Riemannian Metrics

**Def.** A **Riemannian metric** is a \((0,2)\)-tensor \(g\), i.e. smoothly varying
\[ g : T_xM \otimes T_xM \to \mathbb{R} \]
\[ v \otimes w \mapsto g(v, w) \text{ or } \langle v, w \rangle \]
that is,
- **Symmetric**: \(g(v, w) = g(w, v)\)
- **Positive Definite**: \(g(v, v) > 0\), equivalent: \(v = 0\).

In coordinates, write \(g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\).

**Def.** A **Riemannian manifold** is \((M, g)\), \(M = \text{smooth manifold}\),
\(g = \text{Riem. metric}\).

When are two such maps the same?

**Def.** An **isometry** between \((M, g)\) and \((N, h)\) is a diffeomorphism
\[ \varphi: M \to N \text{ such that } g = \varphi^* h, \text{ i.e. : } \]
\[ h(\varphi_* v, \varphi_* w) = g(v, w) \quad \forall v, w \in T_pM \]

\(\varphi: M \to N\) is a **local isometry** at \(p \in M\) if \(\text{and only if } \varphi|_U \text{ isometry} \) with \(\varphi|_U: U \to \varphi(U)\) isometry.
Ex 1. \( \mathbb{R}^n \), \( g = \) standard inner product. \( g_{ij} = \delta_{ij} \) Kronecker delta.

Euclidean space.

Ex 2. Lie groups.

Def. A Riemann metric \( <, > \) on \( G \) is left invariant if \( L_h \) is an isometry for \( h \in G \).

\( \forall h \in G, \forall v, w \in T_g G, \quad <v, w> = <(L_h)_* v, (L_h)_* w> \)

at \( g \)

at \( h g \)

Similarly: right invariant. If both, then biinvariant.

Easy to construct left invar metric: given an inner product \( <, > \) on \( g = T_e G \), define \( <v, w> = <(L_g)_* v, (L_g)_* w> \).

Fact: any compact Lie group has a biinvariant metric

(see do Carmo p. 46 #7). (not true in general)

Proof. Let \( < , > \) be the left invariant metric on \( G \) induced by \( <, > \) on \( g \).

Then \( <, > \) is biinvariant \( \iff \)

\( 0 = <[x, y], z> + <y, [x, z]> \quad \forall x, y, z \in g. \)

PE. \( \Rightarrow \). Let \( x, y, z \in g \). For \( t \in \mathbb{R} \), recall \( \exp(tx) = (\phi_e^t)(e) \);

and \( \left[ x, y \right] = \frac{d}{dt} \bigg|_{t=0} \text{Ad}(\exp(tx)) \cdot y \) where

\( \text{Ad}(L_h)^* = (L_h)_\ast \).

Then \( <y, z> = <(\text{Ad}(\exp(tx)) \cdot y, (\text{Ad}(\exp(tx)) \cdot z) \quad h = \exp(tx) \)

\( = <(\text{Ad}(\exp(tx)) \cdot y, (\text{Ad}(\exp(tx)) \cdot z) \quad h = \exp(tx) \)

and \( \frac{d}{dt} \bigg|_{t=0} 0 = <[x, y], z> + <y, [x, z]> \).

\( \Leftarrow \) : \text{Trivially.}
Ex 3. Given an immersion $f: M \to N$ (often use $N = \mathbb{R}^{n+k}$),
a Riem. metric $\langle \cdot, \cdot \rangle_N$ on $N$ induces one $\langle \cdot, \cdot \rangle_M$ on $M$:
\[ \langle V, W \rangle_M = \langle f_\ast V, f_\ast W \rangle_N. \]

Note immersion $\Rightarrow df = f_\ast : T_pM \to T_{f(p)}N$ is injective.
Thus $\langle \cdot, \cdot \rangle_M$ is pos def since if $\langle V, V \rangle_M = 0$ then $f_\ast V = 0 \Rightarrow V = 0$.

So e.g. embedded submanifolds $M \subset \mathbb{R}^{n+k}$ have metric induced from
Euclidean metric on $\mathbb{R}^{n+k}$.

ex: $S^n \subset \mathbb{R}^{n+1}$ unit sphere inherits the "round metric".
\[ \sum x_i^2 = 1 \] (for $S^1$, this agrees with flat metric on $\mathbb{R}/2\pi$)

More general way to get submanifolds:

$h : N \to P^k$ smooth.
- $p \in N$ is a critical point if $dh_p$ is not surjective.
- $p \in P$ is a critical value if $h^{-1}(p)$ has critical $p$.
- $q \in P$ is a regular value if $h^{-1}(q)$ is smooth.

Prop $q = \text{regular value} \Rightarrow h^{-1}(q) \subset N$ is a smooth $n$-manifold.

So a metric on $N$ induces one on $M \subset N$.

If $p \in M \Rightarrow dh_p : T_pN \to T_{f(p)}P$ is surjective: can choose charts $x_1, \ldots, x_n, y_1, \ldots, y_n$ near $p$ s.t. $h = (h_1, \ldots, h_k)$,
\[ \frac{\partial (h_1, \ldots, h_k)}{\partial (y_1, \ldots, y_k)} \] is non-singular.

Implicit Function Thm $\Rightarrow \exists g : \mathbb{R} \to \mathbb{R}^k$ with $h(x, g(x)) = q$,
and this gives a chart $\mathbb{R} \to M$.
\[ x \mapsto (x, g(x)). \]
Ex. 4. Product manifolds.

\((M_1, g_1), (M_2, g_2)\) Riem manfs \(\to M_1 \times M_2\).

Recall \(T_{(p_1, p_2)} (M_1 \times M_2) = T_{p_1} M_1 \otimes T_{p_2} M_2\).

If \((v_1, v_2), (w_1, w_2) \in T_{(p_1, p_2)} (M_1 \times M_2)\) then define

\[ g((v_1, v_2), (w_1, w_2)) = g_1(v_1, w_1) + g_2(v_2, w_2). \]

Easy to check: this is a metric.

So e.g. \(S^n \times \cdots \times S^1\) has a metric induced from \(S^1\)
(the book calls it the "flat metric").

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Prop Any smooth mf has a Riem metric.

Def \(\{ V_\alpha \subset M \}\) is locally finite if \(\forall p \in M, p \in V_\alpha\) for only finitely many \(\alpha\).

Def \(f \in C^0(M)\). The support of \(f\) is \(\text{supp } f = \{ p \in M | f(p) \neq 0\}\).

Def \(\{ V_\alpha \}\) locally finite open cover of \(M\). A partition of unity subordinate to \(\{ V_\alpha \}\) is a collection \(f_\alpha \in C^0(M)\) st.

1. \(f_\alpha(p) \geq 0 \forall p\)

2. \(\text{supp } f_\alpha \subset V_\alpha\)

3. \(\sum_\alpha f_\alpha(p) = 1 \forall p\) (Note finite sum for each \(p\)).

Thus \(M\) smooth mf, \(\{ V_\alpha \}\) open cover. Then \(\exists\) locally finite
open cover \(\{ V_\alpha' \}\) subordinate to \(\{ V_\alpha \}\), i.e. \(\forall \alpha \exists V_\alpha' \subset V_\alpha\) for some \(\alpha\),
and a partition of unity subordinate to \(\{ V_\alpha' \}\).
PROOF. Let $\{(F_a, U_a, \mathcal{V}_a)\}$ be an atlas for $M$. By hypothesis, assume $\mathcal{V}_a$ is locally finite and $F_a$ is a partition of unity subordinate to $\mathcal{V}_a$.

Each $a$ determines a metric $\langle , \rangle_a$ on $U_a$: $\langle v, w \rangle_a = \langle F_a^* v, F_a^* w \rangle_{U_a}$.

Then define $\langle , \rangle$ on $M$ by $\langle v, w \rangle = \sum_a f_a(p) \langle v, w \rangle_a$. (Note: finite sum.)

Easy to check: symmetric, bilinear, positive definite. $\square$

Uses for metrics:

- Length of curves. $\gamma: [a, b] \to M$ piecewise smooth.
  $\Rightarrow$ Length $L(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2} \, dt$.
  Usual pf: indep of parametrization.

- Isom between tangent + cotangent.
  $p \in M \rightarrow g(p): T_p M \otimes T^*_p M \rightarrow \mathbb{R}$
  $\Rightarrow$ $\tilde{g}(p): T_p M \rightarrow T^*_p M$.
  $g$ is positive definite $\Rightarrow$ isomorphism. So get $\tilde{g}: TM \cong T^* M$ bundle isom.
  Note: works for any nondeg $(0, 2)$-tensor.

- Volume form. Assume $M$ oriented.
  $g \rightarrow \omega = \Omega^n(M)$ given in local coords by
  $\omega = \sqrt{\det(g_{ij})} \, dx^1 \wedge \cdots \wedge dx^n$.
  $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$. 
Check well-defined: change of coords \( x_1 \cdots x_n \rightarrow y_1 \cdots y_n \):
\[
\frac{\partial}{\partial y_i} = \sum \frac{\partial x_k}{\partial y_j} \frac{\partial}{\partial x_k}.
\]
Write Jacobian \( (\frac{\partial x_j}{\partial y_i}) = M \): then if \( \tilde{g}_{ij} = g(\frac{\partial x_i}{\partial y_j}, \frac{\partial x_j}{\partial y_j}) \), the
\[
(\tilde{g}^{ij}) = M (g_{ij}) M^T.
\]
\[
\tilde{g}_{ij} = \sum \frac{\partial x_k}{\partial y_i} \frac{\partial x_k}{\partial y_j} g_{kl}.
\]
Also \( \left( \frac{\partial x_i}{\partial y_k} \right) = M^{-1} (\frac{\partial y_j}{\partial x_m}) \Rightarrow dx_1 \cdots dx_n = (\det M) dy_1 \cdots dy_n.
\[
\Rightarrow \sqrt{\det \tilde{g}_{ij} \; dy_1 \cdots dy_n} = (\det M) \sqrt{\det g_{ij} \; dx_1 \cdots dx_n}.
\]
(If \( \det M > 0 \)) \[
\sqrt{\det \tilde{g}_{ij} \; dy_1 \cdots dy_n} = \sqrt{\det g_{ij} \; dx_1 \cdots dx_n}.
\]

Affine Connections

Motivation:

1. How to take directional derivative of vector fields?

   Would like: \( X \) tangent vector, \( Y = \sum b_i \frac{\partial}{\partial x_i} \)

   \[ \Rightarrow " \text{ directional derivative } " \; \; X(Y) = \sum X(b_i) \frac{\partial}{\partial x_i}. \]

   Problem: not coordinate independent.

   \( \text{Ex.} \; M = \mathbb{R} \)

   \( \chi = \frac{\partial}{\partial x}, \; \; Y = \frac{\partial}{\partial x} \Rightarrow "X(Y)" = 0. \)

   But: another coord system \( Y = x^3 \).

   \( \frac{\partial}{\partial x} = \frac{dy}{dx} \frac{\partial}{\partial y} = 3y^2 \frac{\partial}{\partial y} \)

   \[ "X(Y)" = 3y^2 \frac{\partial}{\partial y} (3y^3) \frac{\partial}{\partial y} = 6y \frac{\partial}{\partial y} \frac{\partial}{\partial y} = 0. \]
2. What's the analogue of a straight line in a manifold $M$? Need notion of "parallel" vector along a curve.

A path in $M$, $\gamma$ = vector field along $\gamma$, i.e. $\gamma(t) \in T_{\gamma(t)}M$.
What does it mean for $V$ to be parallel? $\frac{\partial V}{\partial t} = 0$.

For $\gamma \subset \mathbb{R}^3$ surface, $\gamma(t) \in \mathbb{R}^3$ so $\frac{\partial V}{\partial t}$ is a vector in $\mathbb{R}^3$ but not nec. in $T_{\gamma(t)}M$.

Maybe define $\frac{\partial V}{\partial t} = \text{orthogonal projection of } V'(t) \text{ to } T_{\gamma(t)}M$.

In general:

**Def.** An affine connection $\nabla$ on $M$ is a map

$$\nabla : \text{Vect} M \times \text{Vect} M \to \text{Vect} M$$

$$(X, Y) \mapsto \nabla_X Y$$

Satisfyings:
1. $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$; $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$
2. $\nabla_{fX} Y = f \nabla_X Y$
3. $\nabla_X (fY) = f \nabla_X Y + Y(f) X$

Note: for fixed $Y$, the map $\nabla_Y : \text{Vect} M \to \text{Vect} M$ is a tensor.
Interlude: cards and indices

Convention: write $x^1, \ldots, x^n$ for cards.

$T^a_M$ satisfies $\frac{\partial}{\partial x^i} T^a_M = \delta^a_i T^a_M$; $T^a_M$ satisfies $\partial x^i$.

Vector field $\Sigma a^i \partial_i = a^i \partial_i$; 1-form $\Sigma b^i dx^i = b^i dx^i$.

Einstein summation notation: sum over repeated indices (the upper and lower)

$X = a^i \partial_i \Rightarrow X f = \Sigma a^i \frac{\partial f}{\partial x^i} = a^i \partial_i f$.

Metric $g_{ij} = \langle \partial_i, \partial_j \rangle$; $g = \sum g_{ij} dx^i \otimes dx^j$.

For future use: write inverse metric as $(g^i_j)$; write $g_{ij} g^{ij} = \delta^i_j$.

Write $D_i = D_{\partial_i}$.

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Connection in coordinates.

Def: $x^i, \ldots, x^n$ cards. The Christoffel symbol $\Gamma^k_{ij}$, $1 \leq i, j, k \leq n$, is defined by

$D_{\partial_{x^i}} (\partial_{x^j}) = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$.

Connections are local operators (exc) and on a chart,

$\Gamma^k_{ij}$ determine the connection.

If $D$ = connection then

$D_{\partial_{x^i}} (\partial_{x^j}) = \partial_{x^i} \partial_{x^j} = \partial_{x^i} (\partial_{x^j} + \partial_{x^i} \partial_{x^j}) = \partial_{x^i} \Gamma^k_{ij} \partial_{x^k} + \partial_{x^i} (\partial_{x^j})$.

Conversely can define $D_{\partial_{x^i}}$ on $x,y$ by this formula.

and check that this gives a connection.
Let $M = \mathbb{R}^n$, $\Gamma^k_{ij} = 0 \forall i, j, k$.

$x = a^i e_i, y = b^i e_j \Rightarrow \nabla_x y = a^i \Gamma^k_{ij} b^j e_k = X(b^j) e_j$.

(Caveat: in a different coordinate system, $\Gamma^k_{ij}$ might not be 0).

Now: given $\delta = \text{curve in } M$, $V = \text{vector field along } \delta$, use $\nabla$ to define $\frac{dV}{dt}$. ("Covariant derivative along a curve")

Prop: $M$ smooth, $\nabla = \text{affine connection}$.

Then $\exists!$ way to associate to a curve $\gamma(t)$ and a vector field $V$ along $\gamma$, another vector field $\frac{dV}{dt}$ along $\gamma$, st.:

1. $\frac{d}{dt} (V + W) = \frac{dV}{dt} + \frac{dW}{dt}$
2. $\frac{d}{dt} (fV) = f \frac{dV}{dt} + \frac{df}{dt} V$
3. If $V$ extends to a vector field $\tilde{V}$ on $M$ (or a nbhd of $\gamma$), then $\frac{d\tilde{V}}{dt} = \nabla_{\dot{\gamma}(t)} \tilde{V}$.

Proof: Write $\gamma(t) = (x^1(t), \ldots, x^n(t)) \Rightarrow \gamma'(t) = (x^1', \ldots, x^n') e_i$.

Uniqueness: for $\frac{d}{dt} (e_j) = \nabla_{x^i} e_j = \Gamma^k_{ij} x^k e_k$.

Then for general vector field along $\gamma$, $V = V^j(t) e_j$,

$$\frac{dV}{dt} = V^j(t) \frac{d}{dt} e_j + \frac{dV^j}{dt} e_j \quad (\text{1 and 2})$$

$$= (x^i') V^j(t) \Gamma^k_{ij} x^k e_k + \frac{dV^j}{dt} e_j \quad (\text{3})$$

Existence: define $\frac{dV}{dt}$ by (4); check (2) (5) obvious.

Note: on overlapping charts, values must agree by uniqueness. $\Box$. 

Parallel transport: \( \gamma(t) \) curve, \( V = \text{vector field along } \gamma \).

**Def:** \( V \) is parallel if \( \frac{\text{D}V}{\text{d}t} = 0 \).

**Prop:** \( \forall \gamma \in \mathcal{T}_{\mathcal{M}} \), \( \exists ! \) parallel vector field \( V(t) \) with \( V(0) = v_0 \), this is called "parallel transport" of \( v_0 \) along \( \gamma \).

PE: Sufficient to prove in a coord chart; then cover \( \gamma \) by overlapping charts.

We want to find

\[
V(t) = V^{i}(t) e_{j}
\]

satisfy:

\[
\left( \frac{dV^{k}}{dt} + \frac{dx^{i}}{dt} V^{j} P_{i}^{k} \right) e_{k} = 0
\]

\[
\implies \frac{dV^{k}}{dt} = -V^{i}(t) \frac{dx^{i}}{dt} P_{i}^{k}
\]

This is a system of 1st order differential equations in variables \( t \), hence given initial conditions. \( \Box \)
Important example: $M \subset \mathbb{R}^{n+2}$. Define affine connection $\nabla$ on $\mathbb{R}^{n+2}$ as before. This induces an affine connection $\nabla$ on $M$.

\[ X, Y \in \text{Vect}(M), \text{ expanded to } X, Y \in \text{Vect}(\mathbb{R}^{n+2}) \implies \nabla_X Y = (\overline{\nabla_X Y})^T : \text{orthogonal projection} \]

$T_x \mathbb{R}^{n+2} \to T_x M$.

Check: this is an affine connection.

Parallel transport: \[ \frac{d}{dt} = 0 \implies \text{if } V(t) \in \mathbb{R}^{n+2} \text{ then } (V'(t))^T = 0. \]

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**Levi-Civita Connection.**

$g = \text{Riemann metric on } M$

\[ \text{Def. A connection } \nabla \text{ on } M \text{ is compatible with } g \text{ if } \forall X, Y, Z \in \text{Vect}(M), \]

\[ X \ g(Y, Z) = g(D_X Y, Z) + g(Y, D_X Z). \]
Prop \( \gamma \) curve on \( M \), \( D \) compatible with \( g \:<,> \).

1. If \( V \), \( W \) are vector fields along \( \gamma \), then \( \frac{d}{dt} \langle V, W \rangle = \langle \frac{DV}{dt}, W \rangle + \langle V, \frac{DW}{dt} \rangle \).

2. If \( V \) = parallel vector field along \( \gamma \), then \( \| V(t) \| = \langle V(t), V(t) \rangle \) is constant.

\[ \begin{align*}
\text{PF} & \text{ 1. Extend } V/W \text{ to vector fields near } \gamma, \text{ and use } \frac{DV}{dt} = D_{\frac{d}{dt}} V \text{ etc.} \\
\text{2. Clear.} \end{align*} \]

Def: A Connection \( \nabla \) on \( M \) is torsion-free ("symmetric") if \( \nabla_x Y - \nabla_y X = [X, Y] \) \( \forall \ X, Y \in \text{Vect} M \).

Note: in coords \( (\Gamma^k_{ij} - \Gamma^k_{ji}) \partial_k = \nabla_i \partial_j - \nabla_j \partial_i = [\partial_i, \partial_j] = 0 \Rightarrow \Gamma^k_{ij} = \Gamma^k_{ji} \).

Thm M Riem. There exists a unique connection that is torsion-free and compatible with \( g \), " Levi-Civita connection ".

\[ \text{PF: Uniqueness:} \begin{align*}
\text{1. } & \langle X, [Y, Z] \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle \\
\text{2. } & \langle Z, [X, Y] \rangle = \langle \nabla_Z X, Y \rangle + \langle X, \nabla_Z Y \rangle \\
\text{3. } & \langle Z, [X, Y] \rangle = 2 \langle \nabla_X Y, Z \rangle + \langle [X, Z], Y \rangle + \langle X, [Y, Z] \rangle - \langle Z, [X, Y] \rangle \\
\langle X, Y, Z \rangle & \text{ is determined } \forall \ X, Y, Z \Rightarrow \nabla_X Y \text{ is determined.}
\end{align*} \]

Existence: HW. \( \square \)
In curved: choose \( X = \partial_x, \ Y = \partial_y, \ Z = \partial_z \).

\[
\langle \nabla X, Z \rangle = \nabla^m \langle \partial_m, \partial_z \rangle = \nabla^m g_{mn} \\
\Rightarrow \quad \nabla^m g_{mn} = -\frac{1}{2} (\partial g_{ij} + \partial g_{ij} - \partial g_{ij}) \\
\Rightarrow \quad \nabla^m g_{mn} = \frac{1}{2} \left( \partial g_{ij} - \partial g_{ij} \right) g^{jk} \\
\]

Ex. \( \mathbb{R}^n \), Euclidean metric. Then \( g_{ij} = \delta_{ij} \) so \( \nabla^m = 0 \) \( \forall i,j,k \).

Geodesics.

Def. \( (M, g) \) Riem. mf., \( \nabla \) Levi-Civita connection.

A curve \( \gamma \) on \( M \) is a geodesic if \( \frac{D}{dt}(\gamma'(t)) = 0 \)

ie. \( \gamma'(t) \) is a parallel vector field along \( \gamma \), "\( \gamma \) has zero acceleration".

Observation: \( \gamma \) = geodesic \( \rightarrow \)

\[
\frac{a}{dt} |\gamma'(t)|^2 = \frac{d}{dt} \langle \gamma'(t), \gamma'(t) \rangle = 2 \langle \frac{D}{dt} \gamma'(t), \gamma'(t) \rangle = 0 \\
\Rightarrow \quad |\gamma'(t)| \text{ is constant. (usually assume not constant mag. \( |\gamma'(t)| > 0 \)).}
\]

Ex. "Straight lines in \( \mathbb{R}^n \)

- for \( M = \mathbb{R}^{n+k} \), \( \gamma \) geodesic if tangential component of acceleration \( \frac{\gamma''(t)}{\langle \gamma'(t), \gamma'(t) \rangle} \) is 0.
In coordinates: \( \gamma(t) = (x^1(t), \ldots, x^n(t)) \), \( \gamma'(t) = \frac{dx^i}{dt} \Delta x. \)

Recall \( \frac{d}{dt} (x^i(t)) = (\frac{d}{dt} x^1, \ldots, \frac{d}{dt} x^n) \Delta x \)

\( \Rightarrow \frac{d}{dt} (\gamma'(t)) = (\frac{d^2}{dt^2} x^1 + \frac{dx^i}{dt} \frac{dx^j}{dt} \Delta x^1 \gamma_{ij} \Delta x) \Delta x \)

So a geodesic satisfies a second order system

\[ \frac{d^2}{dt^2} x^k + \frac{dx^i}{dt} \frac{dx^j}{dt} \Delta x^i \gamma_{ij} \Delta x^k = 0 \]

\( k = 1, \ldots, n. \)

**Note** 1. This is homogeneous: \( \gamma(t): (\mathbb{R}, 0) \to M \) is a geodesic then so is the reparametrization \( \tilde{\gamma}(t) = \gamma(t): (\mathbb{R}, 0) \to M. \)

Note \( \tilde{\gamma}'(0) = C \gamma'(0) \).

2. Can reformulate in terms of the tangent bundle.

\( U = \text{Coord chart on } M \), \( \Delta x^1, \ldots, \Delta x^n \)

\( \Rightarrow U \times \mathbb{R}^n = \text{Coord chart on } TM \), \( \Delta x^1, \ldots, \Delta x^n, \gamma^1, \ldots, \gamma^n \).

\( \gamma = \text{Curve in } M \Rightarrow \tilde{\gamma} = \text{Curve in } TM \) given by

\( \tilde{\gamma}(t) = (\gamma(t), \gamma'(t)). \)

Then if \( \tilde{\gamma} \) is geodesic, \( \gamma(t) = (\Delta x^1(t), \ldots, \Delta x^n(t), \gamma^1(t), \ldots, \gamma^n(t)) \) is given by:

\[
\begin{align*}
\gamma^1(t) &= \frac{d}{dt} \Delta x^1(t) \\
\vdots \\
\gamma^n(t) &= \frac{d}{dt} \Delta x^n(t) \\
\frac{d}{dt} \gamma^1(t) &= -\gamma^j \gamma^i \gamma_{ij} \\
&\vdots \\
\frac{d}{dt} \gamma^n(t) &= -\gamma^j \gamma^n \gamma_{ij}.
\end{align*}
\]

**Def.** The geodesic vector field on \( TM \) is the vector field given in coordinates by

\[
(y^1, \ldots, y^n, -y^j \gamma^i \gamma_{ij}, \ldots, -y^j \gamma^n \gamma_{ij}).
\]

(from the following discussion: coord independent).
The geodesic vector field is constructed so that $\tilde{\gamma}$ is a flow of $\xi$. This implies short-time existence of geodesics:

For $(x,v) \in TM$, let $\tilde{\gamma}(t)$ be the time $t$ flow of the geodesic vector field $\xi|_M$.

Then $\gamma(t) = \pi \circ \tilde{\gamma}(t)$ is a geodesic in $M$ with $\gamma(0) = x$, $\gamma'(0) = v$.

And conversely, so $\exists$! geodesic $\gamma : (-\varepsilon, \varepsilon) \to M$ with

$\gamma(0) = x$, $\gamma'(0) = v$, small $\varepsilon$.

Prop. Let $M$, $\xi$, $p$, and smooth map

$\gamma : (-\varepsilon, \varepsilon) \times \Omega \to M$

where $\Omega = \{ (x,v) | x \in U, \|v\| < \varepsilon \}$

$s.t. t \mapsto \gamma(t,x,v)$ is the unique geodesic $\gamma$

with $\gamma(0) = x$, $\gamma'(0) = v$.

\[\text{Short-time existence}:\]

Can find $\tilde{\sigma}$, $h$ so $\tilde{\sigma} \circ h \circ \xi = \xi|_M$, the time $t$

flow of the geodesic v.f. starting at $(x,v)$ is defined for $|t| < \delta$: write this flow as

$\tilde{\sigma}(t,x,v)$.

Might as well assume $\tilde{\sigma}$ is of the form

$\tilde{\sigma} = \{ (x,v) | x \in U, \|v\| < \varepsilon \}$ for some, some $\varepsilon > 0$.

Write $\gamma = \pi \circ \tilde{\sigma}$. Then:

$t \mapsto \gamma(t,x,v)$ is a geodesic for $|t| < \delta$, $\|v\| < \varepsilon_0$

$\gamma(\varepsilon, x,v)$ (reparametrization - homogeneity)

$t \mapsto \gamma(t,x,v)$ is a geodesic for $\|v\| < \frac{\varepsilon_0}{2}$.

Choose $c = \frac{\varepsilon_0}{2}$.

$t \mapsto \gamma(t,x,v)$ is a geodesic for $|t| < 2$, $\|v\| < \frac{\varepsilon_0}{2}$. \( \square \)
Def \( \Omega \) as above: \( \Omega = \{ \xi(x, v) \mid x \in U, |v| < \epsilon \} \)

The exponential map \( \exp: \Omega \to M \) is defined by
\[
\exp(x, v) = \gamma(1, x, v) = \gamma(t, x, \frac{v}{t}). \quad x \in U, |v| < \epsilon.
\]

For fixed \( x \), get \( \exp_x: B_\epsilon(0) \to M \), \( \exp(0) = x \).

**Proof.** Fix \( x \in M \). \( \exists \epsilon > 0 \) s.t. \( \exp_x: B_\epsilon(0) \to M \)

is a diffeomorphism onto an open subset of \( M \).

\[
\frac{d}{dt} \bigg|_{t=0} \exp_x(tv) = \frac{d}{dt} \bigg|_{t=0} \gamma(t, x, tv) = \frac{d}{dt} \bigg|_{t=0} \gamma(t, x, v) = v
\]

So \( d(\exp_x)(0) = \text{isomorphism} \). Now use Inverse Function Thm. \( \Box \)

**Ex.** \((S^n, \text{round metric})\).

Geodesics are (arcs of) great circles, parametrized w/ constant speed.

\[
\exp: \mathbb{R}^n \to S^n. \quad \exp_x(v) :
\]

\[
\text{Speed} = |v| \quad \text{arc length} = tv
\]
Ex. \( SO(n) \) with bi-invariant metric. By HJM, geodesics are 1-parameter subgroups. If \( M = \exp(M) \), define
\[
\exp(M) = I + M + \frac{1}{2!} M^2 + \frac{1}{3!} M^3 + \ldots \in SO(n).
\]
Then \( \exp(tM) \) is a 1-parameter subgroup = geodesic, and
\[
\frac{d}{dt} \bigg|_{t=0} \exp(tM) = M.
\]
So
\[
\exp(tM) = \exp(tM).
\]
\( e \in SO(n), t \in \mathbb{R} \)

(\( \in \) particular, continuous)

If \( \gamma : [a,b] \to M \) is a piecewise smooth path in \( M \), define
length
\[
\ell(\gamma) = \int_a^b \sqrt{g^i_j(\dot{\gamma}(t), \dot{\gamma}(t))} \, dt.
\]

Claim: if the endpoints of \( \gamma \) are sufficiently close, and \( \gamma \) is a geodesic, then a piecewise smooth \( \tilde{\gamma} \) with same endpoints,
\[
\ell(\tilde{\gamma}) \geq \ell(\gamma).
\]

Note: not true in general:
\[ \text{Reviewed} \quad \text{Geodesic} = \gamma(t) : D, t' = 0. \]
\[ \exp : T_pM \rightarrow M \]
\[ v \rightarrow \gamma(1, p, v) \quad \text{when} \quad \gamma(t, (p, v)) = \text{geodesic with} \]
\[ \gamma(0) = p, \quad \gamma'(0) = v. \]

Note if we fix \( v \), then \( \exp(tv) = \gamma(1, p, tv) = \gamma(t, p, v) \)
so \( \exp(tv) \) is the geodesic with initial conditions \( \gamma(0) = p, \gamma'(0) = v. \)

**Normal Neighborhoods**

Eventual goal: Geodesics are length minimizers.

**Def** \( V = \text{hd of } p \in M \) is a normal neighborhood of \( p \) if \( \exists \)
\[ U = \text{hd of } 0 \in T_pM \quad \text{such that} \quad \exp : U \rightarrow V \text{ is a diffeomorphism.} \]

**Prop** \( p \in M, \exists \text{ hd } W \text{ of } p \text{ and } \epsilon > 0 \text{ such that} \)
1. \( \forall x \in W, \quad \exp_x : B_\epsilon(O_0) \rightarrow M \) is a diffeo onto its image and \( \exp_x(B_\epsilon(O_0)) \subset W \) (so \( W \) is normal nd of \( \epsilon \) each \( x \in W \))
2. \( \forall x, y \in W, \exists! \text{ } v \in T_xM \text{ with } |v| < \epsilon \text{ st. } y = \exp_x v \quad \text{i.e. } \exists! \text{ geodesic of length } \epsilon \text{ joining } x \text{ and } y. \)
W is called a totally normal neighborhood of p.

\[ \Omega = \{(x,u) | x \in U, \|v\| < \varepsilon \} \subset TM \text{ as define} \]
\[(u = e_u \circ f(x)). \]

\[ \text{exp}: \Omega \to M \text{ write } F: \Omega \to M \times M \]
\[(x,u) \mapsto (x, \text{exp}_x(u)). \]

Then \[ dF(p,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \] so \( F \) is a local diffeo near \((p,0)\).
Thus \( \exists U' \subset U \) of \( p \), \( \varepsilon' < \varepsilon \), st. \( F \) is a diffeo on \( U' = \{(x,u) | x \in U', \|v\| < \varepsilon' \} \).

There is a \( \varepsilon \)-net \( W \) of \( p \) with \( \overline{W} \subset F(U') \).
If \( x \in W \) then \( F(x) \in F(U) = (x) \times U' \) \( \Rightarrow \) \( \text{exp}_x(B_{\varepsilon'}(0)) \supset W \).
This proves \#1.

\#2 follows directly from \#1. \( \square \)

**Notation**: a normal ball with center \( p \) and radius \( r \) is \( \text{exp}_p(B_{r}(0)) \) whose closure lies in a normal neighborhood \( V \) of \( p \):
\( \overline{B}_r(0) \subset U \) where \( \text{exp}_p: U \to V \) is a diffeo.

Then \( \text{exp}_p(S_r(0)) \) is called the normal sphere with center \( p \) and radius \( r \).
Now: Suppose $B_{r_0}(p)$ is a normal ball. For $v \in T_p \mathcal{M}$ with $|v| = 1$, let $\gamma_v : [0, r_0) \to \mathcal{M}$ be the geodesic $\gamma_v(r) = \exp_p(vr)$. (“radial geodesic”)

Gauss Lemma $\gamma_v$ is normal + $\Sigma_r(p) \cap r \in (0, r_0)$.

Proof: Let $\frac{\partial}{\partial r}$ be the radial vector field on $B_r(0) - \{0\} \subset T_p \mathcal{M}$.

Define $\tilde{Z} := (\exp_p)_* \frac{\partial}{\partial r}$. Then $\tilde{Z}_{\exp_p(v)} = \gamma_v'(r)$. (note $\|\tilde{Z}\| = 1$)

Want: any tangent vector to $\Sigma_r(p)$ at $\gamma_v(r)$ is $\perp$ to $\tilde{Z}_{\exp_p(v)}$.

Let $X$ be any vector field defined on $\Sigma_(0)$; extend this vector field:

$X$ on $B_r(0) - \{0\}$ by $X_{\exp_p(v)} = X_v$, $v \in \Sigma_r(0)$

$\tilde{X}$ on $B_r(0)$ by $\tilde{X}_{\exp_p(v)} = r \tilde{X}_v$.

It suffices to show that $Y := (\exp_p)_* X$ and $\tilde{Y}$ are orthogonal along $\gamma_v$. 
Fix $v$, and write $f(t) = \langle Y_{tv}(t), z_{tv}(t) \rangle$. Want $f(t) = 0$.

\[
\frac{df}{dt} = 2\langle Y, z \rangle = \langle \nabla_z Y, z \rangle + \langle Y, \nabla_z z \rangle
\]

\[
= \langle \nabla_z z, z \rangle + \langle z, \nabla_z Y \rangle
\]

\[
= 0 \text{ since } \langle \nabla_z z, z \rangle = \langle z, z \rangle = 0
\]

So $\frac{df}{dt} = 0$ since $Y$ is a geodesic.

\[
[2, Y] = [\exp_p \frac{2}{t} \cdot (\exp_p \frac{t}{2} \cdot X) \cdot X] = \exp_p \frac{2}{t} \cdot X
\]

\[
= \frac{1}{t} \exp_p \frac{1}{t} \cdot X
\]

\[
= \frac{1}{t} Y
\]

\[
\Rightarrow \frac{df}{dt} = \frac{1}{t} f(t)
\]

\[
f(t) = ct \text{ for some constant } c.
\]

\[
c = \frac{1}{t} \langle Y, z \rangle_{tv(t)} = \langle \exp_p \frac{1}{t} \cdot Y, \exp_p \frac{1}{t} \cdot \frac{2}{t} \cdot X \rangle_{tv(t)}
\]

Now along the arc $[tv, t]$, we can extend $X$ and $\frac{2}{t} \cdot X$ to $0$, and the metric varies continuously. So

\[
c = \lim_{t \to 0} \langle \exp_p \frac{2}{t} \cdot X, \exp_p \frac{2}{t} \cdot \frac{2}{t} \cdot X \rangle_{tv(t)} = \langle \exp_p \frac{2}{t} \cdot X, \exp_p \frac{2}{t} \cdot \frac{2}{t} \cdot X \rangle_0
\]

\[
= \langle X, \frac{2}{t} \cdot X \rangle_0
\]

\[
= 0
\]

Since (as we saw last time) $(\exp_p) \ast (0): TM \to TM$ is the identity. So

\[
f(t) = 0. \quad \square
\]
Prop. \( p \in M \), \( B = B_r(p) \) normal ball centered at \( p \).

\( \gamma \in B \), \( \gamma : [0,1] \to B \) is the geodesic with \( \gamma(0) = p, \gamma'(0) = \eta \).

\( \gamma(t) = \exp_p(t \exp_p(\eta)) \).

If \( \tilde{\gamma} : [0,1] \to M \) is piecewise smooth with \( \tilde{\gamma}(0) = p, \tilde{\gamma}(1) = \eta \),
then \( \ell(\tilde{\gamma}) \geq \ell(\gamma) \), with equality if and only if \( \tilde{\gamma} \) is reparametrization of \( \gamma \).

PF. First suppose \( \tilde{\gamma} \) lies in \( B \), and define \( \tilde{\gamma} : [0,1] \to B_r(0) \) by \( \exp_0 \circ \tilde{\gamma} = \gamma \). Write \( \tilde{\gamma}(t) = v(t) \nu(t) \), \( |v(t)| = 1 \).

Then

\( \tilde{\gamma}'(t) = (v'(t) \nu(t) + v(t) \tilde{\nu}(t)) \).

\( \Rightarrow \tilde{\gamma}'(t) = v'(t) \nu(t) + (Tangents to sphere) \)
and \( \tilde{\nu} \perp \text{normal sphere} \) by Gauss Lemma.

\( \Rightarrow |\tilde{\gamma}'(t)| \geq |v'(t)| \geq |v'(t)| \).

So

\[ \ell(\tilde{\gamma}) = \int_0^1 |\tilde{\gamma}'(t)| dt \geq \int_0^1 |v'(t)| dt \geq \int_0^1 |v'(t)| dt = \ell(\gamma) \]
with equality if and only if no normal component and \( v' \geq 0 \) \( \Rightarrow \tilde{\gamma} = \gamma \).
Now: if \( \tilde{\gamma} \) doesn't lie in \( B \), let \( t_0 \) be first time \( \gamma(t) \notin B \). Then
\[
\ell(\tilde{\gamma}) = \int_{t_0}^t |\gamma'(u)| \, du \geq \int_{t_0}^{t_0^+} |\gamma'(u)| \, du \geq r_0 > \ell(\gamma). \quad \Box
\]

Converse? Are length minimizing geodesics?

**Prop.** \( \gamma: [a, b] \to M \) piecewise smooth, constant speed. If \( \ell(\tilde{\gamma}) \leq \ell(\gamma) \) for any \( \tilde{\gamma} \) with same ends, then \( \gamma \) is a geodesic.

If \( t \in [a, b] \), \( W \) = totally normal neighborhood of \( \gamma(t) \); so \( \gamma \) maps
\( [a, b] \) to \( W \) for some \( a < t < b \). Then \( \gamma(a) \to \gamma(b) \) is a curve in a normal ball. If \( \tilde{\gamma} = \) geodesic joining \( \gamma(a) \) to \( \gamma(b) \)
\[
\ell(\tilde{\gamma}) \leq \ell(\gamma) \Rightarrow \ell(\tilde{\gamma}) = \ell(\gamma) \Rightarrow \tilde{\gamma} = \gamma \] up to reparametrization \Rightarrow \gamma \mid [a, b] \) is a geodesic.
This is true for all \( t \). \( \Box \)

---

**Geodesic convexity** see do Carmo, ch.3 sec.4

Recall: any \( p \) has a totally normal neighborhood \( W \ni p \) and \( \epsilon > 0 \) such that any \( x, y \in W \) can be connected by a geodesic of length \( < \epsilon \); but this could go outside \( W \).

**Def.** A subset \( S \subset M \) is geodesically convex if \( \forall x, y \in S \), \( \exists \! ! \) length minimizing geodesic \( \gamma \) between \( p \) and \( q \) such that the interior of \( \gamma \subset S \).
Proof: \( p \in M \). \( \exists r \) so \( \text{any normal ball } B_r(p) \) with \( p \in \mathbb{R} \) is geodesically convex.

Idea: Lemma. For any suff. small \( \varepsilon \), any geodesic tangent to \( S_r(p) \) lies outside \( B_\varepsilon(p) \).

Then: Connect \( x,y \) by geodesic. If this stays outside \( B_\varepsilon(p) \), then it's tangent to \( B_r(p) \) for some \( \varepsilon \) but lies inside.

Geodesics and Topology

Def: \( M \) Riem. The distance between two points \( p,q \in M \) is

\[
d(p,q) = \inf \{ l(\gamma) \text{ over all piecewise smooth } \gamma \text{ from } p \text{ to } q \}.
\]

(Note: if \( \gamma \) achieves \( \inf \), then \( \gamma \) is a geodesic).

Prop: \((M,d)\) is a metric space, and the topology on \( M \) agrees with the metric topology.

If recall if \( \gamma \in B_r(p) \) then \( d(p,\gamma) = d(\gamma) < r \) where \( \gamma = \text{geodesic from } p \text{ to } \gamma \).

Then the normal ball \( B_r(p) \) is \( \{ \gamma \mid d(p,\gamma) < r \} = \text{metric ball. \( \square \)} \)

When is there always a minimal geodesic between two points? \( \square \)

not here \( \rightarrow \)
Def M is geodesically complete if all geodesics can be extended to have domain \( \mathbb{R} \) i.e., \( \forall p \in M \), exp\(_p\) is defined on all of \( T_pM \).

Then \( M \) connected, \( p \in M \). If exp\(_p\) is defined on all of \( T_pM \), the \( \forall q \in M \) \( \exists \) geodesic \( \gamma \) joining \( p \) to \( q \) with \( d(\gamma) = d(p,q) \).

**Lemma** \( p, q \in M \). For suit small \( s \), \( \exists p_0 \in Ss(p) \) with 
\[
d(p, p_0) + d(p_0, q) = d(p, q).
\]

**PF** Choose \( s \) such that \( B_s(p) \) is closed ball, and choose \( p_0 \in Ss(p) \) minimizing \( d(p_0, q) \) (\( \exists \) since \( d(\cdot, q) : Ss(p) \to \mathbb{R} \) is continuous).

If \( \gamma \) joins \( p \) to \( q \) then if \( p' \in \gamma \cap Ss(p) \), \( b \cdot s \leq \delta \)
\[
d(\gamma) = d(\gamma, p') + d(p', q) \geq d(\gamma, p_0) + d(p_0, q) 
\]
So \( d(p, q) = d(p, p_0) + d(p_0, q) \geq d(p, q) \). \( \square \)

**PF of Thm** Suppose \( d(p, q) = r \). Choose \( s, p_0 \) as in lemma and write \( p_0 = \text{exp}_p(s\nu), \ (\nu) = 1. \)

Let \( \gamma \) be the geodesic \( \gamma(t) = \text{exp}_p(s\nu) \). Claim: \( \gamma(r) = q \).

Define \( I = \{ t \in [0, r], d(\gamma(t), q) = r - t \} \). Note \( 0 \in I \):
\[
d(\gamma(0), q) = d(p_0, q) = d(p, q) - s = r - s.
\]
Also \( I \) is closed\( \Rightarrow \) \( \exists T = \max I \) s.t. \( T \in I \), \( s \leq T \leq r \).
If $T = r$ then $d(x(t), q) = 0 \Rightarrow$ done.
If $T < r$ then apply lemma to $x(t), q$.
\[ \exists \delta', \ p_i \in S_{\delta'}(x(t)) \text{ with} \]
\[ d(x(t), p_i) = \delta', \ d(p_i, q) = r - T - \delta' \]
\[ \Rightarrow \]
\[ T = d(p_i, q) \leq d(p_i, p) + d(p, q) \]
\[ \Rightarrow d(p, q) \geq r - (r - T - \delta') = T + \delta'. \]
Now the path $\gamma$ is a geodesic has length $T + \delta'$ \Rightarrow it's a geodesic \Rightarrow $T$ isn't maximal. \qed

Thus (Hopf-Rinow) $(M, g)$ connected liem. TFAE:
1. $(M, g)$ is geodesically complete, i.e. $\forall p \in M$, exp is defined on all of $B_r(p)$
2. For some $p$, exp is defined on all of $B_r(p)$
3. All closed bounded (w.r.t. $d$) subsets of $M$ are compact
4. $(M, d)$ is complete as a metric space.

Moreover (by previous item), any of these imply that $\forall p, q \in M$, $\exists$ geodesic $\gamma$ between $p$ and $q$ s.t. $d(\gamma) = d(p, q)$.

$\text{Proof}$: 1 $\Rightarrow$ 2 obvious.

2 $\Rightarrow$ 3: $K \subset M$ closed, bounded. Then $\exists R$ with $K \subset B^d_R(p) =$ ball centered at $p$ with radius $R$ in $d$ metric.

$\Rightarrow K \subset \exp_r(B^d_R(p)) \forall q \in K, d(p, q) < R$, so by previous item $\exists$ geodesic of length $< R$ between $p$ and $q$.

$\Rightarrow K \subset \exp_r(B^d_R(p)) = \text{compact since it's the continuous image of compact}$

$\Rightarrow K$ is compact since closed.

3 $\Rightarrow$ 4: $\{p_i\}$ Cauchy $\Rightarrow$ bounded $\Rightarrow$ sits in some compact $B^d_R(p)$

$\Rightarrow$ has a convergent subsequence $\Rightarrow$ converges.
\[ \text{Prop 4.} \quad \text{Let } \gamma \text{ be a geodesic, assume speed } 1. \text{ Say its maximal domain is } T \subset \mathbb{R}. \text{ Local existence } \Rightarrow T \text{ is open.} \]

Claim: \( I \) is closed. Let \( \{ t_n \} \subset I \) satisfy \( t_m \leq t_n \). \( \Rightarrow \) \( \text{length of } \gamma \text{ between } t_m \text{ and } t_n = |t_n - t_m| \).

\[ \Rightarrow \left\{ \gamma(t_n) \right\} \text{ is Cauchy } \Rightarrow \text{ has a limit point } p \in M. \]

Let \( W \ni p \) be a totally normal nbhd, and \( \varepsilon > 0, W \subset \text{Ball}(x) \cup \varepsilon \cdot W. \)

Any geodesic of speed 1 starting at any \( p \in W \) is defined at least \( m \in (\varepsilon, \varepsilon). \) Choose \( n \) \text{ st. } \gamma(t_n) \in W \text{ and } |t - t_n| < \varepsilon. \)

Then \( \gamma \) is defined at least on \( (t_n - \varepsilon, t_n + \varepsilon) \) and thus at \( t. \) \( \Box \)

Cor. The following are geodesically complete:

- any compact \( M \)
- any closed submanifold of a geodesically complete \( M \) (e.g. Euclidean \( \mathbb{R}^n \)).

---

**Curvature**

Gauss: defined "Gaussian Curvature" for surfaces.

Then can extend to 2D slices of a mfd: "Sectional Curvature."

Modern formulation: Curvature tensor, measures deviation from being flat (isometric to Euclidean space).
Def. The curvature tensor of \((M, g)\) is the \((1, 1)\) tensor
\[ \nabla : \text{Vec}(M) \to \text{Vec}(M) \]
defined by
\[ \text{R}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z \]
where \(\nabla\) = Levi-Civita connection.

\[\Box\text{Caution:}\] many people use the opposite sign convention.

Note on name: fixing \(X, Y\), the map \(Z \mapsto \text{R}(X, Y)Z\) is a \((1, 1)\) tensor:
think of \(\text{R}(X, Y)\) as being an endomorphism of \(T_M\), \(\text{R}(X, Y) : \text{Vec}(M) \to \text{Vec}(M)\).

Check tensor: \(\text{R}(fX, fY)Z = \nabla_Y \nabla_X (fZ) - \nabla_X \nabla_Y (fZ) + \nabla_{[X, Y]} (fZ) \)
\[ = f \text{R}(X, Y)Z + f \nabla_Y f \nabla_X Z - f \nabla_X f \nabla_Y Z \]
\[ + \nabla_Y (\nabla_X f Z) - \nabla_X (\nabla_Y f Z) + \nabla_{[X, Y]} f Z \]
\[ = f \text{R}(X, Y)Z. \]

Prop. (First Bianchi identity) \(\text{R}(X, Y)Z + \text{R}(Y, Z)X + \text{R}(Z, X)Y = 0\).

\(\Box\) LHS = \(\nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z\)
\[ + \nabla_Y \nabla_X f - \nabla_X \nabla_Y f + \nabla_{[X, Y]} f \]
\[ + \nabla_X \nabla_Y f - \nabla_Y \nabla_X f + \nabla_{[Y, X]} f \]
\[ = \[(X, Y, Z)\] + cyclic permutations = 0. \]
We can turn the $(1,3)$ tensor $R(x, y) w$ into a $(0,4)$ tensor by using the metric:

\[ R(x, y, z, w) := \langle R(x, y) z, w \rangle. \]

**Prop** 1. $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$

2. $R(x, y, z, w) = -R(y, x, z, w)$

3. $R(x, y, z, w) = -R(x, y, w, z)$

4. $R(x, y, z, w) = R(z, w, x, y)$

**Ref.**

1. Bianchi; 2. Obreux

3. Equivalently: $R(x, y, z, w) = 0$.\[
R(x, y, z, w) = \langle \partial_y \partial_x z, w \rangle - \langle \partial_y \partial_y z, w \rangle + \langle \partial_x \partial_x z, w \rangle - \langle \partial_x \partial_y z, w \rangle = \frac{1}{2} x y (x, z) - \frac{1}{2} x y (x, z) + \frac{1}{2} (x, y) (z, z) = 0.
\]

4. $R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0$

\[R(y, z, w, x) + R(z, w, y, x) + R(w, y, z, x) = 0\]

\[R(z, w, x, y) + R(w, z, y, x) + R(x, z, y, w) = 0\]

\[R(w, y, z, x) + R(x, y, w, z) + R(y, w, x, z) = 0\]

\[2 R(z, x, y, w) + 2 R(w, y, z, x) = 0\]

\[\Rightarrow R(z, x, y, w) = -R(w, y, z, x) = R(y, w, z, x). \quad \Box \]

**In coordinates:** Write $R_{ijk\ell} = \langle R(x, y) z, w \rangle$, $R_{ij} = R_{ijk\ell} e^k e^\ell$.\[
R_{ijk\ell} = R_{ijk\ell} g_{m\ell} \quad \Rightarrow R_{ijk\ell} g_{m\ell} = R_{ijk\ell} g_{m\ell}
\]

1. $R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0$

2. $R_{ijk\ell} = -R_{ik\ell j} = -R_{i\ell jk} = R_{i\ell kj}$. \[\]
Formula for $R_{ijk}^l$:

$$R_{(x_i, x_j)k} = \nabla_j \nabla_i x_k - \nabla_i \nabla_j x_k$$

$$= \nabla_j (\Gamma^l_{ik} x_k) - \nabla_i (\Gamma^l_{jk} x_k)$$

$$= \Gamma^l_{jk} \Gamma^m_{ik} x_k + \delta_j \Gamma^m_{ik} x_k - \delta_i \Gamma^m_{jk} x_k$$

$$\Rightarrow R_{ijk}^l = \Gamma^l_{ik} \Gamma^m_{jk} - \Gamma^l_{jk} \Gamma^m_{ik} + \delta_j \Gamma^m_{ik} - \delta_i \Gamma^m_{jk}$$

For $\mathbb{R}^n$, Euclidean metric: $R_{ijk}^l = 0$ for $i, j, k, l$.

Sectional curvature

Idea: if $n=2$ then curvature tensor is determined by one number, $R_{1212}$. In general, slice by a plane to get a 2-d map.

**Def:** $\sigma \subset T_p M$, 2-dimensional subspace. Then the sectional curvature $K(\sigma)$ at $p$ is, for any basis $\{X, Y\}$ of $\sigma$,

$$K(\sigma) = \frac{R(X, Y, X, Y)}{|X||Y|^2}$$

where $|X||Y|^2 = \det \begin{bmatrix} g(X, X) & g(X, Y) \\ g(Y, X) & g(Y, Y) \end{bmatrix}$.

Check: if we replace $X, Y$ by another basis, this doesn't change.

Just need to check 3 elementary changes of $\sigma$:

1. $(X, Y) \rightarrow (Y, X)$: $R(X, Y, X, Y) = R(Y, X, Y, X)$ and $|X||Y|^2 = |X||Y|^2$.
2. $(X, Y) \rightarrow (P, Y)$: $R(X, Y, Y, Y) = R(P, Y, Y, Y)$ and $(X, P)^2 = X^2$.
3. $(X, Y) \rightarrow (X, Y)$: $|X+Y||Y|^2 = |X||Y|^2$ and $|X+Y||Y|^2 = |X||Y|^2$.

Prop. The sectional curvature at a point determines the curvature tensor.

\[ \frac{\partial^2}{\partial x \partial y} \bigg|_{x=y=0} \left[ \begin{array}{c} \text{determined by } \xi \text{ (plane through } x+a \xi y+b \xi, x+y \xi, x+y \xi, x+y \xi) \\ \text{determined by } \xi \text{ (plane through } x+a \xi y+b \xi, x+y \xi, x+y \xi, x+y \xi) \\ \end{array} \right] \\
= \frac{\partial^2}{\partial x \partial y} \left( R(z,w,x,y) + R(z,w,x,y) + R(x,w,z,y) + R(x,w,z,y) \\
- R(w,z,x,y) - R(w,z,x,y) - R(x,z,w,y) - R(x,z,w,y) \\
+ R(x,z,w,y) + R(x,z,w,y) - R(x,w,y,z) - R(w,y,z,x) \right) \\
= 4 R(x,y,z,w) \quad \text{def (Bianchi)} \\
= 6 R(x,y,z,w). \quad \square \\

Def. \( (M,g) \) has constant sectional curvature if \\
\[ k(\sigma) = \text{Constant } K_0 \quad \forall \sigma \in M \text{ and } \forall \sigma = 2\text{-plane at } p. \]

Ex. \( (\mathbb{R}^n, \text{flat}) \) \quad \( (\mathbb{R}^n/\Gamma, \text{flat}) \)
\( (S^n, \text{round}) \)
\( (H^n, \text{hyperbolic}) \)

Prop. Constant sectional curvature \( k_0 \)

\[ R(x,y,z,w) = K_0 \left( \langle x,z \rangle \langle y,w \rangle - \langle x,w \rangle \langle y,z \rangle \right). \]

PF. Call \( R(z,w,x,y) \). Easy to check, this satisfies all some properties
\( R. \) Also "sectional curvature" \( k'(x,y) = \frac{R'(x,y,x,y)}{\langle x,y \rangle} = \frac{K_0 (\langle x,y \rangle^2 - \langle x,z \rangle \langle y,w \rangle - \langle x,w \rangle \langle y,z \rangle)}{\langle x,y \rangle^3} = K_0. \)

So since sectional curvature determines Rieman curv., \( R = k' \). \quad \square
Two more curvatures

Several approaches.

First, in terms of an orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_PM$:

Def $X, Y \in T_PM$. The

\[
\text{Ricci tensor } \quad \text{Ric}_p(X, Y) = \frac{1}{n-1} \sum_{i=1}^{n} R(X, e_i, Y, e_i) \quad (\text{0,2) tensor})
\]

\[
\text{Ricci curvature } \quad \text{Ric}_p(X) = \text{Ric}_p(X, X) \quad \text{quadratic form}
\]

[Note: $\text{Ric}_p(X)$ is a symmetric bilinear form so it's determined by $\text{Ric}_p(X)$]

\[
\text{Scalar curvature } \quad S(p) = \frac{1}{n} \sum_{i=1}^{n} \text{Ric}_p(e_i) = \frac{1}{n(n-1)} \sum_{i,j} R(e_i, e_j, e_i, e_j).
\]

These are independent of the choice of $e_1, \ldots, e_n$.

Define $\varphi_{X, Y}: T_PM \to T_PM$ by $\varphi_{X, Y}(Z) = R(X, Z, Y)$. Then

\[
\text{Ric}_p(X, Y) = \frac{1}{n-1} \text{Tr} \varphi_{X, Y} \quad \text{indep. of basis}.
\]

Now $\text{Ric}_p(X, Y)$ is a bilinear form so $\exists T: T_PM \to T_PM$ linear s.t.

\[
\text{Ric}_p(X, Y) = \langle T(X), Y \rangle.
\]

Then

\[
S(p) = \frac{1}{n} \text{Tr} T : \text{wth } \text{Ric}_p(e_i, e_i) = \langle T(e_i), e_i \rangle = (i, i) \text{ entry of } T.
\]

Local coordinates: note $\{e_i\}$ isn't usually orthonormal.

$R_{ij} = \text{Ric}_p(e_i, e_j)$.

The map $\varphi_{e_i, e_j}: \mathbb{R} \to R(e_i, e_j, e_j)$ sends $\lambda \to R_{ij}\lambda$.

so the $(k, l)$ entry of $\varphi_{e_i, e_j}$ is $R_{ij}^k$

\[
\Rightarrow R_{ij} = \frac{1}{n-1} \text{Tr} \varphi_{e_i, e_j} = \frac{1}{n-1} R_{ikl} g^{ij} = R_{ij}^k
\]
\[ R_{ij} = \langle T(x_i, x_j), x_j \rangle = \langle T^k_i x_k, x_j \rangle = T^k_i g_{kj} \Rightarrow T^k_i = R_{ij} g^{jk} \]

\[ S = \frac{1}{n} \text{Tr} T = \frac{1}{n} T^i_i = \frac{1}{n} R_{ij} g^{ij} = \frac{1}{n(n-1)} \text{Ric}(x_i, x_j) g^{ij} = S \]

**What are these curvatures?**

Why?: view the curvature tensor algebraically.

Let \( V = T_p M \). Recall a \( 0,2 \) tensor is in \( V^* \otimes V^* \).

Antisymmetric is in \( \Lambda^2 V^* \). The curvature tensor \( R \) is in:

\[ V^* \otimes V^* \otimes V^* \otimes V^* \rightarrow (L^2 V^*) \otimes (L^2 V^*) \rightarrow \text{Sym}^2 L^2 V^* \]

Define the Bianchi map \( b : \text{Sym}^2 L^2 V^* \rightarrow \text{Sym}^2 L^2 V^* \)

\[ b(T)(x_i \otimes w) = T(x_i \otimes w) + T(x_j \otimes w) + T(x_x \otimes w) \]

Then \( R \in \text{ker } b = C(V) \subset \text{Sym}^2 L^2 V^* \)

Note \( O(n) \) acts on \( V, V^*, \text{Sym}^2 L^2 V^* \). \( C(V) \).

**Fact:** we can decompose \( C(V) \) as an \( O(n) \)-module:

\[ C(V) = \mathbb{R} \oplus \text{Sym}^2 V^* \oplus W(V) \]

- Traceless symmetric forms
- All symmetric forms in \( \text{Sym}^2 V^* \)

When the map \( C(V) \rightarrow \text{Sym}^2 V^* \) is \( (x, y) \rightarrow T(x_i, y_j) \)

and the map \( \text{Sym}^2 V^* \rightarrow \mathbb{R} \) is \( T \rightarrow \text{tr } T \).

So \( R \rightarrow \text{Ric}(x, y) \rightarrow S(0) \in \mathbb{R} \)
What about $\mathbb{W}(X)$? The component of $R$ in $\mathbb{W}(X)$ is the \textit{Weyl tensor} of $R$. Facts:

- if $n \geq 5$, $\mathbb{W}(X)$ is irreducible
- if $n \leq 3$, $\mathbb{W}(X) = 0$. ($R$ is determined by $\text{Ric}$!)

\textbf{Review:}

- (0.3) Curvature tensor
  \[ R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z \]
- (0.4) Curvature tensor
  \[ R(X,Y)Z = \langle R(X,Y)Z, W \rangle \]

\textbf{Sectional curvature:}

\[ \sigma \in T_\Sigma M \text{ 2-plane, } e_1, e_2 \text{ on } \Sigma \text{ for } \sigma \]

\[ \Rightarrow \quad K(\sigma) = R(e_1, e_2, e_1, e_2) \]

\textbf{Ricci tensor:}

\[ \text{Ric}_p(X,Y) = \frac{1}{n} \sum_{i=1}^{n} R(X, e_i, Y, e_i) \quad (e_1, ..., e_n) \text{ orthonormal basis of } \Sigma, \]

\[ \text{Ric}_p(X) = \text{ric}_p(X, X) \]

\[ \text{Scalar curvature:} \quad S(p) = \frac{1}{n} \sum_{i=1}^{n} \text{ric}_p(e_i). \]

If $X$ is a unit vector in $T_p M$, then

\[ \text{Ric}_p(X) \text{ is average sectional curvature of planes through } X. \]

Complete $X$ to an ONB $X, e_2, ..., e_n$, and let $\sigma_i = \langle X, e_i \rangle$.

\[ K(\sigma_i) = R(X, e_i, X, e_i), \]

\[ \text{Ric}_p(X) = \frac{1}{n-1} \sum_{i=2}^{n} K(\sigma_i) \quad \text{(note } R(X, X, X) = 0). \]

Nice computation: parametrize plane through $X$ by

$\nu \in \Sigma$, unit vectors \perp \nu \text{ with } X^2 = S^{n-2}$. With the usual measure $\text{meas } S^{n-2}$,

the average $K(\sigma_0)$ over all $\nu$ is $\text{Ric}_p X$.

In particular, for a space of constant sectional curvature $K_0$,

\[ \text{Ric}_p(x) = K_0 |X|^2, \quad \text{Ric}_p(x, y) = K_0 \langle x, y \rangle, \quad S = K_0. \]
**Derivative on Surface**

**Def.** A **parametrized surface** is a smooth map \( F: U \to M \).

Write \( \frac{\partial F}{\partial s} = F_s(\frac{\partial}{\partial s}), \frac{\partial F}{\partial t} = F_t(\frac{\partial}{\partial t}) \).

Note \( F(s_0, \cdot), F(\cdot, t_0) \) are curves in \( M \) for fixed \( s_0, t_0 \).

A vector field \( V \) along the surface is in particular a vector field along each of these curves: can define \( \frac{d}{dt} \) for \( V \) along these curves.

\[ F(s, t_0) \to \frac{d}{ds} V \]

\[ F(s_0, t) \to \frac{d}{dt} V. \]

So get vector fields \( \frac{d}{ds} V, \frac{d}{dt} V \) on the surface.

**Proof.** \( \frac{d}{ds} \frac{\partial F}{\partial t} = \frac{d}{dt} \frac{\partial F}{\partial s} \).

**Proof.** \( \frac{d}{dt} \frac{\partial F}{\partial s} + \frac{d}{ds} \frac{\partial F}{\partial t} = \left[ \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] = F_* \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0 \).

**Proof.** \( \frac{d}{dt} \frac{d}{ds} V = \frac{d}{ds} \frac{d}{dt} V = F_* \left[ \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] V. \)

**Proof.** More or less by definition: \( \frac{d}{ds} V = \nabla_{\frac{\partial F}{\partial s}} V, \frac{d}{dt} V = \nabla_{\frac{\partial F}{\partial t}} V \) and \( \left[ \frac{\partial F}{\partial s}, \frac{\partial F}{\partial t} \right] = F_* \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right] = 0. \)

\( \square \)
Jacobi vector fields

Let $\gamma = \text{geodesic}$. Say we have a $1$-parameter family of geodesics $\gamma_s(t) = \gamma_s(-\epsilon, \epsilon)$, smoothly varying, $\gamma_0 = \gamma$. Define $F(s, t) = \gamma_s(t)$. The infinitesimal change in geodesic is $\frac{d}{ds} F(s, t)$. At $s = 0$ this is a vector field along $\gamma_0$. Suppose $\gamma_s(0) = p \times s$. Then $\frac{d}{ds}$ measures the “spread” of the geodesic.

The difference between these is curvature.

Special case

Let $\gamma_s(t)$ be a family of geodesics with $\gamma_s(0) = p$ as $s$. Each $\gamma_s$ is determined by $\gamma_s(0) = v(s) = \gamma_s(t) = \exp(tv(s))$.

Then the infinitesimal change at $s = 0$ is $J(t) = \frac{d}{ds}(0, t) = (d(\exp_p))_{tv(0)}(tw)$ where $v = v(0) = \gamma(0)$, $w = v'(0) \in T_pM$. 

\[ J(t) \]
Differential equation for $J(t)$:

Fixing $s = s_0 \Rightarrow F(s,t)$ is a geodesic $\Rightarrow \frac{d}{dt} \frac{\partial F}{\partial s} = 0$

$\Rightarrow 0 = \frac{D}{ds} \frac{d}{dt} \frac{\partial F}{\partial s} + R \left( \frac{\partial F}{\partial s}, \frac{\partial F}{\partial s} \right) \frac{\partial F}{\partial s} \frac{d}{dt} \frac{\partial F}{\partial s}$

$= \frac{D}{ds} \frac{d}{dt} \frac{\partial F}{\partial s} + R \left( \frac{\partial F}{\partial s}, \frac{\partial F}{\partial s} \right) \frac{\partial F}{\partial s}$

Plug in $s = 0$:

$0 = \frac{D^2}{dt^2} J(t) + R(\gamma', J) \gamma'$ (1)

Def: A vector field $J$ along a geodesic $\gamma: [0, \infty) \rightarrow M$ is a Jacobian field if (1) holds.

So, if $F(s,t) = \gamma_s(t)$ is a family of geodesics (no assumption on $\gamma_s(0)$) then $\frac{\partial F}{\partial s}(0, t) = J(t)$ is a Jacobian field along $\gamma_0 = \gamma$.

Prop 3: Jacobian field for sufficiently small $a$ and specified initial conditions

$J(0), J'(0) \in T_p M$, 
$\frac{D}{ds} J(0)$

If we choose ONB $e_1, \ldots, e_n$ of $T_p M$ and by parallel transport get $e_1(t), \ldots, e_n(t)$ ONB along $Y$. Want to find

$J(t) = e(t)$, want $e(t)$

$\Rightarrow \frac{D}{dt} e(t) = \frac{D}{dt} e(t)$ since $e(t)$ parallel

$\Rightarrow \frac{D^2}{dt^2} e(t) = \frac{D}{dt} e(t)$.
Write \( v_{ij}(t) = \langle R(\gamma'(t), e_i(t)) \gamma''(t), e_j(t) \rangle \). Then:

\[
\frac{dx_i}{dt} e_i(t) + f_3(t) \langle R(\gamma', e_j) \gamma', e_j(t) \rangle = 0
\]

This is a 2nd order linear system of ODEs.

Cor. The vector space of Jacobi v.f.s along a geodesic is 2n-dimensional.

Two "trivial" Jacobi v.f.s:
- \( J_1(t) = \gamma'(t) \quad \frac{d}{dt} \gamma' = 0 \quad J_1(0) = \gamma'(0), J_1'(0) = 0 \)
- \( J_2(t) = t \gamma'(t) \quad \frac{d}{dt} (t \gamma'(t)) = 0 \quad J_2(0) = 0, J_2'(0) = \gamma'(0) \)

This corresponds to \( F(s, t) = \gamma_s(t) = \gamma((s + 1)t) \).

Note: \( \{ \text{Jacobi along } \gamma(t) \} \rightarrow \mathbb{R}^2 \\
J(t) \rightarrow (\langle J(0), \gamma'(0) \rangle, \langle J'(0), \gamma'(0) \rangle) \)

This is a surjective linear map because of \( J_1, J_2 \). The kernel is \( \{ \text{Jacobi with } J(0), J'(0) \perp \gamma'(0) \} = (2n-2) \)-dimensional vector space.

Proof: \( \langle J(4), \gamma'(t) \rangle = \langle J(0), \gamma'(0) \rangle t + \langle J(0), \gamma'(0) \rangle \)

so \( J(0), J'(0) \perp \gamma'(0) \iff J(t) \perp \gamma'(t) \forall t \).

Proof. Write \( \mu(t) = a \). Then \( e_i(t) = \frac{\gamma_i(t)}{a} \).

\[
\langle J(4), \gamma'(t) \rangle = a f_4(t), \quad \langle J_i(4), \gamma'(t) \rangle = 0
\]

\[
\implies \frac{d^2 f_4}{dt^2} = 0 \implies \langle J(4), \gamma'(t) \rangle = At + B
\]

\( t = 0 \Rightarrow \langle J(0), \gamma'(0) \rangle = A \) \quad \( \langle J, \gamma' \rangle = \langle J', \gamma' \rangle \).
Break up [Jacobi v.f.] into subspace:

<table>
<thead>
<tr>
<th>dim</th>
<th>w restriction</th>
<th>J normal to y</th>
</tr>
</thead>
<tbody>
<tr>
<td>J(0)=0, n</td>
<td>2n</td>
<td>2n-1 cut at Ji, Ji</td>
</tr>
<tr>
<td>J(0)=0</td>
<td>n-1</td>
<td>cut at J2</td>
</tr>
</tbody>
</table>

There are achieved by the special case \( F(s,t) = \exp_p (tv(s)) \)
(infinite variation \( \gamma_s \) with \( \gamma_s(0) = p + t \)). In this case:

\( J(0) = 0 \) since \( F(s,0) = p \). Also:

\( J(t) = (d \exp_p)_{\gamma_t}(t \gamma_s) \) where \( \gamma = v'(0) \)

\( \Rightarrow \frac{DJ}{dt}(0) = \frac{d}{dt} \bigg| _{t=0} (d \exp_p \circ \gamma(t)) = \frac{d \exp_p}{dt} \circ \gamma(0) = \gamma' \)

So: \( J(0) = 0, J'(0) = \frac{DJ}{dt}(0) = \gamma' \).

Conversely:

Prop: \( \gamma: [0,a] \to M \) geodesic, \( \mathbf{J} = \text{Jacobi vector field along } \gamma \) with \( \mathbf{J}(0) = 0 \).

Then \( \exists \) family of geodesics \( F(s,t) = \gamma_s(t) \) with \( \gamma_0 = \gamma \)

st. \( J(t) = \frac{\partial F}{\partial s}(0,t) \). actually holds without this too.

PF Try \( F(s,t) = \exp_p (tv(s)) \). Choose paths \( \{v(t)\} \subset T_p M \)
with \( v(0) = \gamma'(0), v'(0) = J'(0) \). Then \( \frac{\partial F}{\partial s}(0,t) = J(t) \) is a Jacobi vector field, and \( J, J' \) have same initial condition.

So: \( \mathbf{J} = \overline{\mathbf{J}} \).

\[ \text{Diagram:} \]
\[ F(s, t) = \exp_t(t \nu(s)), \quad J(t) = \frac{\partial F}{\partial s}(0, t), \quad \nu(s) = v = \pi(0) \]

\[ J(0) = 0, \quad J'(0) = \omega = \nu(0). \quad \text{Assume } |w| = 1. \]

**Prop.** With these assumptions:

\[ |J(t)|^2 = t^2 - \frac{1}{5} R(v, \omega, v, w) t^4 + o(t^4) \]

\[ \lim_{t \to 0} \frac{o(t^{4})}{t^4} = 0. \]

**PF**

\( J(0) = 0, \quad J'(0) = \omega, \quad J''(0) = -\langle \mathcal{R}(\gamma', J') \gamma' \rangle(0) = 0 \)

\[ \langle J, J' \rangle(0) = 0 \]

\[ \langle J, J' \rangle'(0) = 2 \langle J(0), J'(0) \rangle = 0 \]

\[ \langle J, J'' \rangle(0) = 2 \langle J, J'' \rangle(0) + 2 \langle J', J' \rangle(0) = 2 \]

\[ \langle J, J'' \rangle'(0) = 2 \langle J, J'' \rangle(0) + 6 \langle J', J'' \rangle(0) = 0 \]

\[ \langle J, J'' \rangle''(0) = 2 \langle J, J'' \rangle(0) + 8 \langle J', J'' \rangle(0) + 6 \langle J', J'' \rangle(0) \]

To calculate \( J''(0) \), note for any vector field \( \xi \):

\[ \langle \mathcal{R}(\gamma', J) \gamma(W) \rangle = \langle \mathcal{R}(\gamma', \omega) \xi^0 \rangle \]

\[ \left. \frac{d}{dt} \right|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} \mathcal{R}(\gamma', J) \gamma(W) = \mathcal{R}(\gamma', \omega) \xi^0 \]

\[ = \langle \mathcal{R}(\gamma', J) \gamma(W) \rangle(0) + \langle \mathcal{R}(\gamma', J') \gamma' \gamma(W) \rangle(0) \]

\[ \Rightarrow \quad J''(0) = -\mathcal{R}(\gamma', J') \gamma'(0) = -\mathcal{R}(v, w)v \]

\[ \Rightarrow \quad \langle J, J'' \rangle(0) = 8 \langle J', J'' \rangle(0) = -8 \mathcal{R}(v, w, v, w). \]

**Cor.** If \( |w| = 1 \) (\( \gamma \) param by arclength), \( |w| = 1 \), \( \langle \nu, w \rangle = 0 \), and \( \sigma \) plane gen'd by \( v, w \), then

\[ |J(t)|^2 = t^2 - \frac{1}{5} K(0) t^4 + o(t^4). \]
Can use this to calculate sectional curvature for \((\mathbb{S}^n, \text{round metric})\). 

\[ x \in \mathbb{S}^n, \ u, v \in T_x \mathbb{S}^n \]

\[ \nu : \gamma \mapsto \nu = \frac{\nu \times \dot{\gamma}}{|\nu \times \dot{\gamma}|} \]

\[ \gamma(t) = x \cos t + v \sin t \]

\[ \gamma_t(t) = x \cos t + (v \cos s + w \sin s) \sin t \]

\[ \Rightarrow J(t) = \frac{d\gamma}{dt} \bigg|_{t=0} = w \sin t \]

\[ \Rightarrow |J(t)|^2 = \sin^2 t = \left(t - \frac{t^3}{3} + \ldots\right)^2 = t^2 - \frac{t^4}{3} + \ldots \]

\[ \Rightarrow K(\sigma(u,v)) = 1 . \]

(Similarly for sphere of radius \(R\): \(K(\sigma(u,w)) = \frac{1}{R^2}\)).

---

**Conjugate Points**

**Def:** \( \gamma : [a,b] \to M \) geodesic. Two points \( p = \gamma(a), q = \gamma(b) \) along \( \gamma \) are \underline{conjugate} if \( \exists \) nonzero Jacobian field \( J \) along \( \gamma \) with \( J(t_0) = J(t_0) = 0 \).

One way to get this: \( \gamma \) part of a family of geodesics between \( p \) and \( q \).

The dimension of the vector space \( \mathbb{F} J(t_0) = J(t_0) = 0 \) is the multiplicity of the conjugate point (think: \( k \)-dimensional family of geodesics).
Ex. \( S^n, \text{round} \): antipodal \( p \) are conjugate along any geodesic, with multiplicity \( n-1 \).

**Def.** \( q \) is the \textbf{first conjugate} to \( p \) along \( \gamma \) if no other conjugate points before \( q \). \textbf{The Conjugate locus of } \( p \) is \{first conjugates\} over all geodesics.

Ex.: \( S^n \): conjugate locus \( \{ -p \} \)

- 2-d ellipsoid is:
  - (more generic)
  - \( \mathbb{R}^n \): no conjugate loci \( \text{(Jacobi fields satisfy } J''(t) = 0) \)

**Prop.** \( \gamma : [0,a] \rightarrow M \text{ geodesic, } \gamma(0) = p, \dot{\gamma}(t) = \exp_p(tv), v = \gamma'(0) \).

Then \( q = \gamma(t_v) = \exp_p(t_v) \) is conjugate to \( p \) along \( \gamma \iff tv \text{ is a critical point of } \exp_p, \text{i.e. } d(\exp_p)_{tv} : T_pM \rightarrow T_qM \text{ is not surjective.} \)

**PF** \( q \) conjugate \( \iff \exists \text{ nonzero Jacobi field } J(t) \text{ with } J(0) = J(t_v) = 0. \)

Recall \( J'(0) = wt(0) \text{ then } J(t) = d(\exp_p)(tv)(tw) \), so

\[ \exists J \text{ with } J(t_v) = 0 \iff \exists w \text{ with } d(\exp_p)_{tv}(w) = 0 \iff \ker d(\exp_p)_{tv} \neq 0. \]
Hadamard (Hadamard-Cartan) Theorem

M complete Riemannian with nonpositive sectional curvature:
\[ K(p) \leq 0 \quad \forall p \in M. \]

Then \( \forall p \in M, \) \( \exp: T_p M \rightarrow M \) is a covering map.

If \( M \) is simply connected, then \( \exp: \mathbb{R}^n \rightarrow M \) is a diffeomorphism.

Ex: \( \mathbb{R}^n, \mathbb{H}^n; \) not \( S^n \) (Cor: \( S^n \) can’t have a metric of \( \leq 0 \) sect. curv.)

Lemma 1: \( M \) (good) complete Riemann with nonpositive sectional curvature.

1. Conjugate locus \( \gamma(p) = \emptyset \quad \forall \gamma \in M \). No conjugate pt to \( p \)
   along any geodesic.
2. \( \exp \) is a local diffeo.

Proof (1) \( J=\) Jacobi field along \( \gamma \), \( \gamma(0) = p, \gamma(t_0) = q, \gamma(0) = J(t_0) \geq 0. \)

\[ \frac{d}{dt} \langle J(t), J(t) \rangle = 2 \langle J', J \rangle \]
\[ \frac{d}{dt} \langle J, J \rangle = 2 \langle J', J \rangle + 2 \langle J'', J \rangle \]
\[ = 2 |J'|^2 - 2 \langle 0(J', J), J \rangle \]
\[ = 2 |J'|^2 - 2 K(p) \frac{|\gamma' \wedge J|^2}{\gamma' \cdot J} \geq 0. \]

But if \( J \) has two zeros then \( |J|^2 \) has a max., where \( \frac{d^2}{dt^2} |J|^2 < 0. \)

(2) Follows from previous proof. \( \square \)

Lemma 2: \( M, N \) Riem, \( M \) good complete, \( f: M \rightarrow N \) surjective local isometry (in particular, local diffeo). Then \( f \) is a covering map.
If \( p \in N \), \( f^{-1}(p) = \{ \hat{q} \} \). Let \( B_r(p) = \text{normal ball in } N, \exp_p B_r(p) \) \( \text{diff.} \)

Write \( U = B_r(p), \ U_i = \exp_p (B_r(0)) \subset M \) \( \exists \text{ since complete}. \)

**Claim:** \( f^{-1}(U) = \sqcup U_i, \ f_U : U_i \rightarrow U \text{ diff.}. \)

1. \( f(U_i) \subset U : \) \( q \in U_i \Rightarrow \exists \text{ geodesic } \gamma \text{ from } p \text{ to } q, L(\gamma) < r. \)

   \[ f(\gamma) \in U_i \Rightarrow \exists \text{ geodesic } \gamma' \text{ from } p \text{ to } f(q), \ L(\gamma') < r \Rightarrow f(q) \in U. \]

2. \( f : U_i \rightarrow U \text{ diff.}. \)

\[
\begin{array}{c}
\begin{array}{c}
B_r(0) \xrightarrow{\exp_p} U_i \xrightarrow{f} U
\end{array}
\end{array}
\]

**Diagram commutes:**

Since \( f, \exp_p \) are diff., \( \exp_p \) is injective \( \Rightarrow \) bijective, \( \Rightarrow f \) is bijective \( \Rightarrow \exp_p, f \) are diff.

3. **Claim:** \( f^{-1}(U) = \sqcup U_i. \) Suppose \( \bar{q} \in f^{-1}(U), q = f(\bar{q}). \)

   \[
   \begin{array}{c}
   \begin{array}{c}
   \hat{q} \xrightarrow{f^{-1}} \bar{q}
   \end{array}
   \end{array}
   \]

   **Reverse geod.** \( q \xrightarrow{\gamma} \hat{q} \text{ to be geod. } \gamma \text{ from } q \text{ to } p. \)

   Write \( v = \gamma'(0) \Rightarrow \exists \text{ geodesic } \tilde{\gamma} \text{ with } \tilde{\gamma}(0) = \bar{q}, \)

   \( \tilde{\gamma}'(0) = (f^{-1})'(v). \) Then \( f \circ \tilde{\gamma} = \gamma \text{ so } \)

   the length of \( \tilde{\gamma} \) is some \( p \Rightarrow \bar{q} \in U_i. \)

   If \( \exists \text{ geodesic } \bar{\hat{q}}, \bar{\hat{q}} \text{ from } \bar{q} \text{ to } i, q \), then they must

   project to the same geodesic from \( q \) to \( p \) by uniqueness. \( \Rightarrow \)
PF of Hadamard \[ \exp_p : T_p M \to M \] well-defined, surjective.

Lemma 1 \[ \Rightarrow \] local diffeos. So can pull back metric on \( M \) to

metric on \( T_p M \); the \( \exp_p \) is a local isometry.

Now straight lines \( \Gamma (t) \) through \( 0 \in T_p M \) are geodesics since

they map to geodesics, so by Hopf–Lina, \( T_p M \) is good.

Complete. Lemma 2 \[ \Rightarrow \] covering map. \( \square \)

**Variations of Energy**

Idea: \( p, q \in M \) \[ \Rightarrow \] let \( \gamma (p, q) = \{ \text{piecewise differentiable paths from } p \text{ to } q \} \).

Length gives a map \[ l : \gamma (p, q) \to \mathbb{R}_{\geq 0} : \]

\[ l(\gamma) = \int_0^1 \langle \gamma(t), \gamma'(t) \rangle dt \quad \gamma : [0, 1] \to M \]

A minimizing geodesic is a global minimum for \( l \).

More generally, a geodesic is a critical pt for \( l \):

\[ d\gamma^* : T \gamma (p, q) \to \mathbb{R} \]

satisfies \( d\gamma^* = 0 \).

We'll make this precise.

First note: if \( \gamma \) is a length-minimizing curve from \( p \) to \( q \),

\[ l(\gamma) \leq l(\tilde{\gamma}) \quad \forall \tilde{\gamma} \in \gamma (p, q) \],

then \( \gamma \) is a reparametrization of a geodesic.

Can get rid of reparam. by considering instead the energy function

\[ E(\gamma) = \int_0^1 \langle \gamma(t), \gamma'(t) \rangle dt \]

which, unlike \( l(\gamma) \), changes under reparam.
Prop. \( \gamma: [a, b] \to M \) length-min geodesic between \( p \) and \( q \). Then \( E(\gamma) \leq E(\tilde{\gamma}) \)

For any \( \tilde{\gamma}: [a, b] \to M \) between \( p \) and \( q \), equality \( \Rightarrow \tilde{\gamma} \) is length-min geodesic.

Lemma \( \ell(\gamma)^2 \leq (b-a) E(\gamma) \) for any path \( \gamma \), equality \( \Rightarrow \) constant speed.

PF \( \ell(\gamma)^2 = \int_a^b \|\gamma'(t)\|^2 \, dt \leq \int_a^b \|\gamma'(t)\|^2 \, dt \leq (b-a) E(\gamma) \), Cauchy-Schwarz equality \( \Leftrightarrow \gamma(t) \) is proportional. \( \square \)

PF of Prop. \( (b-a) E(\gamma) = \ell(\gamma)^2 \leq \ell(\tilde{\gamma}) \leq (b-a) E(\tilde{\gamma}) \),

equality \( \Leftrightarrow \ell(\gamma) = \ell(\tilde{\gamma}) \) and \( \tilde{\gamma} \) has constant speed. \( \square \)

Next: define "\( T_\gamma \mathcal{P}(q, g) \)" = infinitesimal deformation of a path.

Def. A variation \( \gamma: [a, b] \to M \) is a smooth map \( F: (-\varepsilon, \varepsilon) \times [a, b] \to M \) for some \( \varepsilon > 0 \)

with \( F(0, t) = \gamma(t) \).

A proper variation is a variation with \( F(s, a) = \gamma(a), F(s, b) = \gamma(b) \).

Variation \( \Rightarrow \) variational field \( V(t) = \frac{\partial F}{\partial s}(0, t) \) vector field along \( \gamma \).

Proper variation \( \Rightarrow \) \( V(t) \) with \( V(a) = V(b) = 0 \).
Prop If \( V(t) \) is a smooth vector field along \( \gamma \), then the tangent variation \( F(s,t) \) (for some \( s \)) s.t. \( V = \frac{d}{dt} \) variational field. If \( V(0) = V(t) = 0 \Rightarrow F \) proper variation.

\[ \frac{d}{ds} \left[ \exp_{\gamma(s)}(sV(t)) \right] \text{ is defined for } |s| < \epsilon(t). \text{ Compactness } \Rightarrow F \text{ is st.} \]

\[ F(s,t) = \exp_{\gamma(s)}(sV(t)) \text{ is defined on } (-\epsilon, \epsilon) \times [0, 1]. \]

- Smooth since geodesic flow is smooth.
- \( \frac{d}{ds} \left[ \exp_{\gamma(s)}(sV(t)) \right] = V(s) \) since \( F(s,t) = \gamma(t) \).

\[ V(t) \text{ vec field along } \gamma, V(0) = V(t) = 0. \]

Now consider \( \gamma : \mathbb{R} \times [0,1] \to \mathbb{R} \): want to find minima.

Want to calculate 1st and 2nd derivatives.

1st: \( dE_{\gamma} : T_{\gamma} \mathcal{P}(p,q) \to \mathbb{R} \).

\[ V(t) \text{ vec field along } \gamma, V(0) = V(t) = 0. \]

Prop (First Variation Formula)

\[ F : (-\epsilon, \epsilon) \times [0,1] \to M \text{ variational } \gamma : [0,1] \to M, \]

\[ \text{Variational field } V(t). \text{ Then} \]

\[ \frac{1}{2} \frac{d}{ds} \bigg|_{s=0} E(\gamma) = \left< V(t), \frac{d}{dt} \gamma'(t) \right|_{t=0} - \int_{t=0}^{t} \left< V(t), \frac{d}{dt} \gamma'(t) \right> dt \]

Notation: Write \( Y(s,t) = \frac{dF}{ds}, X(t,t) = \frac{dF}{dt} \).

\[ Y(0,t) = V(t), X(0,t) = \gamma'(t), X(1,t) = \gamma'(t). \]
\[ E(s) = \int_a^b \langle X, X \rangle \, dt \]
\[ \Rightarrow \quad \frac{1}{2} E'(s) = \int_a^b \frac{d}{ds} \left[ \frac{2}{s} \right] \, dt \]
\[ = \int_a^b \left\langle \frac{\partial}{\partial s} X, X \right\rangle \, dt \]
\[ = \int_a^b \left( \frac{d}{ds} \left[ \frac{2}{s} \right] \cdot X, X \right) \, dt \]
\[ = \int_a^b \left( \frac{2}{s} \cdot \langle X, X \rangle - \langle X, \frac{d}{ds} \frac{2}{s} \rangle \cdot X \right) \, dt \]
\[ = \langle X, X \rangle \bigg|_{t=a}^{t=b} - \int_a^b \langle X, \frac{d}{ds} \frac{2}{s} \rangle \cdot X \, dt. \]

Now plug in \( s = 0 \). \quad \Box

**Claim:** \( \gamma \) is a geodesic \( \iff \) a proper variation of \( \gamma \), \( E'(0) = 0 \).

**Proof:**

\( \leftarrow \) follows from Prop.

By Prop. 3, a vector field \( V(t) \) along \( \gamma \) with \( V(a) = V(b) = 0 \),
\[ \int_a^b \left\langle V(t), \nabla \gamma, \gamma' \right\rangle \, dt = 0. \]

Choose \( V(t) = f(t) \gamma'(t) \) where \( f(a) = f(b) = 0 \), \( f > 0 \) on \( (a, b) \).

\[ \int_a^b f(t) \left| \nabla_{\gamma', \gamma'} \right|^2 \, dt = 0 \Rightarrow \nabla_{\gamma', \gamma'} = 0. \quad \Box \]

So geodesics = critical pt of \( E \). To determine if they're minimizing, need 2nd derivative test.

**Prop. (Second Variation Formula)**

\( \gamma : [a, b] \to M \) geodesic, \( F: (c, \varepsilon) \times [a, b] \to M \) variation. Then
\[ \frac{1}{2} E''(0) = \left. \left\langle \frac{d}{ds} \frac{\partial}{\partial s} (0, t), \gamma' \right\rangle \right|_{t=a}^{b} + \int_a^b \left( |V|^2 - F(V, \gamma', V, \gamma') \right) \, dt. \]
As before: \( \frac{1}{2} \frac{\partial^2}{\partial s^2} \langle x, x \rangle = \langle \frac{D}{Ds} \frac{D}{Dx}, x \rangle \)

\[ \frac{\partial}{\partial s} \left( \frac{D}{Dx} \frac{D}{Dx}, x \right) = \frac{D}{Dx} \left( \frac{D}{Dx}, x \right) + \left( \frac{D}{Dx}, \frac{D}{Dx}, x \right) \]

\[ \frac{D}{Dx} \frac{D}{Dx} \frac{D}{Dx} \frac{D}{Dx}, x \right) - \frac{D}{Dx} \left( \frac{D}{Dx}, x \right) \]

\[ = 0 \text{ at } s=0 \text{ since } y = \psi \text{ is} \]

\[ \frac{1}{2} \frac{\partial^2}{\partial s^2} \langle x, x \rangle \bigg|_{s=0} = \frac{2}{2} \langle \frac{D}{Ds}, \psi' \rangle - R(\psi', \psi', \psi, \psi') \]

\[ \geq 0 \text{ at } s=0 \text{ since } y = \psi \text{ is} \]

Now integrate from \( t=a \) to \( t=b \). \( \square \)

Cor. If \( F \) is proper variation then

\[ \frac{1}{2} E''(0) = \int_a^b \left( |V'|^2 - R(V, V', V, V') \right) dt \]

depends only on \( V \).

Link: \( E''(0) = \text{"Hessian" of } E(V, V) \).

If \( \psi \) is a local min for \( E \) then \( E''(0) \geq 0 \) for all \( V \).

Two applications: Myers' Theorem and Synge's Theorem.

Myers' (Barrett-Mueter) Theorem

\( M \) complete. Suppose there is \( r \geq 0 \) such that

\[ \text{Ric}_g(v) \geq \frac{1}{r} \text{ for all } v \in M \text{ and all } v \in T_M \text{ with } |v| = 1. \]

Then \( M \) is compact and \( \text{diam}(M) \leq \text{diam}(S^r_1). \)
Proof. Sufficient to show $\forall p,q \in M$, $\gamma=\min$ geodesic between $p$ and $q \Rightarrow \kappa(\gamma) \leq \pi r$.

Then $\text{diam}(M) \leq \pi r \Rightarrow M$ bounded, complete $\Rightarrow$ compact.

Suppose $\kappa(\gamma) = \kappa$ and assume $|\gamma'| = 1$. Choose ONB $\{e_i = \gamma'(0), e_{i+1}, \ldots, e_n\}$ of $T_p M$ and extend by parallel transport to ONB $\{e_i(t) = \gamma'(t), e_{i+1}(t), \ldots\}$ along $\gamma$.

Along $\gamma$, define vector field $V_i(t) = \sin\left(\frac{\pi t}{\kappa}\right)e_i(t)$, $2 \leq i \leq n$,

$F_i = \text{proper variation of } \gamma$ with variational field $V_i$.

$\gamma$ minimizes energy $\Rightarrow \gamma$ 2nd variation

$0 \leq \frac{1}{\kappa} E''(0) = \int_0^\kappa \left(\|V_i\|^2 - R(e_i, \gamma', e_i, \gamma')\right) dt$

$= \int_0^\kappa \left(\frac{\pi^2}{\kappa^2} - \sin^2\left(\frac{\pi t}{\kappa}\right)\right) R(e_i, \gamma', e_i, \gamma') dt$

Average over $i$:

$0 \leq \frac{\pi^2}{2\kappa^2} - \int_0^\kappa \sin^2\left(\frac{\pi t}{\kappa}\right) R(e_i, \gamma', e_i, \gamma') dt$

$= \frac{\pi^2}{2\kappa^2} - \int_0^\kappa \frac{1}{\kappa} \sin^2\left(\frac{\pi t}{\kappa}\right) dt$

$= \frac{\pi^2}{2\kappa^2} - \frac{\kappa}{2\pi^2}$

$\Rightarrow \kappa \leq \pi r$. $\square$

Cor. If $M$ complete then, $\text{Ric}(\gamma) \geq \frac{1}{r}$.

Then the universal cover of $M$ is compact and $\pi_1(M)$ is finite.

Proof. Let $\tilde{M} \to M$ be the universal cover. Pull back metric on $M$ to $\tilde{M}$; $\tau$ = local isometry. Then $\tilde{M}$ is complete; apply Myers to $\tilde{M}$

$\Rightarrow \tilde{M}$ cpt, # of sheets = finite. $\square$

Note. Complete, $\text{Ric} > 0 \neq$ cpt, finite diameter. Ex: $\{x^2 + y^2 + z^2 < r^2\} \subset \mathbb{R}^3$. 
Snyge's Theorem: M compact, even-dimensional, orientable, strictly positive sectional curvature. Then M is simply connected.

Rule 1. Statement in odd dim: M compact, odd-dim, K > 0. Then M is orientable. (see book)
2. Can't remove even-dim or orientable assumption: R^2n has K > 0 with metric induced from S^2n. Can't weaken to K ≥ 0: T^n = R^n/Z^n.

Key to Snyge: closed geodesics.
A geodesic γ : [0, a] → M is closed if γ(0) = γ(a) and γ'(0) = γ'(a). Think of this as a smooth map S^1 → M.

Def. A homotopy class of free loops in M is a map S^1 → M up to homotopy.

Proof M compact. In any homotopy class of free loops in M, there is a closed geodesic.

PF: Compact ⇒ E ∈ ℝ with Be(p) normal ∀ p ∈ M. Consider l = inf {E(γ) | γ ∈ homotopy class}.
If E(γ) = l then γ locally minimizes length ⇒ geodesic. Otherwise, assume E(γ) with E(γ) > l. Reparameterize γ to γ_i : [0, 1] → M, ∥γ_i∥ = E(γ_i).
Choose \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) with \( t_k - t_{k-1} \leq \frac{\epsilon}{\text{max} \|\dot{x}\|} \).

Then \( x(t_k), \dot{x}(t_k) \) lie in a normal \( \epsilon \)-ball.

So we can replace \( x_i \) by piecewise smooth curve \( s \).
\( \dot{x}_i \mid_{[t_k, t_{k+1}]} \) is a geodesic.

By passing to a subsequence, we can assume \( \forall k, \quad \dot{x}_i(t_k) \to \dot{p}_k \).
\( x_i(t_k) \to v_k \).

Now define \( y : [0,1] \to M \) by \( y(t_k) = x_i(t_k) \) with \( \dot{y}(t_k) = \dot{x}_i(t_k) = \dot{p}_k \).

Then \( y \mid_{[t_k, t_{k+1}]} \to \dot{y} \mid_{[\cdot, \cdot]} \) so \( l(y) = \lim l(y_i) = l \). \( \square \)

**Proof.** Suppose \( M \) not simply connected. Then \( \exists \) closed geodesic \( \gamma : [0,2\pi] \to M \) in a nontrivial homotopy class of minimum length.

Say \( \gamma(0) = \gamma(a) = p \).

Parallel transport along \( \gamma \) gives \( \gamma : T_p M \to T_p M \), orientation preserving, and \( P(\gamma'(0)) = \gamma'(0) \).

Let \( T_p^\perp M \) orthogonal complement to \( \gamma'(0) \subset T_p M \).

Thus \( P : T_p M \to T_p M \to T_p M \).

Let \( V = \text{parallel vector field along } \gamma \) with \( V(0) = V(a) = V \),
and let \( \gamma_V = \text{corresponding variation} \).

\( V = \gamma_0 \) minimizing energy \( E(0) = E(\gamma_0) \) has local min at \( s = 0 \).

Then
\[
\frac{1}{2} E''(0) = \left( \frac{d}{ds} \frac{d}{ds} (0, t), \gamma \right)_{t=0} + \int_0^a \left( |V'|^2 - R(V, V', V, V') \right) dt
\]

\( \frac{1}{2} E''(0) = -\int_0^a R(V, V', V, V') dt \)

\( E''(0) < 0 \)

\( \Rightarrow \) \( \square \)
Constant Sectional Curvature

Three manifolds of constant sectional curvature

- $\mathbb{R}^n$, flat: $K = 0$
- $S^n$, round: $K = 1$
- $H^n$, hyperbolic: $K = -1$ (See II).

If $(M, g)$ has constant $K$, we can rescale to get $K \in \{0, 1, -1\}$:

If $K > 0$, then $\tilde{g} = \lambda g$ is a metric with $\tilde{\nabla} = \nabla$, $\tilde{R} = \lambda R$, $\tilde{\kappa} = \lambda^{\frac{1}{2}} K$.

Can get more by quotienting by a group of isometries.

Then $M^n$ complete Riem mfd, constant sectional curvature $K \in \{0, 1, -1\}$. Then its universal cover is:

1. $\mathbb{R}^n$, flat if $K = 0$
2. $S^n$, round if $K = 1$
3. $H^n$, hyperbolic if $K = -1$,

i.e. any complete mfd with constant $K$ is a quotient of one of them by isometries.

Proof: $\forall v \in T_p M \leadsto \exp_p T_v M \to M$ is a local isometry if $K = 0$, almost a local isometry if $K = \pm 1$. Choose $v \in T_p M$ unit vector.
Look at \((d \exp)_v : T_v(T_p M) \to T_{\exp v}(v)\),
\[ T_p M \]

Note this maps \[ v \mapsto \gamma'(x) \]
\[ v^\perp \mapsto (\gamma'(x))^\perp \]
by Gauss lemma.

So it suffices to see what this does on \(v^\perp \subset T_p M\).

Choose \(u \in T_p M\) with \(u \perp v\).
\[ \Rightarrow U(t) = \text{parallel vector field along } \gamma \text{ with } U(0) = u; \]
\[ J(t) = \text{Jacobi field with } J(0) = 0, J'(0) = u; \]
\[ J(t) = (d \exp)_v (tu). \]

How are these related? Jacobi equation \(J'' = -R(\gamma', J) \gamma'\),
constant sectional curvature \(R(X, Y)Z = K \langle \langle X, Y \rangle Y, Z \rangle \Rightarrow R(\gamma', J) \gamma' = K \langle \langle \gamma', \gamma' \rangle J - \langle J, \gamma' \rangle \gamma' \rangle = KJ \)
\[ \Rightarrow J'' = -KJ. \]

\[ \text{Case } 1. \quad K \neq 0: \text{ already done in HW.} \]
\[ J''' = 0 \Rightarrow J(t) = t(U(t)) \Rightarrow (d \exp)_v (u) = U(t) \]
and parallel transport preserves inner product, so \((d \exp)_v \) is an isometry on \(v^\perp \) \(\Rightarrow\) on all of \(T_v(T_p M)\).

Thus \(\exp\) is a local isometry \(T_p M \to M\), so by the proof of
Hadamard, it's a covering map.
Case 2: $K = -1$. $J'' = -J \Rightarrow J(t) = (\sinh t) U(t)$.

Choose act $\Phi$ and pick a linear isometry $\varphi: T_p M \to T_{\Phi p} M$.

Claim: $\Phi := \exp_p \circ (\varphi^{-1} \circ \exp_p)^{-1}$ is a local isometry $H^n \to M \Rightarrow$ covering map.

Proof: Let $v \in T_{\Phi p} M$ be any unit vector, $v = \varphi \tilde{v}$ unit vector in $T_p M$.

Want to show: isometry at $\exp_p(t\tilde{v})$. Sufficient to show on $H^n$.

Choose $u \perp v$, $\tilde{u} = \varphi(u) \perp \tilde{v}$ denote Jacobi fields: $J(t) = d(\exp_p)_u(t\tilde{u})$, $\bar{J}(t) = d(\exp_p)_{\tilde{u}}(t\tilde{v})$. Note $J(t) = \Phi \cdot \bar{J}(t)$.

Now if we have $u_1, u_2 \perp v \Rightarrow J_1(t), J_2(t)$ on $M$, $\bar{J}_1(t), \bar{J}_2(t)$ on $H^n$.

Then sufficient to show $\langle J_1(t), J_2(t) \rangle = \langle \bar{J}_1(t), \bar{J}_2(t) \rangle$.

$J_1(t) = (\sinh t) U(t) \Rightarrow \langle J_1(t), J_2(t) \rangle = (\sinh t)^2 \langle U(t), U_2(t) \rangle = (\sinh t)^2 \langle u_1, u_2 \rangle$

and similarly $\langle \bar{J}_1(t), \bar{J}_2(t) \rangle = (\sinh t)^2 \langle \tilde{u}_1, \tilde{u}_2 \rangle$ as desired.
Case 3 \( k = 1 \). \( J^c = J \Rightarrow J(t) = (\sin t) M(t) \).

\[\exp \quad \mathbb{R}^n \quad \mathbb{R}^n \quad \mathbb{R} \]

\[\exp = \text{diffeo from } B_\pi(0) \to S^n - \{ \pi \} \]

Same argument as before: \( \Phi := \exp \circ \Phi^{-1} \circ \exp : S^n - \{ \pi \} \to M \)

is a local isometry.

Now choose \( b \neq a \) in \( S^m \). Define

\[\psi := \exp_b \circ \Phi \circ \exp_{\pi}^{-1} : S^m - \{ \pi \} \to M \]

Claim: \( \Phi = \psi \). Why? A geodesic from \( b \) to \( c \) with tangent vector \( \omega \) maps under both \( \Phi \) and \( \psi \) to a geodesic from \( \Phi(b) \) with tangent vector \( \Phi_{*b} \omega \), so \( \Phi \) and \( \psi \) map \( c \) to the same thing.

So \( \Phi \) and \( \psi \) together give \( \Phi : S^m \to M \) local isometry

\[\Rightarrow \text{covering map, as before.} \]

\[\uparrow \]

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Cut locus

$M$ complete $\Rightarrow \exp_p : T_p M \to M$ surjective. Can we always model $M$ by part of $T_p M$? E.g., for $S^n$, $\exp_p : B^n(0) \to S^n \setminus \{p\}$ different.

For a geodesic $\gamma(t)$, $0 \leq t < \infty$, starting at $p$, we know that $\gamma|_{[0,t]}$ is a minimizing geodesic for small $t$. Define

$I = \{ t_0 \geq 0 | \gamma|_{[0,t_0]} \text{ is a minimizing geodesic} \}.
\Rightarrow t_0 = d(p, \gamma(t_0))$

$I$ is closed.

If $t_0 \in I$ and $t_1 > t_0$ then $t_1 \in I$.
So either $I = [0, T]$ for some $T > 0$ or $I = [0, \infty)$.

Def. The cutpoint of $p$ along $\gamma$ is $\gamma(T)$ (if $I = [0, \infty)$, no cut point).
The cut locus of $p$ is $\text{Cut}(p) = \{ \text{cut points overall all } \gamma \}$.

Ex. Round sphere: $\text{Cut}(p) = S^2 \setminus \{p\}$

$\mathbb{R}^2/\mathbb{Z}^2$: $\text{Cut}(p) = \mathbb{R}^2$.

Complete, simply connected, $K \leq 0 \Rightarrow \text{Cut}(p) = \emptyset$. (Hadamard)

Now let the geodesic vary: $v \in T_p M, |v| = 1 \Rightarrow \exp_p (tv)$

Cut point
Define \( T(W) = \begin{cases} T & \text{if } I = [0, T] \\ \infty & \text{if } I = [0, \infty) \end{cases} \).

The preimage of the geodesic up to the cut point is the ray to \( T(W) \) v.
Write \( U(p) = U(\text{these open rays}) \), i.e.

\[
U(p) = \{ etv \mid v \in S^{n-1}, 0 \leq t \leq T(W) \}\\
= \{ etv \mid N=1, \exp_p(tv) \text{ is minimizing past } t \}.
\]

- \( U(p) \) is star-shaped
- Can show \( T : S^{n-1} \to \mathbb{R}^+ \cup \{ \infty \} \) is continuous \( \Rightarrow U(p) \cong D^n \) (in fact, diffeo)
- \( \text{Cut}(p) = \exp_p(\partial U(p)) \)

Prop M (complete) is the disjoint union \( \exp(U(p)) \sqcup \text{Cut}(p) \).

If \( g \in M \Rightarrow \exists \text{ min geodesic from } p \text{ to } g \) (Horpg-lines).
Either this stops being minimizing past \( g \Rightarrow g \in \text{Cut}(p) \)
or not \( \Rightarrow g \in \exp(U(p)) \).

If \( g \in \exp(U(p)) \cap \text{Cut}(p) \) then \( \exists 2 \text{ minimizing geodesics between } p \text{ and } g \) one minimizing past \( g \), one not. Proof now follows from the next result. \( \square \)
Lemma: If $\exists$ 2 minimizing geodesics between $p$ and $q$, then neither minimize $p$ or $q$.

**PF:**

\[ \gamma(0) \neq \gamma(1) \text{. Extend } \gamma \text{ past } \gamma. \]

A shorter path is $\gamma \cup \text{Cut}(\gamma)$.

**Note:** this argument actually shows:

- $\exp_p$ is an injection $U(p) \rightarrow M$
- $\exp_p$ is a bijection $U(p) \rightarrow M \setminus \text{Cut}(p)$.

**Prop:** $\exp_p : U(p) \rightarrow M \setminus \text{Cut}(p)$ is a diffeomorphism.

**PF:**

\[ \forall u \in U(p), \exp_p \gamma(u) \text{ is minimizing at } t = 1. \text{ Then } \gamma \text{ has no conjugate points to } p \text{ between } p \text{ and } \exp_p (at + v) \text{ (from } \exp_p \gamma) \text{ so } \exp_p \gamma \text{ is not conjugate to } p \Rightarrow \exp_p \text{ is a local diffeomorphism at } \gamma. \]

$\Rightarrow \exp_p \text{ is a local diffeo on } U(p)$

$\Rightarrow$ diff-eo since bijective.

All the interesting topology in $M$ is in $\text{Cut}(p)$:

$\exp_p : \tilde{T}^p \cong U(p) \xrightarrow{\sim} M \setminus \text{Cut}(p)$

$\exp_p : \tilde{U}(p) \xrightarrow{\sim} \text{Cut}(p)$.

**Weird examples:** $S^1$:

$T^2 = \mathbb{R}^2 / \mathbb{Z}^2 = \mathbb{C} / \langle 1, e^{2\pi i/3} \rangle$.

Write $M_p$ deformation retract onto $\text{Cut}(p)$.
Prop (see do Carmo Ch 13 prop 2.2) 
\( q \) is cut point of \( p \) along \( \gamma \) \iff \( p \) first point along \( \gamma \) where either
- Conjugate to \( p \)
- \( \exists \) two minimizing geodesic between \( p \) and \( q \).

\( q \in M \setminus \text{Cut}(p) \implies \exists ! \) min geodesic from \( p \) to \( q \).
\( \Rightarrow \) \( \exp : B_r(c) \to B_r(p) \) is injective \( \iff r \leq d(p, \text{Cut}(p)) \).

Def The **injectivity radius** \( i(M) := \inf_{p \in M} d(p, \text{Cut}(p)) \).

(If \( r \leq i(M) \) then \( \exp : B_r(c) \to M \) is injective \( \forall p \).

Intuitively: if \( k \) is small then \( i \) is large. Sample this.

Thm (Ch 13 prop 2.13) If \( 0 < a \leq k \leq k_{\text{max}} \) for some \( a, k_{\text{max}}, \)
then either \( \exists \) closed geodesic \( \gamma \) with \( i(M) = \frac{-1}{2} \ell(\gamma) \) or \( i(M) \geq \frac{\pi}{\sqrt{k_{\text{max}}}} \).
(Note by Myers that \( M \) is cpt.)

Thm (Klingenberg) \( M \) simply connected, cpt, \( \dim \geq 3 \),
\( \frac{K_0}{q} \leq k \leq K_0 \) for constant \( K_0 > 0 \).
Then \( i(M) \geq \frac{\pi}{\sqrt{K_0}} \). (idea: rule out short geodesics)

Sphere Theorem \( M \) simply connected, cpt,
\( 0 < \frac{K_0}{q} \leq k \leq K_0 \).
Then \( M \) is homeomorphic to \( S^n \).
Submanifolds

\( M \subset \overline{M} \) submanif. Riem metric on \( \overline{M} \rightarrow \) Riem metric on \( M \).

Levi-Civita connections \( \nabla, \overline{\nabla} \).

Recall: \( X, Y \in \text{Vect}(M) \) extending to \( \overline{X}, \overline{Y} \in \text{Vect}(\overline{M}) \)

\[ \nabla_X Y = (\overline{\nabla}_X \overline{Y})^+ \]

independent of the extensions \( \overline{X}, \overline{Y} \).

where \( p \in M \rightarrow T_p \overline{M} = T_p M \oplus (T_p M)^{\perp} \)

\[ \nabla_X Y \rightarrow \nu \rightarrow \nu^{\perp} \]

Def: \( X, Y \in \text{Vect}(M) \)

\[ B(X, Y) := (\overline{\nabla}_X \overline{Y})^+ = \overline{\nabla}_X \overline{Y} - \overline{\nabla}_X \overline{Y} \]

"vector valued second fundamental form" (in \( T_p \overline{M} \)).

Fact:

1. Independent of extensions \( \overline{X}, \overline{Y} \).

as in HW: if \( \overline{X}, \overline{X}' \) are extensions of \( X \), \( \overline{\nabla}_X \overline{Y} - \overline{\nabla}_{X'} \overline{Y} = \overline{\nabla}_{X - X'} \overline{Y} = 0 \)

since \( \overline{X} - \overline{X}' = 0 \) on \( M \)

\[ \overline{X}, \overline{Y} \] extensions of \( X, Y \)

\[ \overline{\nabla}_X \overline{Y} - \overline{\nabla}_{X'} \overline{Y} = \overline{\nabla}_X (\overline{Y} - \overline{Y'}) = 0 \]

since \( \overline{Y} - \overline{Y}' = 0 \) on \( M \)

\( \overline{X} \) target to \( M \).

2. Tensor: clearly tensorial in \( X \).

\[ B(X, fY) = \overline{\nabla}_X (f \overline{Y}) - \overline{\nabla}_X (f \overline{Y}) = fB(X, Y) + \overline{\nabla}_X (f \overline{Y}) - \overline{\nabla}_X (f \overline{Y}) \]

\[ = fB(X, Y) \]

3. Symmetric:

\[ B(X, Y) - B(Y, X) = [\overline{X}, \overline{Y}] - [\overline{X}, \overline{Y}] = 0 \] on \( M \).

Choose a unit normal vector field \( \nu \) along \( M \).

(if \( M, \overline{M} \) orientable, then \( \nu \) is well-defined globally up to \( \pm \)).
The second fundamental form is the symmetric bilinear form 
\[ \Pi : T_p M \otimes T_p M \to \mathbb{R} \] given by 
\[ \mathcal{B}(X,Y) = \Pi(X,Y) v. \] (Note: 1st fundamental form is just the metric)

**Def** \( \sigma = 2 \)-plane in \( T_p M \) generated by \( X,Y \).

The **Gaussian curvature** \( G(\sigma) \) is 
\[ G(\sigma) = \frac{\Pi(X,X) \Pi(Y,Y) - \Pi(X,Y)^2}{\langle X,X \rangle \langle Y,Y \rangle - \langle X,Y \rangle^2}. \]

This is independent of choice of \( X,Y \) (as in sectional curvature).

**Def** The **shape operator** \( S : T_p M \to T_p M \) is defined by 
\[ \langle S(X), Y \rangle = \Pi(X,Y). \]

Alternate def for \( S \): note 
\[ \Pi(X,Y) = \langle B(X,Y), v \rangle = \langle \overline{\nabla}_X Y, v \rangle = \langle \overline{\nabla}_{\overline{\nabla}} Y, v \rangle = \langle \overline{\nabla}_Y \overline{\nabla} Y, v \rangle = -\langle Y, \overline{\nabla}_X v \rangle \]
\[ \Rightarrow S(X) = -\langle \overline{\nabla}_X v \rangle^T. \]

Eigenvalues of \( S \) are principal curvatures.

For surfaces in \( \mathbb{R}^3 \), 
\[ G(\sigma) = \lambda_1 \lambda_2 \] product of eigenvalues.

Then \( G_{\text{Gauss}}(X,Y,U,V) \in T_p M \). The
\[ \mathcal{B}(X,Y,U,V) = \mathcal{B}(X,Y,U,V) + \Pi(X,U) \Pi(Y,V) - \Pi(X,V) \Pi(Y,U) \]
\[ K(X,Y) = \overline{K}(X,Y) + G(X,Y). \]
Extend $X, Y, U, V$ at $p$ to vector fields $X, Y, U, V$ on $M$, $X, Y, U, V$ on $\mu$.

$$
\bar{\nabla}_X Y = \nabla_X Y + II(Y, U)\nu \quad \text{on } M
$$

$$
= \bar{\nabla}_X (\nabla_Y U + II(Y, U)\nu)
$$

$$
= \nabla_X \nabla_Y U + II(X, \nabla_Y U)\nu + II(Y, U)\bar{\nabla}_X \nu + (\bar{X} II(Y, U))\nu
$$

$$
\Rightarrow \langle \bar{\nabla}_X \nabla_Y U, \nu \rangle = \langle \nabla_X \nabla_Y U, \nu \rangle + II(Y, U)\langle \bar{\nabla}_X \nu, \nu \rangle
$$

$$
= \langle \nabla_X \nabla_Y U, \nu \rangle - II(Y, U) II(Y, U).
$$

Important special case:

Then if $\mu = (\mathbb{R}^n, \text{flat})$ then $K(X, Y) = G(X, Y)$.

Note: For $n=3$ this is Gauss's Theorema Egregium:

in his language: the extrinsic quantity $G(X, Y)$ (depends on the isometric embedding of $M$ in $\mathbb{R}^3$) is actually an intrinsic quantity.

Ex.

$S^n(r) \subset \mathbb{R}^{n+1}$.

$x \in S^n(r) \Rightarrow y(x) = \frac{x}{r} = \frac{1}{r}(x, e_i)$. \hspace{1cm} \text{For } y \in T_x(S^n(r)),

$$
\bar{\nabla}_y y = \nabla_y y = \frac{1}{r} (y(x), e_i) = \frac{y}{r}
$$

$\forall r = 0$

$S(y) = -\frac{y}{r} \Rightarrow II(X, y) = -\frac{1}{r} \langle X, y \rangle$

And $G(X, y) = \frac{1}{r^2} \Rightarrow K = \frac{1}{r^2}$. 

Totally geodesic submanifold:

Def: $M \subset \overline{M}$ is totally geodesic if every geodesic in $M$ is also a geodesic in $\overline{M}$.

Proof: $M$ is totally geodesic $\iff$ $B = 0$ (mean value 2nd fund. form).

RE: Let $\gamma$ be a path in $M$, $\gamma(0) = p$, $\gamma'(0) = v$. Let $N$ be a normal vector field to $M$, $X = \text{exterior of } N$.

$$\langle B(x, x), N \rangle = \langle D_x X - D_x X, N \rangle = \langle D_x X, N \rangle$$

So $B(x, x) = 0 \iff D_x X$ has no normal component

$\iff$ geodesics in $M$ (satisfying $D_x X = 0$)

are also geodesics in $\overline{M}$ (satisfying $D_x X = 0$). $\square$

Eilenberg's original definition of Sectional Curvature:

$S$ is geodesic at $p$ (finite version of totally geodesic)

$\rightarrow$ by above Prop, $B(p) = 0$

$\rightarrow$ Give $S$ the metric induced by $M$; then $R = \overline{R}$

and $K(S) = K(\overline{S}).$

Original sectional curvature of $S$ $\rightarrow$ usual sectional curvature.