1. (10 points) Let $0 \leq a < b < 2\pi$ and let $f$ be the function on the unit circle $\{ |z| = 1 \}$ defined by
\[
f(e^{i\theta}) = \begin{cases} 
1 & a \leq \theta \leq b \\
0 & 0 \leq \theta < a \text{ or } b < \theta < 2\pi.
\end{cases}
\]
Find an (integral-free) formula for the harmonic function $u : \Delta \to \mathbb{R}$ that limits to $f$ on the unit circle, and find an analytic function on $\Delta$ whose real part is $u$. Then show directly (without using Schwarz's theorem) that $\lim_{z \to z_0} u(z) = f(z_0)$ for $z_0 = e^{i\theta}$, $\theta \neq a, b$.

2. (15 points)
(a) Prove that every analytic, one-to-one, onto function $f : \Delta \to \Delta$ can be uniquely written as $f = c\phi_a$ for some $a, c$ with $|c| = 1$ and $a \in \Delta$, where $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$.
(b) Let $U$ be a connected open set containing $B(0, 1)$. Find all analytic functions $f : U \to \mathbb{C}$ that satisfy $|f(z)| = 1$ whenever $|z| = 1$. (Note similarity to a problem from HW 2.)
(c) Consider the following three regions in $\mathbb{C}$: $\{ r_1 < |z| < r_2 \}; \{ 0 < |z| < r \}; \mathbb{C} - \{ 0 \}$. (Here $r_1, r_2, r$ are any fixed numbers.) These are all homeomorphic; prove that they are not conformally equivalent to each other.

3. (10 points; a mashup of Ahlfors section 4.3.4 exercises 1, 6, and 7.)
(a) Use the Schwarz Lemma to show that if $f$ is an analytic map from $\Delta$ to $\Delta$, then
\[
\frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2}.
\]
(b) If $\gamma$ is a piecewise differentiable path in $\Delta$, the integral
\[
\int_{\gamma} \frac{|dz|}{1 - |z|^2}
\]
is called the hyperbolic length of $\gamma$. Show that if $f$ is an analytic map from $\Delta$ to $\Delta$, then $f$ maps every $\gamma$ to a path with smaller or equal hyperbolic length. Deduce that a Möbius transformation of the form in 2(a) preserves hyperbolic length.
(c) Prove that the path of smallest hyperbolic length that joins two given points in $\Delta$ is a circular arc that is orthogonal to the unit circle. (Hint: use a Möbius transformation sending one point to 0.)