1. (a) First note any rotation of the sphere fixes two antipodal points $P_1, P_2$, and stereographic projection sends these to $a, b \in \mathbb{C} \cup \infty$ with $b = -\frac{1}{a}$.

Next let $C$ be a “latitude” of the sphere w.r.t. $P_1, P_2$, i.e., the intersection of $S^2$ with a plane $\perp \overline{P_1 P_2}$. Then $C$ is the locus of points $Q \in S^2$ with $\frac{d(Q, P_1)}{d(Q, P_2)}$ fixed. Thus by (23), p. 20, the stereographic projection of $C$ is the locus of points $z \in C$ with

$$\frac{2(z-a)}{2(z-b)} = \sqrt{\frac{1+|b|^2}{1+|a|^2}} \frac{2-a}{2-b}$$

fixed; so $C$ maps to a circle of Apollonius for $a, b : \{|z-a| = \text{const.}\}$.

A rotation fixing $P_1, P_2$ must also fix $C$, so it corresponds to a map fixing each circle of Apollonius:

$$z \mapsto w \text{ with } \frac{w-a}{w-b} = k \frac{z-a}{z-b}, \quad |k|=1.$$  

Thus for $a > 0$ this is $z \mapsto kz$, $(k=1)$.

(4)

These are circles of Apollonius for some $a, b \in \mathbb{R}$.

Solve for $a, b$ by noting

$$\frac{1-a}{1-b} = \frac{|a-1|}{|b-1|} = \frac{|a+1|}{|b+1|} = \frac{a+1}{1-b}$$

and

$$\frac{5-a}{5-b} = \frac{|a+5|}{|b+5|} = \frac{|a+3|}{|b+3|} = \frac{a+3}{5-b}$$

$$\Rightarrow a = -7 + 4\sqrt{3}, \quad b = -7 - 4\sqrt{3}.$$
1. (b) If we map \( C_{u, v} \) by a Möbius transformation \( T \) with \( T(a) = 0 \), \( T(b) = \infty \), then the circles become concentric circles around 0, and the circles that are \( \perp \) to both are exactly lines through 0. Thus the circles \( \perp \) to \( \{ |z-1| = 1 \} \) and \( \{ |z-4| = 1 \} \) are the circles passing through the two points \(-7 \pm 4\sqrt{3}\). 

2. (a) The inequality implies \( f(z) \notin (-\infty, 0] \). Thus if \( \log \) principal branch, then \( \log f(z) \) is well-defined and gives a primitive for \( \frac{f'(z)}{f(z)} \) on \( \Omega \).

(b) \( \int_C p(z) \, dz = 0 \) (\( p \) has a primitive).

If \( p(z) = x_n (z-a)^n \) then
\[ \int_C \overline{p(z)} \, dz = \int_C \overline{a_n (z-a)^n} \, dz = \int_C \overline{a_n} \, r^n \, e^{-i\pi n} (i r e^{i\theta}) \, dt = \left\{ \begin{array}{ll} 0 & , n \neq 1 \end{array} \right. \]

Thus in general \( \int_C \overline{p(z)} \, dz = 2\pi i r^2 \overline{p'(a)} \).

(c) If \( \gamma \) is path from \( 1+i \) to \(-1+i \) then it's parametrized by \( \gamma(t) = i - t , \quad -1 \leq t \leq 1 \) \( \sqrt{\text{principal branch}} \)
\[ \Rightarrow \int_{\gamma} \frac{dz}{z} = \int_{-1}^{1} \frac{dt}{i-t} = \log (i-t) \bigg|_{-1}^{1} = \frac{\pi i}{2}. \]

Similarly each side contribute \( \frac{\pi i}{i} \) (note we need a non-principal branch of \( \log \) for the side from \(-1+i \) to \(-1-i \)), so the answer is \( \frac{\pi i}{2} \).
3. Write $a = re^{i\theta}$; then

$$\int \frac{dt}{\sqrt{z - a}} = \int_0^{2\pi} \frac{i e^{it} dt}{Re^{it} - re^{i\theta}} = iR \int_0^{2\pi} \frac{e^{it} dt}{Re^{it} - r}.$$ 

Parametrize by $Re^{it}, \theta \leq t \leq 2\pi + \theta$.

Now

$$\frac{e^{it}}{Re^{it} - r} = \frac{e^{it}(Re^{it} - r)}{(Re^{it} - r)(Re^{it} - r)} = \frac{R - re^{it}}{(R \cos t - r)^2 + (R \sin t)^2} = \frac{(R - r \cos t) - ir \sin t}{R^2 \cos^2 t - 2R \cos t + 1},$$

and

$$\int_0^{2\pi} \frac{r \sin t}{R^2 \cos^2 t - 2R \cos t + 1} dt = 0$$

since the integrand, $f(t)$, satisfies $f(2\pi - t) = -f(t)$.

The other integral is

$$\int_0^{2\pi} \frac{R - r \cos t}{R^2 \cos^2 t - 2R \cos t + 1} dt$$

and derivative is $\frac{1}{R} \left( \frac{t + \tan^{-1} \left( \frac{r - R}{r + R} \cos \left( \frac{t}{2} \right) \right)}{2} \right)$.

The graph of $\tan^{-1} \left( \frac{r - R}{r + R} \cos \left( \frac{t}{2} \right) \right)$ looks like:

\[\begin{array}{c|c|c}
\pi & \frac{\pi}{2} & \frac{\pi}{2} \\
\pi & \frac{\pi}{2} & \frac{\pi}{2} \\
\pi & \frac{\pi}{2} & \frac{\pi}{2} \\
\pi & \frac{\pi}{2} & \frac{\pi}{2} \\
\end{array}\]

So

$$\int_0^{2\pi} \frac{R - r \cos t}{R^2 \cos^2 t - 2R \cos t + 1} dt = \frac{1}{R} \left( \frac{\pi + \pi}{2} \right) = \frac{\pi}{R},$$

and

$$\int_0^{2\pi} \frac{R - r \cos t}{R^2 \cos^2 t - 2R \cos t + 1} dt = \begin{cases} \frac{\pi}{R} & r < R \\ 0 & r > R \end{cases}.$$