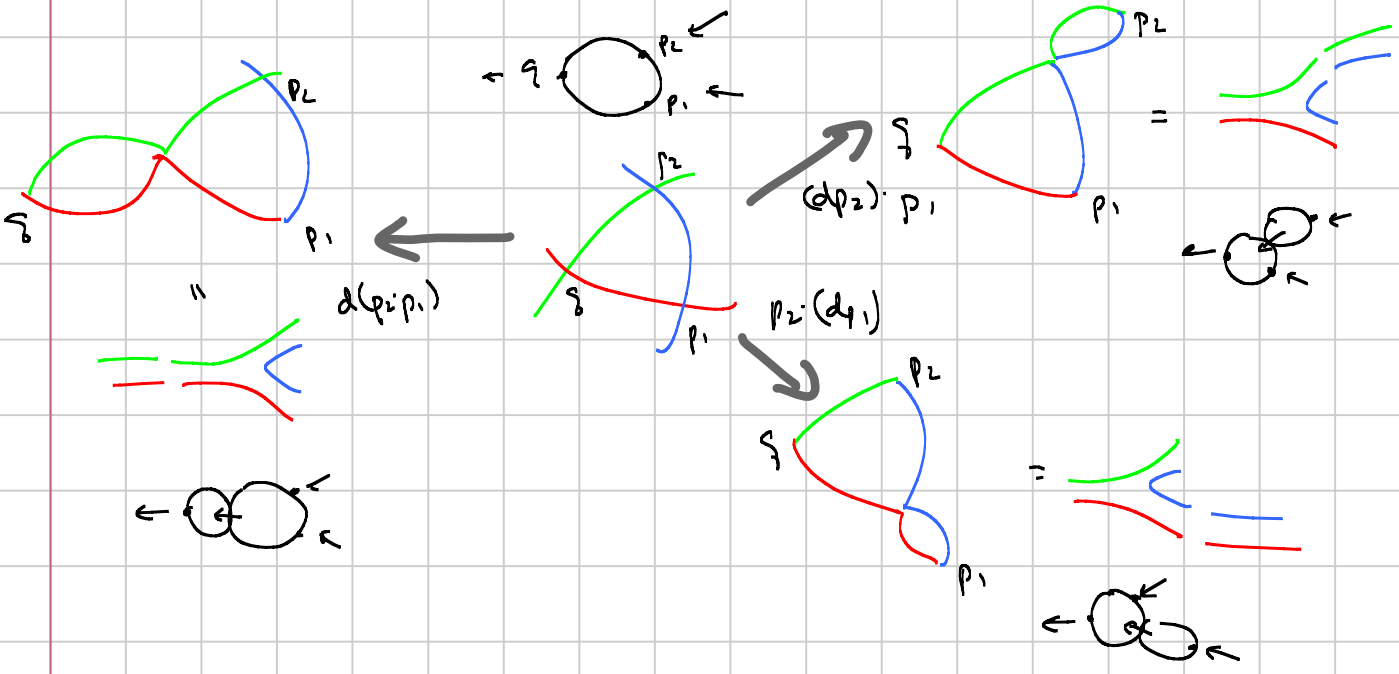


$\partial p_1 = T^{A_1+A_2} p_4$ $\quad$ $p_5 \cdot p_4 = T^{A_3} p_3$ $\quad$ $\partial p_5 = T^{A_2+A_3} p_2$ $\quad$ $p_2 \cdot p_1 = T^{A_1} p_3$

$$\partial(p_5 \cdot p_1) = 0 = (\partial p_5)\cdot p_1 + p_5 \cdot (\partial p_1)$$

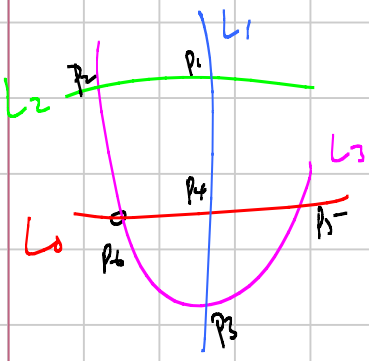
Idea of proof that ∂ satisfies Leibniz:

Look at $\bar{\mathcal{M}}(p_1, p_2, \xi; [u])$ for $\operatorname{ind}[u] = 1$. This is a cpt 1-dim mfd with boundary. Three types of boundary:



and so $\partial(p_2 \cdot p_1) + (\partial p_2)\cdot p_1 + p_2 \cdot (\partial p_1) = 0$ (up to signs).

Next: associativity. Product is not associative on chain level.



$p_2 \cdot p_1 = p_3$

$p_3 \cdot p_4 = p_5$ $\Rightarrow$ $(p_2 \cdot p_1) \cdot p_4 = p_5$ but $p_1 \cdot p_4 = 0$!

However! $p_5 = d\, p_6$ so $(p_2 \cdot p_1) \cdot p_4 = p_2 \cdot (p_1 \cdot p_4)$ in cohomology.

There is a "triple product"

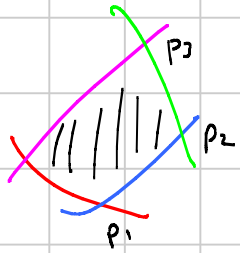
$$m_3 : CF(L_2, L_3) \otimes CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_3)$$

such that (mod 2)

(*) $$(p_3 \cdot p_2) \cdot p_1 + p_3 \cdot (p_2 \cdot p_1) = d\, m_3(p_3, p_2, p_1) + m_3(d p_3, p_2, p_1) + m_3(p_3, d p_2, p_1) + m_3(p_3, p_2, d p_1)$$

So the product is associative on cohomology.

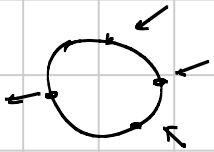
m_3 counts 0-dim moduli spaces of 4-gons

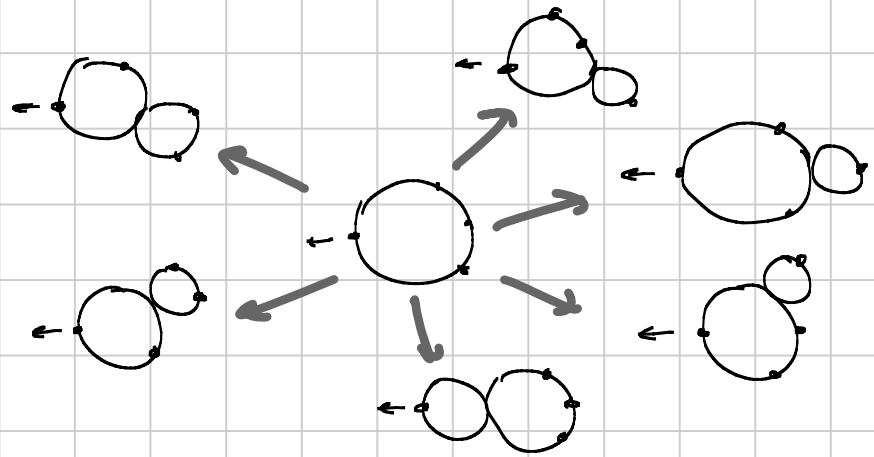


$m_3(p_3, p_2, p_1) = q + \cdots$

(in example, $m_3(p_2, p_1, p_4) = p_6$)

Pf of (*): look at degenerations of





A_∞ Categories

Def k = field. An **A_∞ category** over k consists of:

- Objects L_i
- morphism spaces $\mathrm{Hom}(L_0, L_1)$ = graded k-vector space
- for $k \geq 1$, a degree $(2-k)$ operation

$$m_k : \mathrm{Hom}(L_{k-1}, L_k) \otimes \cdots \otimes \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_k)$$

satisfying the **A_∞ relations**:

$$\sum_{l=1}^{k} \sum_{j=0}^{k-l} (\mathrm{Sgn})\, m_{k+1-l}(p_k, \ldots, p_{j+l+1}, m_l(p_{j+l}, \ldots, p_{j+1}), p_j, \ldots, p_1) = 0.$$

Pictorially:

A_∞ relations for $k = 1, 2, 3$:

$m_1(m_1(p_1)) = 0$: $m_1^2 = 0$ — $m_1 : \mathrm{Hom}(L_0, L_1) \circlearrowright$ is a differential

$$m_1(m_2(p_2, p_1)) + m_2(m_1(p_2), p_1) + m_2(p_2, m_1(p_1)) = 0$$

m_1 satisfies Leibniz w.r.t. $m_2 : \mathrm{Hom}(L_1, L_2) \otimes \mathrm{Hom}(L_0, L_1) \to \mathrm{Hom}(L_0, L_2)$

$$m_1(m_3(p_3, p_2, p_1)) + m_2(m_2(p_3, p_2), p_1) + m_2(p_3, m_2(p_2, p_1)) + m_3(m_1(p_3), p_2, p_1) + m_3(p_3, m_1(p_2), p_1) + m_3(p_3, p_2, m_1(p_1)) = 0$$

m_2 is associative up to error terms.

Rmks 1. An A_∞ category with one object is an A_∞ algebra.

2. If A is an A_∞ category, then define the cohomology category HA by

- $\mathrm{Ob}\, HA = \mathrm{Ob}\, A$
- $\mathrm{Hom}_{HA}(L_0, L_1) = H^*(\mathrm{Hom}_A(L_0, L_1), m_1)$
- Composition in HA is induced by m_2

$$m_2 : H^*\mathrm{Hom}(L_1, L_2) \otimes H^*\mathrm{Hom}(L_0, L_1) \to H^*\mathrm{Hom}(L_0, L_2)$$

(note associative by m_3 relation).

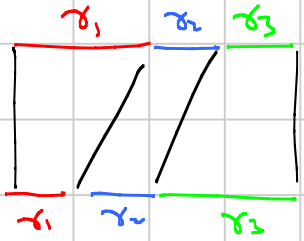
3. If $m_k = 0$ for $k \geq 3$, an A_∞ category is a dg-category: m_1 = differential, m_2 = multiplication. (eg. A_∞ algebra $\Rightarrow$ dg algebra).

4. Motivation: based loop space $\Omega X = \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = *\}$.

Stasheff '63: $C_*^{\mathrm{sing}}(\Omega X)$ has the structure of an A_∞ algebra.

Note m_2 = concatenation: $C_0(\Omega X) \otimes C_0(\Omega X) \to C_0(\Omega X)$ isn't associative,

but



$$m_2(m_2(\gamma_1,\gamma_2),\gamma_3) + m_2(\gamma_1, m_2(\gamma_2,\gamma_3)) = \partial\, m_3(\gamma_1,\gamma_2,\gamma_3)$$

$$\leftarrow\ m_3(\gamma_1,\gamma_2,\gamma_3) \in C_1(\Omega X)$$

5. Can recast A_∞ relations in terms of bar construction.

Let A = A_∞ algebra = graded vector space with $m_k : A^{\otimes k} \to A$. (degrees shifted by 1)

Define $\bar{T}A := \bigoplus_{k \geq 1} (A[1])^{\otimes k}$: this is the (reduced) tensor coalgebra of $A[1]$,

comultiplication $\Delta(\rho_1 \otimes \cdots \otimes \rho_k) = \sum_j (\rho_1 \otimes \cdots \otimes \rho_j) \cdot (\rho_{j+1} \otimes \cdots \otimes \rho_k)$.

Prop There is a 1-1 correspondence (this summarizes A_∞ relations)

$$\left(A_\infty \text{ algebra structure } \{m_k\} \text{ on } A\right) \longleftrightarrow \left(\begin{array}{c}\text{degree 1 coderivations } \delta : \bar{T}A \to \bar{T}A \text{ with } \delta^2 = 0 \\ \Delta\delta = (\delta \otimes \mathrm{id} + \mathrm{id} \otimes \delta)\Delta\end{array}\right).$$

(by co-Leibniz, δ is determined by components $\delta_k : A[1]^{\otimes k} \to A[1]$).

Fukaya category

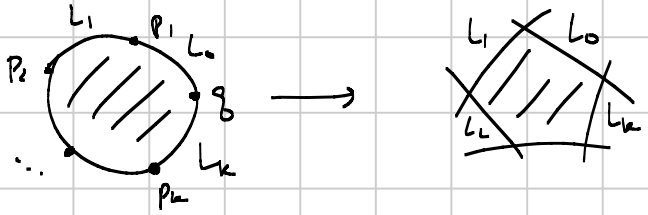
(M,ω) symplectic. The Fukaya category is the A_∞ category $Fuk(M,\omega)$:

- Objects = compact, closed, oriented, spin lagrangians $L \subset M$ with $[\omega]\cdot\pi_2(M,L)=0$.
- Morphisms $\quad Hom(L_0,L_1) = CF^*(L_0,L_1)$
- $m_1 = d: \; CF^*(L_0,L_1) \to CF^*(L_0,L_1)$
- $m_k: \; CF^*(L_{k-1},L_k)\otimes\cdots\otimes CF^*(L_0,L_1) \to CF^*(L_0,L_k)$

  Counts holomorphic k-gons:

$$m_k(p_k,\dots,p_1) = \sum_{ind[u]=2-k} \#\mathcal{M}(q,p_1,\dots,p_k;[u])\, T^{\omega([u])} q$$

$\mathcal{M}(q,p_1,\dots,p_k;[u])$ = finite energy hol. disks



These satisfy the A_∞ relations:

the moduli space $\mathcal{M}_{0,k+1}$ of conformal structures on $(D^2, k+1 \text{ boundary pts})$ has a compactification: $\bar{\mathcal{M}}_{0,k+1}$: Stasheff associahedron.

Top-dim'l faces are nodal degenerations $D^2 \to D^2 \cup D^2$.

each degeneration contributes to A_∞ relations.

⚠ WARNING: for this def, need all Lagrangians to intersect transversely. But we want to describe eg. self-Hom $Hom(L,L)$. So: choose:

- $\forall L_0, L_1$, Hamiltonian perturbations L_0', L_1' that are transverse
- $\forall L_0, \dots, L_k$, perturbations $L_0', \dots, L_k'$ consistent with the pairwise perturbation

(in fact: instead of perturbing Lagrangians, perturb $\bar\partial$ equation by inhomogeneous Hamiltonian term corresponding to (L_0, L_1) etc.)

Thm $Fuk(M,\omega)$ is a (cohomologically unital) A_∞ category, well-defined up to quasi-equivalence (A_∞ functor inducing $\cong$ on cohomology) indep of choices + perturbation data.

Rmk:
- Can generalize to other situations
- For Homological Mirror Symmetry, need to enrich Fukaya category by adding local systems to each Lagrangian.

To make contact with HMS, need to produce a triangulated category out of $Fuk(M,\omega)$. Standard technique: twisted complexes.

Def $A = A_\infty$ category. A twisted complex consists of objects $L_1, \dots, L_n$ with a strictly lower triangular differential $\delta \in End(L_1 \oplus \cdots \oplus L_n)$, i.e. $\delta_{ij} \in Hom(L_i, L_j)$ for $i<j$ such that $\sum_{k, i_1, \dots, i_{k-1}} m_k(\delta_{i_{k-1} i_k}, \dots, \delta_{i_0 i_1}) = 0 \ \forall i_0, i_k$:

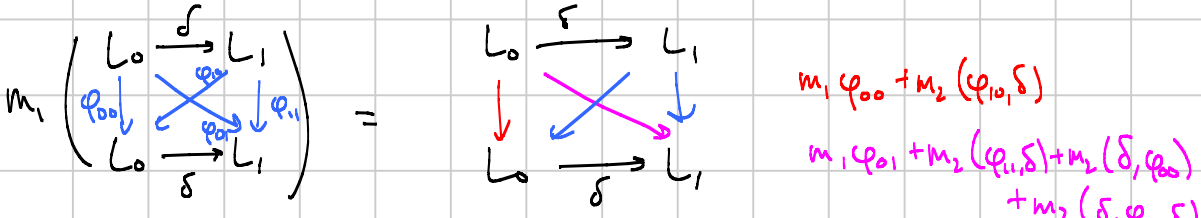
$L_0 \xrightarrow{\delta_{01}} L_1$ $\qquad m_1(\delta_{01}) = 0$

$L_0 \xrightarrow{\delta_{02}} L_2$, $L_0 \xrightarrow{\delta_{01}} L_1 \xrightarrow{\delta_{12}} L_2$ $\qquad m_1(\delta_{01}) = 0$, $m_1(\delta_{12}) = 0$, $m_2(\delta_{12}, \delta_{01}) + m_1(\delta_{02}) = 0$.

$\dots$

Def $\mathrm{Tw}\mathcal{A}$ = A_∞ category with

- objects = twisted complexes
- morphisms = maps between twisted complexes
- $m_1 : \mathrm{Hom}(E, E') \to$ defined by $m_1(\varphi) = \sum_{k \geq 1} m_k(\delta, \ldots, \delta, \varphi, \delta, \ldots, \delta)$



$m_1\varphi_{00} + m_2(\varphi_{10}, \delta)$

$m_1\varphi_{01} + m_2(\varphi_{11}, \delta) + m_2(\delta, \varphi_{00}) + m_3(\delta, \varphi_{10}, \delta)$

- m_k defined similarly for $k > 1$.

This is a triangulated A_∞ category.

Def $\mathcal{A}$ = A_∞ category $\rightsquigarrow$ the derived category is

$$D\mathcal{A} = H(\mathrm{Tw}\mathcal{A}).$$

When $\mathcal{A} = \mathrm{Fuk}(M, \omega)$, this produces the (bounded) derived Fukaya category $D^b\mathrm{Fuk}(M, \omega)$.