1. (a) First: a Möbius transformation sending the unit disk to the upper half plane:

\[ f(z) = \frac{z - i}{z + i} \]

\[ f(1) = w \text{ when } (2, i, 1, -1) = (w, 1, 0, \infty) = w \]

\[ \Rightarrow f(z) = -i \frac{z - 1}{z + 1} \]

Now compose:

(b) Send \( z \to \infty \) to transform the circles into parallel lines:

Then compose:

and note each map is \( 1 - 1 \) (and onto).
1. Let $C = \mathbb{C}$.

\[ C - \{ -1, 0 \} \rightarrow \{ \text{Im } z > 2 \} \rightarrow \{ |z| < 1 \} \rightarrow \{ |z| < 1, |\text{Re } z| < \frac{1}{2} \} \]

Note: $z$ is missing because it's the image of $\infty$.

\[ z \rightarrow \exp(\theta \log z) \]

\[ \{ \text{Re } z > 0 \} \rightarrow \{ \text{Im } z = 0 \} \rightarrow \{ |z| > 1 \} \rightarrow \{ |z| > 1, |\text{Re } z| < \frac{1}{2} \} \]

2. Lemma: \( \varphi_a(z) = \frac{z-a}{1-\bar{a}z} \) sends \( \{ |z| = 1 \} \) to itself if \( a \in \mathbb{C}, |a| \neq 1 \).

Proof: \( |z| = 1 \implies |z-a|^2 = (z-a)(\bar{z}-\bar{a}) = 1 - a\bar{z} - \bar{a}z + |z|^2 = (1-\bar{a})z(1-a\bar{z}) = |1-\bar{a}z|^2 \)

\[ |\bar{a}z| = |1-\bar{a}z| \implies \frac{|\bar{a}z|}{|1-\bar{a}z|} = 1 \quad \text{since } z, 1-\bar{a}z \neq 0. \]

Now suppose \( T = \text{Möbius sending } \{ |z| = 1 \} \) to itself. Then \( T^{-1} \) sends symmetric points to symmetric points, so it has to send \( \infty, 0, a, \bar{a}^{-1} \) for some \( a \in \mathbb{C} \setminus \{0\} \), \( |a| \neq 1 \).

If \( a = \infty \) then \( T(0) = \infty, T(\infty) = 0 \implies T(z) = \frac{k}{z} \) for some \( k \in \mathbb{C} \).

Then \( T \) sends \( \{ |z| = 1 \} \) to itself \( \implies |k| = 1 \).

Otherwise \( a \in \mathbb{C} \) and \( T(0) = a, T(\infty) = \bar{a}^{-1} \implies T(z) = k \frac{z-a}{1-\bar{a}z} = k \varphi_a(z) \) for some \( k \in \mathbb{C} \). By the lemma, \( T \) sends \( \{ |z| = 1 \} \) to itself \( \implies |k| = 1 \).

So the answer is: \[ \begin{cases} k \varphi_a(z) & \text{for some } a, k \in \mathbb{C} \text{ with } |k| = 1 \text{ and } |a| \neq 1, \\ \frac{k}{z} & \text{for some } k \in \mathbb{C} \text{ with } |k| = 1. \end{cases} \]
3. Suppose \( R(z) \) is rational and it sends \( \{ |z| = 1 \} \) to itself.
Then for all \( z \) with \( |z| = 1 \),
\[
1 = R(z) \overline{R(z)} = R(z) \overline{R(\overline{z})}.
\]
But \( R(z) \overline{R(\overline{z})} \) is a rational function (just multiply it out),
so the only way \( R(z) R(\overline{z}) - 1 \) can have infinitely many roots
is if it's identically 0:
\[
R(z) \overline{R(\overline{z})} = 1 \quad \text{for all } z \in \mathbb{C} - 0.
\]
(omit 0 because of \( \overline{z}^{-1} \))

Now write \( R(z) = a e^k \frac{(z-r_1) \cdots (z-r_n)}{(z-s_1) \cdots (z-s_m)} \) for \( k \in \mathbb{Z}, a \in \mathbb{C} - 0, r_i, s_j \neq \infty, ri \neq sj, i, j \).

Then
\[
R(z) \overline{R(\overline{z})} = |a|^2 e^{-2k} \frac{(z-r_1) \cdots (z-r_n)}{(z-s_1) \cdots (z-s_m)} \frac{(1-\overline{r}_1) \cdots (1-\overline{r}_n)}{(1-\overline{s}_1) \cdots (1-\overline{s}_m)}.
\]
For this to be \( 1 \), we must have:
\[
l = m, \quad \{ s_1, \ldots, s_m \} = \{ \overline{r}_1, \ldots, \overline{r}_n \} \quad \text{(i.e. } s_i \text{'s are } \overline{r}_i \text{'s in some order),}
\]
and \( a e^k \prod_{i=1}^n r_i \cdots \overline{r}_m = s_1 \cdots s_m \).

\[\text{For } r \neq 0, \quad r \text{ is zero of } R(z) \iff \overline{r}^{-1} = \text{pole of } R(z).\]

We can then write
\[
R(z) = a e^k \frac{(z-r_1) \cdots (z-r_n)}{(z-s_1) \cdots (z-s_m)} = (-1)^m a e^k \frac{\prod_{i=1}^n (z-r_i)}{\prod_{j=1}^m (z-s_j)} \phi_1(z) \cdots \phi_n(z),
\]
and \( |z|^m \phi_1(z) \cdots \phi_n(z) \).

\[
R(z) = k \times \phi_1(z) \cdots \phi_n(z)
\]
for \( k \in \mathbb{Z}, l = 1, m > 0, r_i, s_j \in \mathbb{C} - 0 - \{1\} \).

Conversely, any such \( R(z) \) sends \( \{ |z| = 1 \} \) to itself since if \( z = 1 \) then
\[
|R(z)| = \frac{|a| |z|^k |\phi_1(z)| \cdots |\phi_n(z)|}{|z|^m} = 1.
\]
\( \phi_i \neq 0 \)
4. Note if any of $z_1, z_2, z_3, z_4$ are equal, then two consecutive ones are equal (w.r.t. $z \equiv z'$) and the result is clear.

Otherwise, divide out by $|z_2-z_1||z_2-z_4|$ then we need to show

$$|z_1, z_2, z_3, z_4| = 1 + |z_1, z_3, z_2, z_4|$$

Let $T = M$ with $T_2 = 1, T_3 = 0, T_4 = 0$. Write $w = Tz_1 = (z_1, z_2, z_3, z_4)$ then WLOG (since it's concyclic with $1, 0, \infty$) and $w > 1$ (because of order of $z_1, z_2, z_3, z_4$). Thus

$$(z_1, z_3, z_2, z_4) = (w, 0, 1, \infty) = 1 - w$$

$$\Rightarrow 1 + |z_1, z_3, z_2, z_4| = 1 + |1 - w| = w = |z_1, z_3, z_2, z_4|$$

as desired.

Geometrically this is Ptolemy's Theorem: the product of the diagonals of a cyclic quadrilateral is the sum of the products of the opposite sides.