A Beginner's guide to A∞ algebras and Fukaya categories

Vague motivation: Homological Mirror Symmetry conjecture

(Kontsevich '94)

Kähler mfd: \((M, J, \omega)\) \(\quad M = \text{even-dim mfd, } J = \text{integrable complex structure, } \omega = \text{symplectic form}\)

Certain Kähler manifolds come in "mirror pairs" \((M, J, \omega) \leadsto (M', J', \omega')\).

Physics: Consider "branes" on Calabi-Yau mfd: come in two flavors:

\(A\)-model: branes = Lagrangian submanifolds \(wrt \omega\)

\(B\)-model: branes = \(\infty\) analytic submanifolds \(wrt J\)

Kontsevich HMC: equivalence of triangulated categories

\[ D^b \text{Fuk}(M, \omega) \cong D^b \text{Coh}(M', J') \]

\[ D^b \text{Coh}(M, J) \cong D^b \text{Fuk}(M', \omega') \]

Coherent sheaf category: objects = coherent sheaves (thick: h.e.v.d.)

morphisms = Hom or Ext of sheaves

Fukaya category: objects = Lagrangian submanifolds

morphisms = intersection between them

Important additional structure given by (Lagrangian intersection) Floer homology

\[ \Rightarrow \text{gives Fuk the structure of an } \mathbb{A}_\infty \text{category.} \]

Goal: define the Fukaya category and examine some properties.

This is the key.

Floer homology = "\(\infty\) dual version of Morse homology"
Outline:

- Morse homology (overview)
- Symplectic geometry intro: Symplectic manifolds
  - Lagrangian submanifolds, Hamiltonian diffeos, Arnold conjecture
- Lagrangian intersection Floer homology
- A∞ algebras, A∞ categories, twisted complexes
- Fukaya categories

Morse homology review. (see: my minicourse notes)

\( M = \) smooth n-dim manifold.
\( f: M \to \mathbb{R} \) is Morse if all critical points are nondegenerate.
\( p \in M, \ (df)_p \neq 0 \)

\( f \) is height function on \( T^2 \):

\[ f: \mathbb{R}^2 \to \mathbb{R} \]

If \( p \in \text{Crit } f \) \( = \) \{critical pt of \( f \)\} then the Morse index
\( \text{ind}_f(p) = \# \) negative eigenvalues of \( (df)_p \).

2. 1. 1. 0

Given a metric \( \langle , \rangle \) on \( M \), define \( Df \) as usual:
\( \langle Df, v \rangle = df(v) \).

For \( x \in M \), the (negative) gradient flow line \( \gamma_x(t) \) is the flow of \( -Df \):
\[ \frac{d}{dt} \gamma_x(t) = -Df(\gamma_x(t)). \]

Note: \( M \) cpt \( \Rightarrow \) every gradient flow line begins and ends at a crit pt:
\( \exists p \in \text{Crit } f \) s.t.
\[ \lim_{t \to -\infty} \gamma_x(t) = p, \ \lim_{t \to +\infty} \gamma_x(t) = q. \]
\[ p, q \in \text{Crit } f \text{ we define } \mathcal{W}(p, q) = \bigcap_{\varepsilon > 0} \bigcup_{T > 0} \left\{ x \in \mathcal{M} \mid \lim_{t \to T} \gamma_{\varepsilon}(t) = p, \lim_{t \to T} \gamma_{\varepsilon}(t) = q \right\}. \]

We can write this as
\[ \mathcal{W}(p, q) = W^u(p) \cap W^s(q), \]
\[ W^u(p) = \left\{ x \mid \lim_{t \to \infty} \gamma(t) = p \right\} \quad \text{disk of dim = ind}_p, \]
\[ W^s(q) = \left\{ x \mid \lim_{t \to \infty} \gamma(t) = q \right\} \quad \text{disk of dim = n - ind}_q. \]

**Def**: \( (f, \langle \cdot, \cdot \rangle) \) is Morse-Smale if \( W^u(p) \cap W^s(q) \neq \emptyset \) for \( p, q \in \text{Crit } f \), \( \forall x \in W^u(p) \cap W^s(q), \ T_x W^u(p) + T_x W^s(q) = T_x \mathcal{M}. \)

Set of Morse-Smale points is dense.

**If** \( (f, \langle \cdot, \cdot \rangle) \) is Morse-Smale, then \( \dim \mathcal{W}(p, q) = \text{ind } p - \text{ind } q. \)

**Note**: \( \mathbb{R} \) acts on \( \mathcal{W}(p, q) \) by time translation:
\[ S \cdot x = \gamma(t). \quad \text{This action is free (if } p \neq q). \]

**Def**: The moduli space of flow lines from \( p \to q \) is
\[ \mathcal{M}(p, q) = \mathcal{W}(p, q) / \mathbb{R}. \]

This is a smooth manifold of dim = \( \text{ind } p - \text{ind } q - 1 \).

(Rank: given orientations on \( W^u(p) \) \& \( \text{Crit } f \), can orient \( \mathcal{M}(p, q) \) \& \( p, q \))
The key to Morse homology: compactification by broken flow line.

\[ M(p_1, p_4) = \bigcup 4 \text{ open intervals}; \text{ can compactify:} \]
\[ \overline{M}(p_1, p_4) = \bigcup 4 \text{ closed intervals.} \]

Thus,
1. If \( \text{ind } p = \text{ind } q + 1 \) then \( M(p, q) \) is compact (= finite union of points).
2. If \( \text{ind } p = \text{ind } q + 2 \) then \( M(p, q) \) has a compactification \( \overline{M}(p, q) = \text{cpt } 1\text{-mfld with } \partial \) (= finite union of closed intervals and \( S^1 \)’s).

\[ \overline{M}(p, q) = \bigcup M(p, r) \times M(r, q). \]

(In fact, in general \( \overline{M}(p, q) \) has a compactification to a mfld with corners whose strata are broken flow lines.)
Note: components of proof.

1. Transversality: moduli space have the expected dimension (built into the def.)
2. Compactness:

3. Any sequence of finitely many M(q,i) has a subsequence converging to a broken smooth line
4. Any broken smooth line can be perturbed to an honest smooth line. ("gluing theorem")

\[ M, (f_i g) \text{ Morse-Smale} \Rightarrow (CM_k(M; f, g), \partial) \]

\[ \text{Def} \quad CM_k = \mathbb{Z}_2 \text{Crit}_k(f) \]
\[ \text{Crit}_k(f) = \{ \text{crit pt} \text{ of index } k \} \]
\[ \mathcal{E} : CM_k \rightarrow CM_{k-1} \text{ defined by} \]
\[ \mathcal{E}(p) = \sum_{q \in \text{Crit}_{k-1}(f)} \left( \# M(q, i) \right) \]
\[ \in \mathbb{Z}_2: \# \text{crit pt in } M(q, i), \text{ counted with sign}. \]

This is the More complex of \( M \) associated to \((f, g)\).

**Prop** \[ \partial^2 = 0. \]

**PF** \[ p \in \text{Crit}_k(f), q \in \text{Crit}_{k-2}(f) \Rightarrow \]
\[ \langle \partial p, q \rangle = \sum_{r \in \text{Crit}_{k-1}(f)} \left( \langle \partial p, r \rangle \langle \partial r, q \rangle \right) \]
\[ = \sum_{r \in \text{Crit}_k(f)} \# M(p, r) \times M(r, q) \]
\[ = \sum_{r \in \text{Crit}_k(f)} \# M(p, r) \times M(r, q) \]
\[ = 0. \quad \square \]

**Def** The Morse homology \( HM_k(M; f, g) = H_k(CM_k(M; f, g), \partial) \).

(Note: depends on \((f, g)) \).
\[ M = T^2, \quad \text{CM}_k = \mathbb{R} \langle p_1, p_2, p_3, p_4 \rangle \]

\[ \delta(p_1) = p_2 - p_3 + p_3 - p_2 \]
\[ \delta(p_2) = p_4 - p_4 \]
\[ \delta(p_3) = p_4 - p_4 \]
\[ \delta(p_4) = 0 \]

\[ \text{HM}_k = \begin{cases} \mathbb{Z} & k = 1, 3, 5 \\ 0 & \text{otherwise} \end{cases} \]

Thus, \( M \) is a smooth manifold, Morse-Smale. Then

\[ \text{HM}_k(M; f, g) \cong H^k_{\text{sing}}(M). \]

**Corollary:**

\[ b_k = k^{th} \text{ betti number of } M = \dim_{\mathbb{Q}}(H_k(M) \otimes \mathbb{Q}). \]

Then \( f \) is Morse function on \( M \). Then

\[ (\# \text{critical points of index } k) \geq b_k \]
\[ (\# \text{total critical points of } f) \geq \sum_{k=0}^{\infty} b_k \]

**Proof:**

Find a metric \( g \) so \( (f, g) \) is Morse-Smale (needs proof).

Then

\[ (\# \text{critical points of index } k) = \text{rank } \text{CM}_k(f, g) \]
\[ \geq \text{rank } \text{HM}_k(f, g) \]
\[ = b_k. \quad \Box \]

**Proof that** \( \text{HM}_k \cong H^k_{\text{sing}} \): Several approaches, but "standard" one involves sublevel sets and seeing how homology changes as we pass thru crit pt.
A priori invariance

Suppose we didn't know $H^s \equiv H^s_{\text{sing}}$. We'd still deduce some interesting info of $M$, provided $H^s(M, (f,g))$ independent of $(f,g)$.

This can be proven directly by contraction (Flan).

Suppose $(g_0, g_\ast)$, $(f, g)$ are Morse-Smale, and let $(g_t, g_t)$ be a path between them.

On $[0, 1] \times M$, let $M_t = \{t\} \times M$; then $g_t$ is a metric on $M_t$.

Write $V_t = -\nabla_{g_t} f$ on $M_t$. Then define a vector field $V$ on $[0, 1] \times M$.

$$V = h(t) \frac{\partial}{\partial t} + V_t$$

with $h(t) > 0$, $h(0) = h(1) = 0$, $h(t) > 0$ for $0 < t < 1$ if $V$ is actually the negative gradient flow for $F = f_t - \int_0^t h(u) du$ with respect to the metric $g_t \otimes dt \otimes dt$ on $[0, 1] \times M$.

(technically, need to perturb so that this is Morse-Smale).

$$\text{Crit}(F) = (\mathbb{R} \times \text{Crit}(f_\ast)) \cup (\mathbb{R} \times \text{Crit}(f_\ast))$$

As usual, $q \in \text{Crit}(F) \implies M(p, q) = \{\text{flowlines from } p \text{ to } q\}$.
For \( p, q \in \text{Crit}(V) \) with \( \text{ind}(q) = k \), define

\[
\Phi(p) = \sum_{\substack{q \in \text{Crit}(V) \\text{ such that} \\text{ind}(q) = k}} (\# M(p, q)) q.
\]

Extend \( \Phi \) linearly to get a map \( \Phi : C_{k}(f_{0}, g_{0}) \to C_{k}(f_{1}, g_{1}). \)

**Prop** \( \Phi \) is a chain map:

\[
\Phi \circ d_{0} = d_{1} \circ \Phi.
\]

**PF** Similar to proof that \( \delta^{2} = 0 \).

For \( p \in \text{Crit}(f_{0}), q \in \text{Crit}(f_{1}) \),

\[
\text{ind } q = \text{ind } p - 1, \quad M(p, q) \text{ has boundary}
\]

\[
\partial M(p, q) = \bigcup_{r \in \text{Crit}(f_{0}) \cap \text{Crit}(f_{1}) \\text{ and } \\text{ind } r = \text{ind } p - 1} M(p, r) \times M(r, q).
\]

Note: This says that the Morse complex \( C_{k}(M, f_{0}, g_{0}) \times M, F, g) \) is a mapping cone, Cone \( (C_{k}(M, f_{0}, g_{0}), C_{k}(M, f_{1}, g_{1})). \)

Next: let \( \Gamma, \Gamma' \) be two generic paths

\[
\sim \Phi_{\Gamma}, \Phi_{\Gamma'} : C_{*}(f_{0}, g_{0}) \to C_{*}(f_{1}, g_{1}).
\]

**Prop** If \( \Gamma, \Gamma' \) are homotopic then \( \Phi_{\Gamma}, \Phi_{\Gamma'} \) are chain homotopic:

\[
\exists K : C_{*}(f_{0}, g_{0}) \to C_{*}(f_{1}, g_{1}) \text{ such that}
\]

\[
\partial_{1} K + K \circ d_{0} = \Phi_{\Gamma} - \Phi_{\Gamma'}.
\]
Idea: instead of using $[0,1] \times M$, use $[0,1]^2 \times M$. Consider gradient flow for $r_t - j_t(\cdot) = (r_t - j_t(S))$ with respect to $g_t$. Let $t \in \mathbb{R}$ and consider homology class from $(S, t) = (0, 0)$ to $(S, t) = (1, 1)$.

$\text{Homology class from } (S, t) = (0, 0) \text{ to } (S, t) = (1, 1)$.

$\text{Boundaries of 1D moduli space}$

$\text{Cor}$

$\text{PF}$

$\text{HM}_k(f, \gamma) \overset{\text{index of } (f, \gamma)}{\longrightarrow} (f_0, \gamma_0) \overset{\phi_0}{\longrightarrow} (f_1, \gamma_1)$

$\text{Pu}(\phi)$ homotopic to constant path.

$\text{hm} = \text{id}$

Symplectic Geometry Background.

Def. $M^n$ smooth. A symplectic form is $\omega \in \Omega^2(M)$ with:

- $d\omega = 0$
- $\omega$ nondegenerate: $\omega^n \in \Omega^{2n}(M)$ is nowhere 0.

$(M, \omega)$ is a symplectic manifold.

Important examples:

- $\Sigma$: Riemann surface, $\omega$ = any area form
- $\mathbb{R}^{2n}$, $\omega = dp_1 \wedge dq_1 + \cdots + dp_n \wedge dq_n$
\[ T^*Q, \quad Q = \text{any smooth manifold}, \quad \omega = -d\lambda \text{ defined as follows.} \]

\[ \iota_1, \ldots, \iota_n \text{ coord on } Q, \quad \pi_1, \ldots, \pi_n \text{ dual coord on } T^*Q \quad (\Sigma \pi_i dq_i \in T^*Q) \]

The Liouville 1-form \( \lambda \in \Omega^1(T^*Q) \) is given in coordinates by

\[ \lambda = \Sigma \pi_i dq_i. \quad (\Rightarrow \omega = \Sigma dq_i \wedge dp_i) \]

**Fact:** \( \lambda \) is indep of coordinates.

Coord-free def of \( \lambda \):

\[
\begin{array}{ccc}
T(T^*Q) & \xrightarrow{\pi^*} & \pi^*(\pi_1, \pi_2) \\
\downarrow & & \downarrow \\
TQ & \xrightarrow{\lambda} & T^*Q
\end{array}
\]

**Def:** A symplectomorphism between \((M,\omega)\) and \((M',\omega')\) is a diffeo \( \varphi : M \to M' \) with \( \varphi^*(\omega') = \omega \).

**Special case:** \( \text{Symp}(M,\omega) = \text{group of symplectomorphisms } (M,\omega) \)

**Infinitesimal symplectomorphisms**

\[ X_t = \text{time-dependent vector field on } M \quad (t \in \mathbb{R}) \]

\[ \text{generate a family of diffeomorphisms } \varphi_t : M \to M : \frac{d}{dt}\varphi_t = X_t \cdot \varphi_t. \]

Conversely, \( \{\varphi_t\} \) determines \( X_t \) by the same equation.

\[ \xi_{\omega}(\varphi_t^{-1})(\varphi_t^*\omega) = \xi_{\varphi_t^*\omega}(t) \quad (\text{for any } \xi, \omega) \]

**Def:** A vector field \( X \) on \( M \) is symplectic if \( i_X\omega \) is closed.

**Prop:** \((M,\omega) \) symplectic. \( \{\varphi_t\} \) family of diffeos trivial by \( (X_t)(\text{time dep. v.f.)} \)

Then \( \varphi_t \in \text{Symp}(M) \quad \forall t \Rightarrow \quad X_t \text{ is symplectic} \forall t \).
pf, cartan's magic formula.  
\( \phi_t \in \text{Sympl}(M) \forall t \Rightarrow \phi_t^* \omega = 0 \forall t \Rightarrow i_{\phi_t^* \omega} \text{ closed.} \)  
\[ \text{d}(i_{\phi_t^* \omega}) = i_{\phi_t^* \omega} \text{ d}\omega \]

special case: \( i_{\phi_t^* \omega} \) is exact.

def \( H \in C^\infty(M) \text{. The Hamiltonian vector field } X_H \in \text{ Vect } M \text{ is determined by } \)  
\[ \text{d}H = i_{X_H} \omega. \quad \Rightarrow \text{ this is symplectic.} \]

rank. by nondegeneracy of \( \omega \), we have an isomorphism \( TM \cong T^* M \)  
and \( X_H \leftrightarrow \text{ d}H \) under this isomorphism.

if instead we used a metric \( g \) to get \( TM \cong T^* M \)  
then \( \text{d}H \leftrightarrow \text{D}H \).

if we have an almost kähler tripe \( (\omega, J, g) \), \( g(., .) = \omega(., J \cdot) \),  
thenn \( X_H = -J \text{D}H \).

if we have \( H: M \times [0, 1] \rightarrow \mathbb{R}, \quad H_t = H(\cdot, t), \text{ then get } X_{H_t}, \quad t \in [0, 1] \), and we get a corresponding \( \phi_t \in \text{Sympl } M \).

def \( \phi : M \rightarrow M \text{ is a Hamiltonian diffeomorphism if } \phi = \phi_t \text{ for some time-dependent Hamiltonian } \) \( H: M \times [0, 1] \rightarrow \mathbb{R}. \)

rank 1. if \( H'(M, P) = 0 \) then all symplectomorphisms are Hamiltonian diffeomorphisms.
2. \( M = T^*Q \), \( H = \text{any function on } M \).
\[ \omega(X_H, \frac{\partial}{\partial q}) = dH \left( \frac{\partial}{\partial q} \right) = \frac{\partial H}{\partial q} \rightarrow X_H = \sum \frac{\partial H}{\partial q} \frac{\partial}{\partial q} = \frac{\partial H}{\partial \phi} \frac{\partial}{\partial \phi} \]

So, if we flow along \( X_H \), get

\[ \dot{q}_i = \frac{\partial H}{\partial \phi_i}, \quad \dot{\phi}_i = -\frac{\partial H}{\partial q_i} \]

Hamilton's equation of motion!

Side consequence: Conservation of energy:
\[ dH (X_H) = (\omega(X_H, \omega)) (X_H) = \omega(X_H, X_H) = 0 \] so Hamiltonian flow is tangent to level sets of \( H \).

Arnold conjecture

Setting: 1-periodic Hamiltonian \( H_t \) on \((M, \omega)\).
\( H_t : M \rightarrow \mathbb{R}, \quad H_{t+1} = H_t \rightarrow \varphi_t = \text{time } t \text{-flow of } X_{H_t} \)

\( \varphi_t \) is a Hamiltonian diffeo and fixed pts \( \gamma \varphi_t \)
are periodic w/ d\( \varphi_t \) of Hamiltonian flow.

Say that a fixed point \( x \) of \( \varphi_t \) is nondegenerate if
\[ \text{det} (1 - d\varphi_t(x)) \neq 0. \]

Arnold conjecture \((M, \omega)\) compact symplectic, \( H_t \) 1-periodic.
Suppose all fixed pts of \( \varphi_t \) are nondegenerate. Then
\[ \# \text{Fix} (\varphi_t) \geq \sum_{k=0}^{\infty} b_k(M). \]
Special case: $H = H_{\text{time-independent}}$. Then $x \in \text{Crit } H \Rightarrow dH(x) = 0$ 
$\Rightarrow X_H(x) = 0 \Rightarrow x \in \text{Fix}(\varphi_t)$. So
$
\text{Fix}(\varphi_t) \supseteq \text{Crit } H \supseteq \Sigma b_k(M)
$
more inequality.

Arnold conjecture has proven:

1979 Eliashberg: Hilbert surface
1983 Conley-Zehnder for $T^2$
1988 Floer for $T^2(M) = \emptyset$
1989 Floer for monotone symplectic $\text{Fr}(\omega) = \omega(M) > 0$

**Lagrangian submanifolds**

$(M,\omega)$ symplectic $\Rightarrow$ $\forall x \in M$, $(T_x M, \omega)$ is a symplectic vector space: $\mathbb{R}^{2n}$ equipped with a skew-symmetric nondegenerate bilinear form.

Given $(V, \omega)$ symplectic, a subspace $W \subset V$ is isotropic if $\omega|_W = 0$.
Linear alg. $\dim W \leq n$. $W$ is Lagrangian if $\omega|_W = 0$ and $\dim W = n$.

**Def** $L \subset M$ is Lagrangian if $T_x L \subset T_x M$ is Lagrangian $\forall x \in L$:

$\omega|_L = 0$, $\dim L = n$.

**Ex.** $M = T^* Q$. $Q$ = zero section in $M$ is Lagrangian.
Lagrangian neighborhood theorem. If \( L_c(M, \omega) \) is Lagrangian then there is a \( \omega \)-bundle \( N(L) \) s.t. \((N(L), \omega)\) is symplectomorphic to \( N(\text{zero section}) \subset T^*L \) where \( L \to \text{zero section}. \)

More general than 0 sections:
\( \alpha \in \Omega^1Q \) induces a section \( \alpha_x \) of \( T_x^*Q \)

Claim: \( \alpha_x \) is Lagrangian \iff \( \alpha \) is closed.

Proof: recall Liouville form \( \lambda \in \Omega^1(T^*Q) \). Then \( \lambda|_{\alpha_x} = \pi^*\alpha : \)

\[ (x, \alpha(x)) \in T^*Q, \quad \lambda(v) = \alpha(x)(\pi^*v) = (\pi^*\alpha)(v). \]

So \( \omega|_{\alpha_x} = 0 \iff \pi^*\lambda|_{\alpha_x} = 0 \iff \pi^*\lambda|_{\alpha_x} = 0 \quad \text{and} \quad \pi^* \text{is an isom. from} \]

\( T_xQ \to T(x, \alpha(x))Q \) \( \alpha_x \) \quad \text{so} \quad \text{iff} \quad \alpha = 0. \quad \Box

In particular, the graph \( \Gamma_\alpha \) of any function \( f \in C^\infty(M) \)

is Lagrangian.

Note for next page: \( \Gamma_\alpha \) from 0 section by flowing along the vector field given in fibers by \( df \) itself: this is the Hamiltonian vector field \( H = f \circ \pi : T^*Q \to \mathbb{R} \).

Note: plenty of other Lagrangians.
- \( \mathbf{ex:} \) cotangent fiber \( T^*Q \)
- \( \mathbb{Q} = S^1, \quad T^*Q = S^1 \times \mathbb{R} = \mathbb{R}^2/(x, y) \sim (\theta, y), \quad \theta = y \, dx, \quad w = dx \wedge dy \). Any 1-dim submanifold is Lagrangian.
Arnold Conjecture for Lagrangian intersections

Let $\varphi = \varphi_t$ be a Hamiltonian diffeomorphism $(H_t : \mathbb{R} \to \mathbb{R}, 0 \leq t \leq 1)$.

Suppose $L$ is compact Lagrangian
- $L \cap \varphi(L)$
- Some additional assumption e.g. $\pi_2(M, L) = 0$.

Then
\[ \#(L \cap \varphi(L)) \geq \sum b_k(L) \varepsilon_{\mathbb{Z}_2} \]

where $b_k$ are the Betti numbers.

Say $M = T^2$, $L = 0$ section, $\varphi(L) = \text{graph} \; \varphi$, $\text{graph} \cap (0 \text{ section}) = \text{Crit}(\varphi)$. Thus $L \cap L$ inequality

Then (Floer 1988)

Let $(M, \omega)$ be symplectic, $L$ be Lagrangian. If
1. $\varphi$ is Hamiltonian diffeomorphism.
2. $L \cap \varphi(L)$
3. $\exists \; \omega : 0^2, \omega^2 \to (M, L)$, $\int_{0^2} \omega = 0$

Then
\[ \#(L \cap \varphi(L)) \geq \sum \dim H^k(L, \mathbb{Z}_2) \]

Note: by Stokes, since $\omega|_L = 0$,
\[ \int_{0^2} \omega \cdot u = \int_{0^2} \omega \] depends only on the class of $u \in \pi_2(M, L)$;

So (3) is satisfied if $\pi_2(M, L) = 0$.

Ex: $S^1 \times \mathbb{R}$. $L = S^1$; Floer says
\[ \#(L \cap \varphi(L)) \geq 2 \]

Claim: the signed area bounded by $L$ and $\varphi(L)$ is 0. Then $\#(L \cap \varphi(L)) \geq 2$ follows.
Claim is true because it's true infinitesimally: \( L = \gamma(t) \),

infinitesimal change in area is

\[
\int \left( \gamma(t) \times \gamma'(t) \right) \cdot \frac{\partial}{\partial t} H(t) \, dt = 0.
\]

Note: not true if:

\( \gamma \) isn't Hamiltonian:
vertical translation

\( L \) is homotopically trivial:

bounds a disk of positive area

Special case: apply Floer to \((\tilde{M} = M \times M, \omega \times \omega)\).

The diagonal \( \Delta = \{(x,x) | x \in M\} \subset \tilde{M} \) is Lagrangian.

More generally, if \( \gamma : M \to \tilde{M} \) is a symplectomorphism then

\( \Gamma_\gamma = \{(x,\gamma(x)) | x \in M\} \subset \tilde{M} \) is Lagrangian.

If \( H_t : M \to \mathbb{R} \) then \( \bar{H}_t : M \times M \to \mathbb{R} \)

\( \bar{H}_t(x,y) = -H_t(y) \)

Hamiltonian flow for \( \bar{H}_t \) is constant in 1st factor, follows Hamilton flow
in 2nd

\( \bar{\phi}_t : \tilde{M} \to \tilde{M} \)

\( \bar{\phi}_t(\Delta) = \Gamma_\gamma \)

\( \Delta \cap \Gamma_\gamma = \{(x,x) | x = \gamma(x)\} \).

Cor (Floer 1988) \((M,\omega)\) opt sympl, \( \pi_2(M) = 0 \), \( \gamma \) Hamiltonian diffeo
with all fixed pts nondegenerate. Then

\[ \# \text{Fix}(\gamma) \geq \Sigma \left( \frac{1}{2} - \text{Betti numbers of } M \right). \]
Setting up Lagrangian Floer theory: Motivation.

$\mathcal{L}$ is symplectic, $L_0, L_1$. Lagrangians, $L_0 \cap L_1$.

(Of special interest: $L_i = \phi(L_0)$ for $\phi \in \text{Ham}(\mathcal{L})$)

Write $P(L_0, L_1) =$ path space from $L_0 \to L_1$

$$= \{ \gamma : [0, 1] \to \mathcal{L} \mid \gamma(0) \in L_0, \gamma(1) \in L_1 \}$$

For $\gamma \in P(L_0, L_1)$, let $P_0(L_0, L_1) =$ connected component of $P(L_0, L_1)$ containing $\gamma_0$

$$= \{ \gamma \mid \gamma(0) \in L_0, \gamma(1) \in L_1, \gamma \text{ homotopic (rel endpts) on } L_0, L_1 \}$$

Define $\tilde{P}_0(L_0, L_1) =$ universal cover of $P_0(L_0, L_1)$

$$= \{ \tilde{\gamma} : [0, 1] \times [0, 1] \to M \mid \tilde{\gamma}(0, 0) \in L_0, \tilde{\gamma}(0, 1) \in L_1, \tilde{\gamma}(1, 0) = \gamma_0 \}$$

Write $\tilde{\gamma}$ for short.

Now define an action functional $A : \tilde{P} \to \mathbb{R}$ by

$$A(\tilde{\gamma}) = \int \int \omega \left( \frac{\partial \tilde{\gamma}}{\partial t}, \frac{\partial \tilde{\gamma}}{\partial s} \right) dt ds$$

Goal: perform Morse theory on $P$ with Morse function $A$.

Simplifying assumption to make notation nicer: $A$ descends to $P(L_0, L_1)$.
(This happens under some topological assumption; else it's well-defined locally up to a constant.)

$$T_\gamma P = \{ \xi : [0, 1] \to TM \mid \xi(0) \in T_{\gamma(0)} L_0, \xi(1) \in T_{\gamma(1)} L_1 \}$$
To define gradient flow on $\mathcal{P}$, need a metric.

**Def.** $J$ is an almost complex structure on $M$ compatible with $\omega$ if
- $J: TM \to TM$ with $J^2 = -I$
- $\omega(Ju, Jv) = \omega(u, v)$ and $g_J(u, v) := \omega(u, Jv)$ is a (symmetric) positive definite bilinear form.

**Prop.** If $J$ compatible with $\omega$, and the space $J(M, \omega)$ of almost complex structures compatible with $\omega$ is contractible.

Given $J$ compatible with $\omega$, get a metric on $\mathcal{P}$ defined by
\[ \langle \xi_1, \xi_2 \rangle = \int_0^1 g_J (\xi_1, \xi_2) \ dt. \]

Calculate gradient $D\xi$:
\[
\begin{align*}
\frac{d}{dt} \xi(t) &= \int_0^1 \omega \left( \frac{d\xi}{dt}, \xi \right) \ dt \\
&= \int_0^1 g_J \left( J \frac{d\xi}{dt}, \xi \right) \ dt \\
&= \langle J \frac{d\xi}{dt}, \xi \rangle.
\end{align*}
\]

So $D\xi = J \frac{d\xi}{dt}$.

(Note: Strictly speaking $\frac{d\xi}{dt}$ is not $\frac{d}{dt} \xi$ because of boundary conditions.)

- Critical points $\xi$ satisfy $\frac{d\xi}{dt} = 0$ so $\xi = \text{const}$;
  these are intersection points $L_0 \cap L_1$.
- Gradient flows:

\[ L_0 \rightarrow L \rightarrow \cdots \rightarrow L_0 \rightarrow L \rightarrow \cdots \]

- $L_1$
A gradient flow \( u : \mathbb{R}_s \rightarrow \mathbb{P} \) with \( \lim_{s \to -\infty} u = p \), \( \lim_{s \to +\infty} u = q \) can be written as

\[
\begin{align*}
\dot{u} : \mathbb{R}_s \times [0, 1]_t &\rightarrow M \\
\lim_{s \to -\infty} u(s, 0, 1) &= p \\
\lim_{s \to +\infty} u(s, 0, 1) &= q \\
\frac{d}{ds} u &= -J \frac{du}{dt}.
\end{align*}
\]

Now, define complex structure \( j \) on \( \mathbb{R} \times [0, 1] \) as usual: \( j \frac{d}{ds} = \frac{2}{\pi i}, j \frac{d}{dt} = -\frac{2}{\pi i} \).

Then

\[
\begin{align*}
\frac{d}{dt} u(\frac{2}{\pi i}) &= du(\frac{2}{\pi i}) = J \frac{du}{dt} = J du(\frac{2}{\pi i}) \\
\frac{d}{dt} u(\frac{2}{\pi i}) &= -\frac{du}{ds} = J du(\frac{2}{\pi i})
\end{align*}
\]

so \( du \circ j = J \circ du : u \) is a holomorphic curve \( (\mathbb{R} \times [0, 1], j) \rightarrow (M, J) \).

Floer: Use these holomorphic curves to define a complex \( \langle CF_L(u_0, L_1) \rangle \).

\( \langle CF_L(u_0, L_1) \rangle = \mathbb{Z} \langle \text{intersections of } L_0 \text{ and } L_1 \rangle \). \( \mathbb{Z} \) counts hol. curves:

\( \langle \mathcal{D}_{p, q} \rangle = \# M(p, q) \) \( \langle \text{hol. strips from } p \text{ to } q \rangle \).

---

**Marco theory**

- \( M \)
- \( F \)
- \( f \) Morse
- \( \text{Crit(} f \text{)} \)
- \( \text{Morse index} \)
- \( \text{Gradient flow} \)
- \( \text{HM}(M; \mathfrak{g}) \)

**Lagrangian intersections-Floer theory**

- \( \langle P(L_0, L_1) \rangle \)
- \( \langle \mathcal{F}_{n, (L_0, L_1)} \rangle \)
- \( \mathcal{A} \)
- \( L_0 \cap L_1 \)
- \( \text{Maslov index} \)
- \( \text{hol. strips} \)
- \( \text{H}_{\text{Floer}}(L_0, L_1; \text{choiro}) \)
Note: "Masur index" in \( \mathbb{R}^d \) setting is usually \( \infty \) since \( \dim W(p) = \infty \).

But we only need relative indices of crit pts: any map \( \text{ind} : \text{Crit } f \to \mathbb{Z} \) will do as long as it satisfies \( \dim M(p_i) = \text{ind } f \cdot \text{ind } p_i - 1 \).

For Lagrangian case, "renormalization" of index is given by Maslov index.

Steps to proving Floer's result:

1. \( HF_x (L_0, L_i) \) well-defined
   - set up grading, check moduli spaces are manifolds of the expected dimension
     \[ \to \exists : CF_v \to CF_x, \]  \text{TRANSVERSALITY}

   - \( \exists^2 = 0 \) : compactification of moduli space
   \[ \to HF_x (L_0, L_i) = H^* (CF_x, \exists) \]  \text{COMPACTNESS}

   - check indeg of all choices made along the way, e.g. \( J \) \text{ INVARINACE}

2. Behavior under Hamiltonian diffeomorphism
   - if \( L, L' \) are Hamiltonian isotopic \( (\exists \varphi \in \text{Ham} (M), \varphi (L) = L') \)

     \[ \text{then } HF_x (L_0, L) = HF_x (L_0, L') \]  \text{also INVARINACE}

   - if \( L_0, L_i \) are Hamiltonian isotopic

     \[ \text{then } HF_x (L_0, L_i) = H^* (L_0) \]

Then: \( L = L_{\text{Leg}}, \varphi \in \text{Ham} (M) \Rightarrow \)

\[ \# (L \cap \varphi (L)) = \text{rk } CF_x (L, \varphi (L)) \geq \text{rk } HF_x (L, \varphi (L)) = \text{rk } H^* (L). \]

Remarks: Don't use \( \infty \) duality Morse homology directly; lots of analytic difficulties.

(eg. long-time existence of gradient flow; transversality, compactness)

- Use cohomology instead of homology (i.e. reverse the arrows, \( d : CF^* \to CF^{**} \) because of product operations (to come).
Lagrangian Intersection Floer Cohomology

Setup as before:
- $(M,\omega)$ symplectic
- $L_0, L_1 \subset M$ cpt lagrangian, $L_0 \pitchfork L_1$ ($\Rightarrow \#(L_0 \pitchfork L_1) < \infty$)
- $J$ almost $C^\infty$ stc compatible with $\omega$.
  (note: the space of such is nonempty and contractible.)

Simplest setting: Floer complex $CF^*(L_0,L_1) = \bigoplus_{p \in L_0, q \in L_1} \mathbb{Z}_{\mathbb{Z}^*}$.
(we'll talk more about coefficients later).

Def: $p,q \in L_0 \pitchfork L_1$. $W(p,q)$ is the set of finite energy holomorphic strips between $p$ and $q$.
  
  - holomorphic strip: $u: \mathbb{R} \times [0,1] \to M$ satisfying
    
    - Cauchy-Riemann equation $\bar{\partial}_J u = \partial u + J \frac{\partial u}{\partial s} = 0$ (i.e. $i du = d\omega(J)$)
    
    - boundary conditions $u(\cdot,0) \in L_0$, $u(\cdot,1) \in L_1$

    - finite energy: $E(u) = \int_{\mathbb{R} \times [0,1]} |u|^2 \omega = \int \int |\frac{\partial u}{\partial s}|^2 ds dt < \infty$.

  $\mathbb{R} \times [0,1]$ is biholomorphic to $\overline{\mathbb{D}}^2 - \{ \pm 1 \}$

  and $u$ extends to $\overline{\mathbb{D}}^2$ with $\pm 1 \to q$, $-1 \to p$.

  - Reaction $W(p,q)$ by translation in $s$ direction:
    $\forall \alpha \in \mathbb{R}$: $u \mapsto u(s-\alpha,t)$.
\[ \mathcal{M}(p, q) = W(p, q) / \mathbb{R} \]

For \([\mathcal{U}] \in \pi_2(M, L_0, L_1)\), \(W(p, q; [\mathcal{U}]) = \{ u \in W(p, q) \mid [u] \equiv 0 \} \)

\[ \mathcal{M}(p, q; [\mathcal{U}]) = W(p, q; [\mathcal{U}]) / \mathbb{R} \]

The condition determining a finite energy holomorphic strip determines a Fredholm problem: if \(U\) is such a strip, then the linearization \(D_{\bar{\partial}_J} u \mid u = u\) is Fredholm on a suitable space of sections of \(U^* TM\) with boundary conditions, e.g.

\[ W^{1,1}(\mathbb{R} \times [0, 1], \mathbb{R} \times \{0, 1\}; U^* TM, U^*_1 L_0, U^*_1 L_1) \]

Then assuming that all solutions to (1) are regular (\(D_{\bar{\partial}_J} u\) is injective, then

\[ W(p, q; [\mathcal{U}]) = \{ u \in \text{class } [\mathcal{U}] \text{ satisfying (1)} \} \]

is a smooth manifold of dimension \(-\text{Fredholm index}\):

\[ \text{ind } [\mathcal{U}] = \dim \ker D_{\bar{\partial}_J} u - \dim \text{im } D_{\bar{\partial}_J} u \]

Hypothesis: we can define degrees on \(L_0, L_1\), so that

\[ \text{ind } [\mathcal{U}] = \deg q - \deg p \]

We'd then like to define

\[ d: \text{CF}^*(L_0, L_1) \to \text{CF}^{*+1}(L_0, L_1) \]

by

\[ d(p) = \sum_{\forall q \in \text{L}^* L_1, [\mathcal{U}] : \text{ind } [\mathcal{U}] = 1} \# \mathcal{M}(p, q; [\mathcal{U}]) q \]

Problem: in general this sum is infinite; but it's not infinite

(Courant compactness).
Def. Let $k$-field. The Novikov ring over $k$ is
\[ \Lambda_0 = \{ \sum \lambda_i T^{x_i} \mid \lambda_i \in k, \lambda_i \geq 0, x_i \rightarrow \infty \} \]
The Novikov field over $k$ is the field of fractions of $\Lambda_0$:
\[ \Lambda = \{ \frac{\lambda}{\mu} \sum \lambda_i T^{x_i} \mid \lambda, \mu \in \Lambda_0, \lambda \cdot \mu \neq 0 \} \]
Then define $\text{CF}(L_0, L_1) = \bigoplus_{p \in \Lambda_0} \Lambda \cdot p$.

Def. The Floer differential $d : \text{CF}(L_0, L_1) \to \text{CF}(L_0, L_1)$ is given by
\[ d(p) = \sum_{p \in \Lambda_0} \left( \# M(p, \gamma; \{ u \}) \right) \Theta \left( u, u^* \right) \]
where $E([u]) = \text{energy} = \int u^* \omega \geq 0$.

* In fact, for transversality, need to consider solutions to perturbed CR eqn.

Gromov compactness: given any $E_0$, there are finitely many $[u]$ with $M(p, \gamma; [u]) \neq \emptyset$ and $E([u]) \leq E_0$. So $d$ is well-defined when we work with Novikov coefficients.

- Special case: if $\omega$ is exact, $\omega = d\theta$ for $\theta \in \text{Ham}(M)$, and $L_0, L_1$ are exact, $\theta|_{L_i} = \text{def}$ for $f_i : L_i \to \mathbb{R}$, then $E([u]) = \int u^* \omega = (f_i(q_i) - f_0(q_i)) - (f_i(q) - f(q)) \text{ where } [u]$.
So can drop $\Lambda$ coefficients and work with $k$-coeff in rescaling $p_{\omega} + f_i(q_i) - f(q)$.

Thus (Floer) $k = \mathbb{Z}_2$. Assume $[\omega] \cdot \mathbb{P}(M, L_i) = 0$ for $i = 0, 1$.

Then $d$ is well-defined and satisfies $d^2 = 0$ and $\text{deg} d = +1$ (for a choice of grading of $\text{CF}(L_0, L_1)$), and $HF^+(L_0, L_1)$ is index $A$ choice of $J$ and inv't under Hamiltonian isotopies of $L_0, L_1$. 
Grading: let \([\mu]\) a homotopy class for a strip between \(p\) and \(q\).

There is a Maslov index \(\text{ind}([\mu]) \in \mathbb{Z}\) such that when transversality holds,
\[
\dim \mathcal{M}(p; \pi, [\mu]) = \text{ind}([\mu]) - 1.
\]
In nice circumstances, can assign degrees to \(p, q\) so that
\[
\text{ind}([\mu]) = \deg q - \deg p + \text{r} u
\]
so that \(\dim \mathcal{M}(p; \pi, [\mu]) = \deg q - \deg p - 1\) and \(\text{d} \) increases degree by 1.

To define degree to \(p \in \mathcal{L}_0 \cap L_i\):

Use the fact that \(\pi_1(\text{LGr}(n)) = \mathbb{Z}\) where
\[
\text{LGr}(n) = \{\text{Lagrangian } n\text{-planes in } (T\mathbb{R}^{2n}, \omega_0)\}.
\]
Pick a base point \(L_0, L_i\). Connect \(T_{p_0}L_0\) to \(T_pL_i\) by a fixed path \(\mathcal{L} \in \text{LGr}(n)\).

This path gives an \(\pi_1 \mathcal{L} = \mathbb{Z}\): grading of 1.

What's needed for this:
- \(\mathcal{L}_0 = \mathbb{C}(TM) = 0\) (to trivialize \(\text{LGr}(TM)\))
- Maslov class of \(L_i \in \text{Hom}(\pi_1(L_i), \mathbb{Z})\) vanish
  (so answer is index of path in \(L_0, L_i\))

Then: \(\dim \mathcal{M}(p, q) = \deg(p) - \deg(q) - 1\). (Calculation of Fredholm index)

(In general: grading lies in \(\mathbb{Z}/2\mathbb{Z}\) for some \(k\); have to define index for \([\mu]\). If Lagrangian are oriented then grading is always at least \(\mathbb{Z}/2\).

Ex: \(L_0, L_1\), oriented curves in a surface.

\[\text{even: } L_0 \leftrightarrow L_1 \text{ odd .} \]
Compactness ($\mathcal{C}^2=0$).

Gromov compactness: any sequence of holomorphic maps $u \to M(p,q)$ with bounded energy has a subsequence converging to:

1. an honest holomorphic strip
2. a broken strip

There exist infinite translated strips $u_n(s-a_n t), u_n(s-b_n t)$ converging to strips between $p,q$ and another intersection, $r$.

3. a strip with a disk bubble: energy concentrates at a point in $\Omega$(strip), and rescaling gives a hol. disk with $\Omega$ on $L_0$ or $L_1$.

4. a strip with a sphere bubble: energy concentrates at an inner point, and rescaling gives a hol. sphere.

The assumption $\mathcal{A} \cdot \bar{\nabla}(M,L) = 0$ rules out disk and sphere bubbles: if $u: (\mathbb{C},d^2) \to (M,L)$ is a disk bubble then $\mathcal{A} \cdot [u] = \int_{\mathbb{C}} u \mathbb{C} = \int_{\mathbb{C}} u^{\bar{\nabla}} d^2 \mathbb{C} > 0$ (unless $u=$ constant).
Then \( p, q \in L_i \), \([w] \in \mathcal{L}_i \). \( \pi_2(M, L_i) = 0 \).

1. If \( \text{ind}[w] = 1 \) then \( M(p, q; [w]) \) is a compact 0-manifold.
2. If \( \text{ind}[w] = 2 \) then \( M(p, q; [w]) \) can be compactified to a compact 1-manifold \( \overline{M}(p, q; [w]) \) with

\[
\mathcal{D}(p, q; [w]) = \bigcup_{\gamma \in L_i \cap \mathcal{L}_i} M(p, q; [w]) \times M(r, q; [w']) .
\]

This allows us to define \( d^* : CF^*(L_0, L_i) \to CF^{*+1}(L_0, L_i) \) and proves \( d^2 = 0 \).

Example: \( M \) is a surface; holomorphic maps are just usual conformal maps.

\( p_1 = T \alpha \alpha \beta \alpha p_1 + T \alpha \alpha \beta \alpha p_4 \), \( dp_1 = T \alpha \alpha \beta \alpha p_4 \), \( dp_4 = T \alpha \alpha \beta \alpha p_3 \) \( \Rightarrow d^2 p_i = 0 \).

Use Riemann mapping theorem.
What if \( \text{Gr} J \cdot \pi_2(M, L_i) \neq 0 \)? Then it's quite possible that \( d \neq 0 \).

Ex. \( M = \text{surface} \), but now \( L_1 \) bounds a disk.

\[
d(p) = -A^1 q, \quad d(q) = -A^2 p.
\]

\[
\Rightarrow d^2(p) = T_{A^1 A^2} p \neq 0!
\]

(Note: in this case degree one only defined mod \( 2 \))

This is a trivial strip plus a disk bubble:

\[
\text{this is all at } p.
\]

**Remark:** Sometimes possible to rule out disk bubbling in other circumstances besides \( \pi_2 = 0 \), e.g., when it's disallowed for degree reasons.

Also \( J \) approaches when disk bubbling happens—e.g., Cernea-Lands Lie branch approach.

**Invariance**

Claim: \( HF^*(L_0, L_i; J) = HF^*(L_0, L_i; J') \).

This uses a continuation argument: construct family of almost \( \alpha \) s

\[
\text{compatible with } \omega, \quad J(s), \quad \text{st. } J(s) = \begin{cases} J' & \text{loc.} \cr J & \text{glob.} \end{cases}
\]

\( s \gg 0 \).
Then look at moduli space of light strips \( \nu: R \times [0,1] \to M \) \( M_{J,J'}(p,s) \).

\[
\begin{align*}
\xi & \to \nu_p \\
& \frac{du}{ds} + J(x) \frac{dx}{ds} = 0.
\end{align*}
\]

We get a map \( \Xi: CF^*(L_0, L_1; J) \to CF^*(L_0, L_1; J') \) by

\[
\Xi(p) = \sum_{M_{J,J'}(y,x) = 0} (\ast M_{J,J'}(y,x)) \cdot q.
\]

This is a chain map: look at 1-dim moduli spaces. Boundaries look like

\[
\begin{align*}
\Xi & = d & \Xi' & = d \\
\end{align*}
\]

Similar argument to show invariance under Hamiltonian isotopy.

\[
HF^*(L, L) = HF^*(L): \text{perturb } L \text{ by a small Hamiltonian isotopy } \phi \text{ to } \phi(L): \text{the } HF^*(L, L) \text{ means } HF^*(L, \phi(L)) \text{ and this is independent of choice of } \phi.
\]

Energy estimate \( \text{strips contribute to } d \) on \( CF^*(L, \phi(L)) \) lie close to \( L \). So we can replace \((M, L)\) by \((T^*L, L)\) using Lagrange.

Choose a Morse function \( f: L \to \mathbb{R} \) and let \( L_0 = L \), \( L_1 = \phi(L) = \Gamma_{\text{crit } f} \subset T^*L \).

Note \( L_0 \cap L_1 = \text{crit } f \).
Prop (Floer) For a suitable choice of almost con J on L, we have a chain isomorphism,

\[(C\mathcal{F}^*(L_0, L_1), d) \cong (CM^*(f), d^*)\,.

\[\Rightarrow \, \operatorname{HF}^*(L, q(L)) \cong \operatorname{HM}^*(L, f).\]

Idea of pf: choose metric g on L. This induces a natural almost con J on T^*L such that at \((q, p) \in T^*L \), J maps \(\nabla f \in T_qL \subset T_{q, p}T^*L\)

\[\frac{df}{dt} \in T_{q, p}T^*L \subset T_{q, p}T^*L.

Now let \(\gamma(s)\) be a gradient flow for \(\nabla f \in L: \gamma'(s) = \epsilon \nabla f\).

Define \(u(s, t) = (\gamma(s), \epsilon t df(s))\).

Note this has boundary on 0 section and \(\partial u\).

This is almost holomorphic:

\[\frac{\partial u}{\partial s} \approx (\gamma(s), t) \Rightarrow (0, \epsilon df(s)) = \frac{\partial u}{\partial t}.

Proof: there is a 1-1 corr. between gradient flows for f and honest holomorphic strips. 0

Ex: \(L = \text{zero section} \subset S^1 \times \mathbb{R}\).

\[d(p) = T^A g - TA g = 0 \]

\[d(q) = 0 \]

\[\operatorname{HF}^*(L, q(L)) = \langle p, q \rangle \cong \mathbb{H}^*(S^1)\]
Product structure. In everything that follows, we need to perturb.

Suppose now we have three Lagrangians $L_0, L_1, L_2$, and suppose bubbling doesn't occur. (In particular, no bubble wind $\exists$ on $L_i$.)

For $p_1 \in L_0, p_2 \in L_1, p_3 \in L_2$, consider holomorphic maps

The domain is conformally equivalent to $\overline{D^2} - \{3 \text{ pts on boundary}\}$.

More precisely, given a homotopy class $[\gamma] \in \pi_2(M, L_0, L_1, L_2)$, define

$\mathcal{M}(p_1, p_2, p_3; [\gamma]) = \text{space of maps } u: D^2 \to M$

Such that:

- $u$ extends continuously to $\partial D^2$, mapping $2_0 \to q_1, 2 \to p_1, 3 \to p_2$, and arcs to $L_0, L_1, L_2$ as drawn
- $u$ is holomorphic: $\tilde{\partial} u = 0$
- $u$ has finite energy: $E([u]) < \infty$.
Assume transversality, \( M(p_1, p_2, q; \mathcal{U}) \) is a smooth manifold with dimension \( \text{dim}(\mathcal{U}) = \deg(q) - \deg(p_1) - \deg(p_2) \) (as before, assuming \( 2c_1(TM) = 0 \) and Maslov classes \( b_1, b_2, b_3 = 0 \)).

**Def:** The product is the map
\[
CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_2)
\]
defined by
\[
P_2 \cdot p_1 = \sum_{g \in \text{Loc}(L_0, L_1), \text{deg}(g) = 0} \left( \# M(p_1, p_2, q; \mathcal{U}) \right) T E(g, q).
\]

Then if \( [g] : T_g(M, L_i) = 0 \) for \( i \), then the product satisfies the Leibniz rule:
\[
d(p_2 \cdot p_1) = (dp_2) \cdot p_1 + p_2 (dp_1).
\]

Thus this induces a product
\[
HF^*(L_1, L_2) \otimes HF^*(L_0, L_1) \rightarrow HF^*(L_0, L_2).
\]

Furthermore, this product in cohomology is associative and independent of choice of almost complex structure.

**Associativity:** later.

**Examples:**
\[
p_2 \cdot p_1 = T^{A_2 + A_3} p_3, \quad dp_2 = T^{A_1 + A_2} p_5
\]
\[
dp_5 = T^{A_1} p_4, \quad p_5 \cdot p_1 = T^{A_2} p_4
\]
\[
d(p_2 \cdot p_1) = T^{A_1 + A_2 + A_3} p_4 = (dp_2) \cdot p_1 + p_2 (dp_1).
\]
Idea of proof that $d$ satisfies Leibniz:

Look at $\mathcal{U}(p_1, p_2, \gamma_i[u])$ for $\text{ind}[u]=1$. This is a cpt 1-dim mfd with boundary. Three types of boundary:

$$d(\phi_2 \cdot p_1) + (\phi_2 \cdot p_1) \cdot p_2 + p_2 \cdot (d\phi_1) = 0 \quad (\text{up to sign})$$

and so $d(\phi_2 \cdot p_1) + (\phi_2 \cdot p_1) + p_2 \cdot (d\phi_1) = 0 \quad (\text{up to sign})$.

Next: associativity. Product is not associative on chain level.
There is a "triple product"

$m_3: CF(L_0, L_3) \otimes CF(L_1, L_2) \otimes CF(L_2, L_0) \to CF(L_0, L_3)$

such that (mod 2)

\[
\langle p_3 : p_0 \rangle \cdot p_1 + \langle p_3 : p_1 \rangle \cdot p_0 = d\ m_2(p_0, p_1, p_1) + m_3(p_0, p_2, p_1) + \sum m_3(p_0, p_2, p_1)
\]

So the product is associative on cohomology.

$m_3$ counts 0-dim moduli spaces of 4-gms

$m_3(p_0, p_2, p_1) = q + \ldots$

(As example,

$m_3(p_1, p_2, p_2) = p_0$)

PF Y(-): look at degenerates of $p$
**Ab Category**

**Def.** Let $k$ be a field. An Ab category over $k$ consists of:

- Object $L_i$
- morphism spaces $\text{Hom}(L_0, L_i) = \text{graded } k$-vector space
- For $k \geq 1$, a degree $(2-k)$ operation
  
  $m_k : \text{Hom}(L_{k-1}, L_0) \otimes \cdots \otimes \text{Hom}(L_1, L_0) \otimes \text{Hom}(L_0, L_i) \to \text{Hom}(L_0, L_i)$

Satisfying the Ab relative:

\[
\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} m_{k-1-j}(p_{j+1}, \ldots, p_k; p_j, \ldots, p_1) = 0.
\]

**Pictorially:**

\[
\sum_{k=1}^{\infty} \sum_{j=0}^{k-1} m_{k-1-j}(\text{Diagram}) = 0
\]

**Ab relations for $k = 1, 2, 3$:**

\[
m_1(m_1(p_i)) = 0 = m_i^2 = 0 \quad m_i : \text{Hom}(L_0, L_i) \otimes \text{Hom}(L_0, L_i) \to \text{Hom}(L_0, L_i)
\]

\[
m_1(m_2(p_1, p_i)) + m_2(m_1(p_2), p_i) + m_2(p_1, m_1(p_i)) = 0
\]

**$m_1$ satisfies Leibniz w.r.t. $m_i: \text{Hom}(L_1, L_i) \otimes \text{Hom}(L_0, L_i) \to \text{Hom}(L_0, L_i)$**

\[
m_1(m_3(p_3, p_2, p_i)) + m_3(m_2(p_3, p_2), p_i) + m_3(p_3, m_2(p_2, p_i)) + m_2(m_1(p_3, p_2, p_i)) + m_3(p_3, m_1(p_2, p_i)) + m_3(p_3, m_1(p_2, p_i)) = 0
\]

**$m_3$ is associative up to error terms.**
Rule 1. An $A_\infty$ category with one object is an $A_\infty$ algebra.

2. If $\mathcal{C}$ is an $A_\infty$ category, then define the cohomology category $\mathcal{H}_\mathcal{C}$ by

- $\text{Ob } \mathcal{H}_\mathcal{C} = \text{Ob } \mathcal{C}$
- $\text{Hom}_{\mathcal{H}_\mathcal{C}}(L_0, L_1) = H^*(\text{Hom}_{\mathcal{C}}(L_0, L_1), \mathbb{Z})$

Composition in $\mathcal{H}_\mathcal{C}$ is induced by $m_2$.

$$m_2 : H^* \text{Hom}_{\mathcal{C}}(L_1, L_2) \otimes H^* \text{Hom}_{\mathcal{C}}(L_0, L_1) \to H^* \text{Hom}_{\mathcal{C}}(L_0, L_2)$$

(note associative by $m_2$ relation).

3. If $m_k = 0$ for $k \geq 3$, a $A_\infty$ category is a $d$-category, $m_i =$ differential, $m_1 =$ multiplication. (eg. $A_\infty$ algebras $\Rightarrow$ $d$-algebras).

4. Motivation: based loop space $\Omega X = \{ \gamma : [0,1] \to X | \gamma(0) = \gamma(1) = * \}$.

Stasheff '63: $C_*^{d}\Sigma(\Omega X)$ has the structure of an $A_\infty$ algebra.

Note $m_2 =$ concatenation: $C_*(\Omega X) \otimes C_*(\Omega X) \to C_*(\Omega X)$ is not associative, but

$$m_2(m_1(\gamma_1, \gamma_2), \gamma_3) + m_2(\gamma_1, m_1(\gamma_2, \gamma_3)) = m_2(m_1(\gamma_1, \gamma_2, \gamma_3))$$

5. Can recast $A_\infty$ relations in terms of bar construction.

Let $A = A_\infty$ algebra = graded vector space with $m_k : A^\otimes k \to A$. [degree shifted]

Define $\overline{T}A := \bigotimes_{k \geq 0} (A[1])^k$. This is the $\mathbb{Z}$-linear coalgebra of $A[1]$.

Comultiplication

$$\Delta(p_1 \otimes \cdots \otimes p_m) = \sum (p_{i_1} \otimes \cdots \otimes p_{i_s} \otimes p_{j_s} \otimes \cdots \otimes p_m)$$

This summarizes $A_\infty$ relations.

Proof: There is a 1-1 correspondence

$$(A_\infty \text{ algebra structure } (m_{ij}) \text{ on } A) \leftrightarrow (\text{cohomomorphism } \delta : \overline{T}A \to \overline{T}A \text{ with } \delta(1) = 0)$$

[by co-leibniz, $\delta$ is determined by components $\delta_i : A[1]^\otimes k \to A[1]$]
Fukaya category

$(M, \omega)$ symplectic. The Fukaya category is the Ab category $\text{Fuk}(M, \omega)$:

- **Objects**: compact, closed, oriented, spin Lagrangians $L \subset M$ with $[\omega]\cdot \tau_2(M, L) = 0$.

- **Morphisms**: $\text{Hom}(L_0, L_1) = \text{CF}^*(L_0, L_1)$

- $M_0 = d: \text{CF}^*(L_0, L_1) \to \text{CF}^*(L_1, L_0)$

- $M_k: \text{CF}^*(L_{i-1}, L_i) \otimes \ldots \otimes \text{CF}^*(L_0, L_1) \to \text{CF}^*(L_0, L_k)$

Counts holomorphic k-genus:

$$M_k(\beta_1, \ldots, \beta_k) = \sum_{\text{g, v}} M_{g, v} \cdot \beta_1^g \cdot \ldots \cdot \beta_k^g$$

$M(g, v) = \text{finite energy \& disks}$

These satisfy the Ab relations:

the moduli space $\mathcal{M}_{0,n}$ of conformed structure on $S^2$, $n+1$ boundary pt.s has a compactification: $\mathcal{M}_{0,n} \cong$ Stasheff associahedra

Top-dim facets are nodal degenerations $S^2 \to S^2 \cup D^2$

Each degeneration contributes to Ab relations.
\section*{Warning:} for this definition, all Lagrangians to intersect transversely.

But we want to describe eg. self-map $\text{Ham}(L_1)$. So: choose:

- A $L_0, L_1$, Hamiltonian perturbation $L', L'$ that are transverse
- A $L_0, \ldots , L_k$, perturbations $L'_1, \ldots , L'_k$ consistent with the pairwise perturbation (in fact: instead of perturbing Lagrangian, perturb $\partial$ equation by inhomogeneous Hamiltonian term corresponding to $(L_0, L_1)$ etc.)

Thus $\text{Fuk}(M, \omega)$ is a (cohomologically unital) $\text{Ab}$ category, well-defined up to quasi-equivalence ($\text{Ab}$ functor inducing $\cong$ on cohomology) indep. of choice + perturbation data.

\section*{Remark:} can generalize to other situations

- For homological Mirror Symmetry, need to enrich Fukaya category by adding local system to each Lagrangian.

To make contact w/ HMS, need to produce a triangulated category out of $\text{Fuk}(M, \omega)$. Standard technique: twisted complexes.

\textbf{Def:} $A$-Ab category. A twisted complex consists of objects $L_1, \ldots , L_n$ with a strictly lower triangular differential $\delta \in \text{End}(L_1 \oplus \ldots \oplus L_n)$, i.e. $\delta_{i,j} \in \text{Ham}(L_i, L_j)$ for $i < j$ such that $\sum \text{w}_k(\delta_{i+k, i}, \ldots , \delta_{i, i}) = 0$ $\forall i, k$:

\begin{align*}
\begin{array}{c}
L_0 \xrightarrow{\delta_{0,1}} L_1 \\
\vdots \\
L_0 \xrightarrow{\delta_{0,1}} L_1 \xrightarrow{\delta_{1,2}} L_2 \\
\vdots \\
\end{array}
\end{align*}

$\text{w}_1(\delta_{0,1}) = 0$ $\text{w}_2(\delta_{1,2}, \delta_{0,1}) + \text{w}_1(\delta_{0,2}) = 0$.
Def \( \text{Tw} \mathcal{A} = \text{A} \text{ss category with} \)
\[ \begin{align*}
\cdot \text{objects} &= \text{twisted complexes} \\
\cdot \text{morphisms} &= \text{maps between twisted complexes} \\
\cdot m_i : \text{Hom}(E, F) \to \text{defined by} m_i(\varphi) &= \sum_{k \neq i} m_k(\delta, \ldots, \delta, \varphi, \delta, \ldots, \delta) \\
\end{align*} \]

\[ m_i \left( \begin{array}{c}
L_0 \\
\delta
\end{array} \right) = \\
\left( \begin{array}{c}
L_0 \\
\delta
\end{array} \right) \begin{array}{c}
\varphi_{01} \\
\varphi_{10}
\end{array} \\
\left( \begin{array}{c}
L_0 \\
\delta
\end{array} \right)
\]

\[ m_1 \varphi_{00} + m_2 (\varphi_{10}, \delta) + m_3 (\delta, \varphi_{01}) + m_4 (\delta, \delta, \varphi_{00}) \]

\[ m_k \text{ defined similarly for } k \geq 1. \]

This is a triangulated \( \text{A} \text{ss} \) category.

Def \( \mathcal{A} = \text{A} \text{ss category and the derived category is} \)
\[ \text{DA} = \mathcal{H}(\text{Tw} \mathcal{A}). \]

When \( \mathcal{A} = \text{Fuk}(M, \omega) \), this produces the (bounded) derived Fukaya category \( \text{DFuk}(M, \omega). \)