Math 621 – Differential Geometry

Note Title 1/11/2017

$\frac{1}{2}$ differential topology: manifolds, vector fields, vector bundles, tensors, diff'l forms

$\frac{1}{2}$ Riemannian geometry: metrics, connections, curvature, geodesics: topics mentioned in qual syllabus
both intrinsic (coord-free) and with coords.

Needs: basic multivar calculus, topology (topl space, subspace topology, covering space, fundamental group)
Analysis: differential; inverse/implicit function thm along the lines of Math 532.

- Text: do Carmo, Riemannian Geometry
  Supplement: Lee Smooth manifolds, Gallot-Hulin-Lafontaine Riemannian Geometry.
- My notes will be posted on course webpage math.duke.edu/~ng/math621/
- Grading based on HW and take-home final.
- Office hrs TBD. New office! 216.

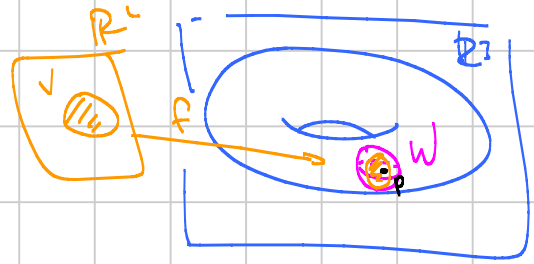
Questions:
- makeup classes on Mondays?
- manifold? tangent space? vector field?
- differential form?
- covering space? fund gp?
- tensor product of vector spaces?

Smooth Manifolds

Ex: surfaces in $\mathbb{R}^3$ or more generally submanifolds of $\mathbb{R}^n$.

Def $M \subset \mathbb{R}^{n+k}$ is a (smooth) <u>submanifold of dimension n</u> if $\forall\, p \in M$, $\exists$ nbd W of p in $\mathbb{R}^{n+k}$, an open set $V \subset \mathbb{R}^n$, and a map $f: V \to \mathbb{R}^{n+k}$ so that

- f is smooth
- $f(V) = M \cap W$ "coord. chart"
- $f|_V$ is a bijection $V \to M \cap W$
- $df(x)$ is injective $\forall\, x \in V$.



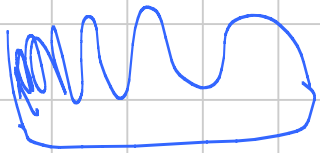
Submfds: $\mathbb{R}^n \subset \mathbb{R}^{n+k}$


$\subset \mathbb{R}^3$: use spherical coords except at N, S.

$(\varphi, \theta) \longmapsto (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$

Not submfds:

-  $\subset \mathbb{R}^2$
- $\{y^2 = x^3\} \subset \mathbb{R}^2$  $t \mapsto (t^3, t^2)$
- image of  $\subset \mathbb{R}^2$

- 
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Prop TFAE.

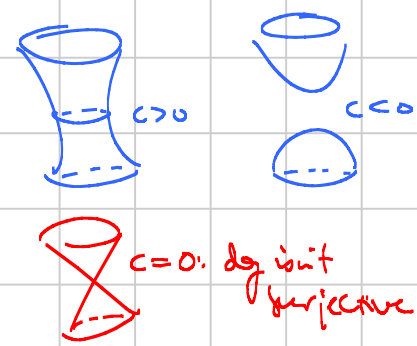
1. M is a smooth submfd of dim n in $\mathbb{R}^{n+k}$
2. $\forall\, p \in M$, $\exists$ nbd W of p in $\mathbb{R}^{n+k}$ and a smooth map $g: W \to \mathbb{R}^k$ such that
   - $W \cap M = g^{-1}(0)$
   - g is a <u>submersion</u>: $dg(x)$ is surjective $\forall\, x \in W$.

$\longrightarrow \mathbb{R}^k$

Pf implicit function theorem.

Ex: $S^n \subset \mathbb{R}^{n+1}$; $\{x^2+y^2-z^2=c\} \subset \mathbb{R}^3$

$\{x_1^2+\cdots+x_{n+1}^2=1\}$ $\qquad g = x^2+y^2-z^2-c$



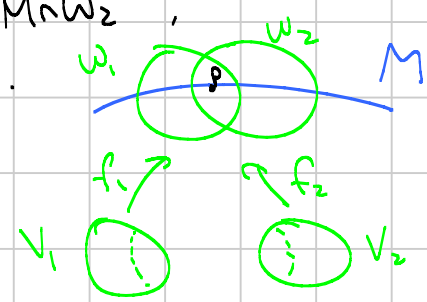
$\{x^3-y^3=0\} \subset \mathbb{R}^2$: $\quad g = x^3-y^3$ doesn't work at $(0,0)$

but this is $= \{x-y=0\}$ which does work.

Prop $M^n \subset \mathbb{R}^{n+k}$ submfd, $p \in M$, two coord charts

$f_1, f_2 : V_1, V_2 \to \mathbb{R}^{n+k}$ mapping onto $M \cap W_1$, $M \cap W_2$,

$\hat{\mathbb{R}}^n$ $\qquad W_1, W_2$ nbds of p.



Then

$$f_2^{-1} \circ f_1 : \underbrace{f_1^{-1}(W_1 \cap W_2)}_{\subset \mathbb{R}^n} \to \underbrace{f_2^{-1}(W_1 \cap W_2)}_{\subset \mathbb{R}^n}$$

is a <u>diffeomorphism</u>: smooth, smooth inverse.

This allows us to dispense with the "ambient space" $\mathbb{R}^{n+k}$.

M = set (!), $n \geq 0$ fixed.

Def A <u>smooth atlas</u> of coord charts on M is a collection

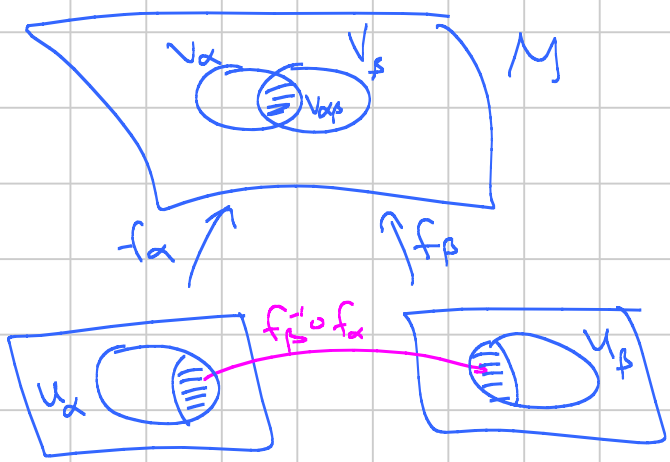
Use capital F for consistency.

$$f_\alpha : \underset{\subset \mathbb{R}^n}{U_\alpha} \longrightarrow V_\alpha \subset M \qquad \text{(coord chart)}$$

(U_α = open set)

such that:
- f_α is a bijection
- V_α cover M: $\bigcup_\alpha V_\alpha = M$
- $\forall \alpha, \beta$ with $\underset{\substack{\| \\ V_{\alpha\beta}}}{V_\alpha \cap V_\beta} \neq \emptyset$,

$$f_\beta^{-1} \circ f_\alpha : \underset{\substack{\cap \\ U_\alpha \\ \cap \\ \mathbb{R}^n}}{f_\alpha^{-1}(V_{\alpha\beta})} \to \underset{\substack{\cap \\ U_\beta \\ \cap \\ \mathbb{R}^n}}{f_\beta^{-1}(V_{\alpha\beta})} \quad \text{is smooth.}$$



Topology on M is induced by atlas:

$V \subset M$ is open $\Leftrightarrow$ $f_\alpha^{-1}(V)$ is open $\forall \alpha$.

Def M is a <u>smooth manifold of dimension n</u> (n-manifold) if it has a smooth atlas, <u>and</u> w.r.t. this topology, M is <u>Hausdorff</u> and <u>second countable</u>.

no

topology has a countable basis.

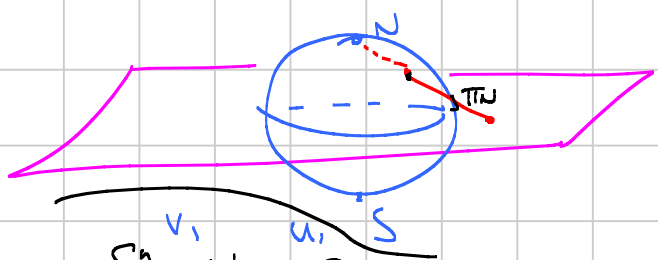
Rmks: · Sometimes useful to consider <u>maximal</u> atlases (not properly contained in any other atlas: any coord chart compatible with the atlas is in the atlas: if $f_\alpha : U_\alpha \to M$ is a chart then $f: U \to M$ is also a chart for $U \subset U_\alpha$, $f = f_\alpha|_U$.)

· intrinsic vs extrinsic: <u>Whitney embedding theorem</u>: any smooth n-mfd M can be smoothly embedded in $\mathbb{R}^{2n}$.

<u>Ex</u> of n-manifolds.

- $\mathbb{R}^n$
- any n-dim submfd of $\mathbb{R}^{n+k}$
- $S^n \subset \mathbb{R}^{n+1}$: here's an atlas.

$N = (0, \dots, 0, 1)$ $\quad S = (0, \dots, 0, -1)$



$\pi_N : \underset{V_1}{S^n \setminus N} \to \underset{U_1}{\mathbb{R}^n}$

$\pi_S : \underset{V_2}{S^n \setminus S} \to \underset{U_2}{\mathbb{R}^n}$

$$\pi_N(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 - x_{n+1}}, \dots, \frac{x_n}{1 - x_{n+1}} \right)$$

$$\pi_S(x_1, \dots, x_{n+1}) = \left(\frac{x_1}{1 + x_{n+1}}, \dots, \frac{x_n}{1 + x_{n+1}} \right).$$

$$\begin{tikzcd}
 & S^n & \\
V_1 \arrow[ur, phantom, "\subset"] \arrow[d, "\pi_N"'] & & V_2 \arrow[ul, phantom, "\subset"] \arrow[d, "\pi_S"] \\
U_1 & & U_2 \\
\pi_N^{-1}(V_1 \cap V_2) = \mathbb{R}^n - 0 \arrow[rr] & & \mathbb{R}^n - 0 = \pi_S^{-1}(V_1 \cap V_2)
\end{tikzcd}$$

this map: $(y_1, \dots, y_n) \mapsto (y_1', \dots, y_n')$

$y_i = \frac{x_i}{1 - x_{n+1}}$, $y_i' = \frac{x_i}{1 + x_{n+1}} \Rightarrow y_i' = \frac{y_i}{y_1^2 + \dots + y_n^2}$

This is a smooth map $\mathbb{R}^n - 0 \to \mathbb{R}^n - 0$.

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· $\underline{\mathbb{RP}^n}$ Points are equiv. classes $\subset \mathbb{R}^{n+1}\setminus 0$

$(x_1,\dots,x_{n+1}) \sim (\lambda x_1,\dots,\lambda x_{n+1})$ $\lambda\neq 0$. Write $[x_1,\dots,x_{n+1}]$ for equiv class.

$1\le i\le n+1$: $V_i = \{[x_1,\dots,x_{n+1}] \mid x_i\neq 0\} \subset \mathbb{RP}^n$

$V_i \longrightarrow U_i = \mathbb{R}^n$ bijective, inverse $f_i : U_i \longrightarrow V_i$

$[x_1,\dots,x_{n+1}] \longmapsto \left(\frac{x_1}{x_i},\dots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\dots,\frac{x_{n+1}}{x_i}\right)$ $\quad (y_1,\dots,y_n)\mapsto(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)$

$i>j$: $f_j^{-1}\circ f_i(y_1,\dots,y_n) = f_j^{-1}(y_1,\dots,y_{i-1},1,y_i,\dots,y_n)$

$$= \left(\frac{y_1}{y_j},\dots,\frac{y_{j-1}}{y_j},\frac{y_{j+1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\dots,\frac{y_n}{y_j}\right).$$

$V_i\cap V_j = \{x_i, x_j\neq 0\}$, $f_i^{-1}(V_i\cap V_j) = \{y_j\neq 0\}$, $f_j^{-1}\circ f_i$ is a map on $\{y_j\neq 0\}$.

Smooth Maps

Def M_1^n, M_2^m smooth mfds. A map $\varphi: M_1\to M_2$ is smooth at $p\in M_1$ if $\exists$ charts $f_1: \underset{\mathbb{R}^n}{\underset{\cap}{U_1}} \to M_1$, $f_2: \underset{\mathbb{R}^m}{\underset{\cap}{U_2}} \to M_2$

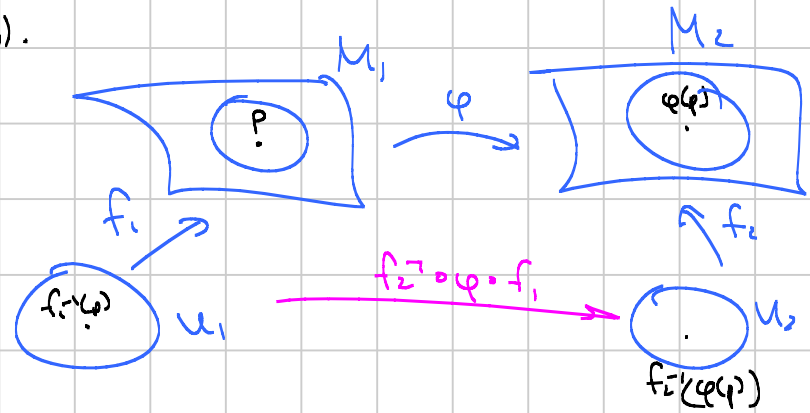
with

$p\in f_1(U_1)$, $\varphi(p)\in f_2(U_2)$, $\varphi(f_1(U_1))\subset f_2(U_2)$,

and

$f_2^{-1}\circ\varphi\circ f_1 : \overset{\subset\mathbb{R}^n}{U_1} \longrightarrow \overset{\subset\mathbb{R}^m}{U_2}$ (or more precisely $\mathrm{nbd}(f_1^{-1}(p))\to\mathrm{nbd}(f_2^{-1}\varphi(p))$)

is smooth at $f_1^{-1}(p)$.



φ is <u>smooth</u> if it's smooth at all points.
Rmk: this is <u>independent</u> of coord charts.

<u>Special cases of smooth maps</u>.

<u>Def</u> $\varphi: M_1 \to M_2$ is an <u>immersion</u> if at any $p \in M_1$, there are coord charts f_1 near p, f_2 near $\varphi(p)$ such that

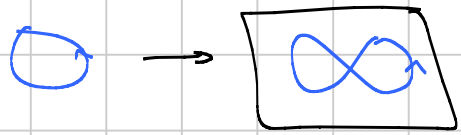
$$f_2^{-1} \circ \varphi \circ f_1 : U_1 \to U_2$$

is an <u>immersion</u> at $f_1^{-1}(p)$: $\quad d(f_2^{-1} \circ \varphi \circ f_1)$ is <u>injective</u>

φ is a <u>submersion</u> ..... <u>submersion</u>: $d(f_2^{-1} \circ \varphi \circ f_1)$ is <u>surjective</u>.

<u>ex</u>: Submersion: $\mathbb{R}^n \to \mathbb{R}^m$ projection, $n \geq m$

immersion:

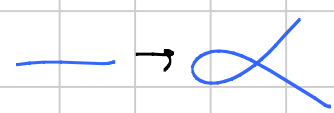


<u>Def</u> cont'd. <u>embedding</u>: immersion and homeomorphism onto its image (with subspace topology); in particular, injective.

<u>diffeomorphism</u>: smooth, smooth inverse
(in particular, immersion and submersion)

<u>Ex</u> cont'd. not immersion: $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^2, t^3)$

immersion, not injective: $\mathbb{R} \to \mathbb{R}^2$



immersion, injective, not embedding:

Rmks. ① Any immersion is locally an embedding:
$\varphi: M_1 \to M_2$ immersion, $p \in M_1 \Rightarrow \exists$ nbd $V \ni p$ with $\varphi|_V : V \to M_2$ embedding.

② A map that's both an immersion and a submersion is a <u>local diffeomorphism</u>: restricted to a nbd of any point, it's a diffeo. (eg. covering maps)

③ if $\varphi: M_1 \to M_2$ is an embedding, then $\varphi(M_1)$ is a <u>submanifold</u> of M_2.

<u>Quotients by group actions</u>

<u>Def</u> A group G <u>acts</u> on a manifold M if each $g \in G$ gives a diffeo. $\varphi_g: M \to M$ st

$$\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2} \qquad (\Rightarrow \varphi_{1_G} = id).$$

G acts <u>properly discontinuously</u> if $\forall p \in M \ \exists$ nbd V of p s.t. $V \cap \varphi_g(V) = \emptyset \ \forall g \neq id$.

G acting on $M \rightsquigarrow$ <u>quotient space</u> M/G of orbits under G $(p \sim \varphi_g(p))$.

<u>Ex</u>. 1(a) $\mathbb{R}^\times$ acts on $\mathbb{R}^{n+1} \setminus \{0\}$ by scalar mult., <u>not</u> properly disc.
$(\mathbb{R}^{n+1} \setminus 0)/\mathbb{R}^\times = \mathbb{RP}^n$.

1(b) $\mathbb{Z}/2 = \{\pm 1\}$ acts on $S^n \subset \mathbb{R}^{n+1}$ by antipodal map.
$S^n/(\mathbb{Z}/2) = \mathbb{RP}^n$.

2. $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by vector addition, $\mathbb{R}^n/\mathbb{Z}^n = T^n = \overbrace{S^1 \times \cdots \times S^1}^{n}$

3. Möbius strip $\mathbb{R} \times (0,1)/\langle g \rangle$ generated by $(x,y) \mapsto (x+1, 1-y)$).

Prop If G acts properly discontinuously on M, then M/G is a smooth mfd and $\pi: M \to M/G$ is a smooth map (in fact, local diffeomorphism). (next page)

Rmk Actually need to either assume M/G is Hausdorff or change the def of properly discontinuous. E.g., add:
$\forall\, p_1, p_2 \in M$ not in the same G-orbit, there are nbhs U_1 of p_1, U_2 of p_2 such that $U_1 \cap \varphi_g(U_2) = \emptyset \quad \forall\, g \in G$.

Otherwise, point-set exercise:

$M = \mathbb{R}^2$, $G = \mathbb{Z}$, $\quad n \cdot (x,y) = (2^n x, 2^{-n} y)$

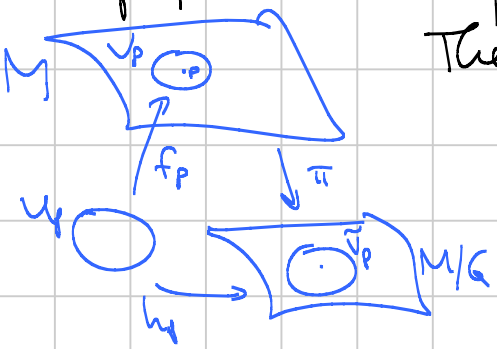
$p_1 = (1,0)$, $p_2 = (0,1)$.

Prop If G acts properly discontinuously on M, then M/G is a smooth mfd and $\pi: M \to M/G$ is a smooth map (in fact, local diffeomorphism).

Pf First construct atlas on M/G.

$p \in M$: choose a coord chart $f_p: U_p \to M$, $f_p(U_p) = V_p \ni p$, such that $V_p \cap \varphi_g(V_p) = \emptyset$ $\forall g \neq id$.

(if V_p doesn't satisfy this, intersect it with V from def.).



Then $\pi|_{V_p}$ is injective so

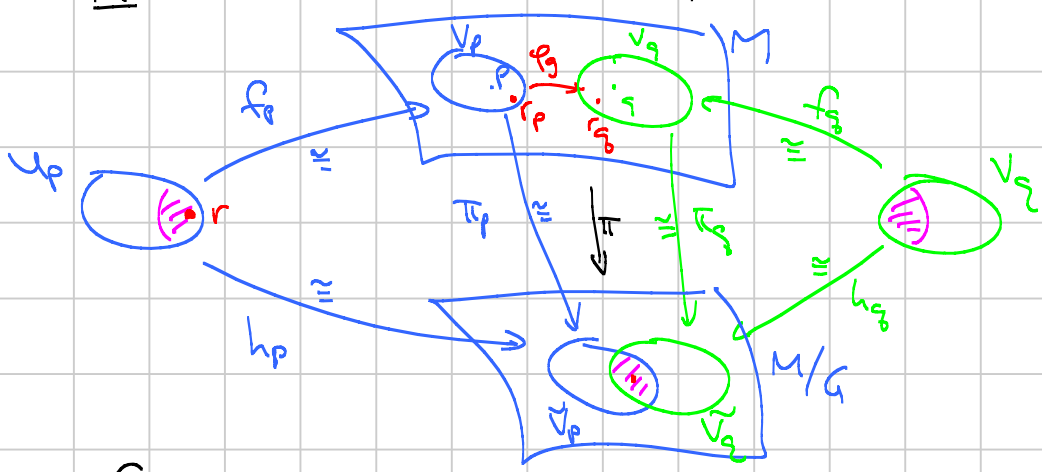
$h_p := \pi \circ f_p : U_p \to M/G$ is injective.

Write $\tilde{V}_p := \pi(V_p) = h_p(U_p)$.

Then $h_p: U_p \to \tilde{V}_p$ is bijective.

Claim $\{(h_p, U_p, \tilde{V}_p)\}$ is an atlas for M/G.

Pf Need to check transition functions.



The following are bijections:

$f_p: U_p \to V_p$ $\qquad$ $f_q: U_q \to V_q$

$\pi_p: V_p \to \tilde{V}_p$ $\qquad$ $\pi_q: V_q \to \tilde{V}_q$

$h_p: U_p \to \tilde{V}_p$ $\qquad$ $h_q: U_q \to \tilde{V}_q$

Near $r \in h_p^{-1}(\tilde{V}_p \cap \tilde{V}_q)$ we have

$$h_q^{-1} \circ h_p = f_q^{-1} \circ \pi_q^{-1} \circ \pi_p \circ f_p .$$

Write $r_p = f_p(r)$, $r_q = \pi_q^{-1}(\pi_p(r_p))$ so $\pi(r_p) = \pi(r_q)$

$\Rightarrow \exists\, g$ with $r_q = \varphi_g(r_p)$.

Near r_p, $\pi = \pi \circ \varphi_g \Rightarrow \pi_p = \pi_q \circ \varphi_g \Rightarrow \pi_q^{-1} \circ \pi_p = \varphi_g$.

So $h_q^{-1} \circ h_p = f_q^{-1} \circ \varphi_g \circ f_p$ is smooth. $\square$

smooth map

coord charts

1/19 (without immersion estz.)

Orientations

$M = n$-mfd, atlas $\{(f_\alpha, U_\alpha, V_\alpha)\}$.

$f_1 : U_1 \to V_1 \subset M$

$f_2 : U_2 \to V_2$

If $V_1 \cap V_2 \neq \emptyset$ then "transition function"

$$f_2^{-1} \circ f_1 : \underset{\mathbb{R}^n}{\underset{\cap}{f_1^{-1}(V_1 \cap V_2)}} \to \underset{\mathbb{R}^n}{\underset{\cap}{f_2^{-1}(V_1 \cap V_2)}}$$

is smooth with smooth inverse.

So for all $x \in f_1^{-1}(V_1 \cap V_2)$,

$d(f_2^{-1} \circ f_1)(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a <u>nonsingular</u> linear map.

$\det\left(d(f_2^{-1} \circ f_1)(x)\right) \neq 0$.

Say f_1, f_2 determine the $\left\{\begin{matrix}\text{same}\\ \text{opposite}\end{matrix}\right\}$ orientation at $p = f_1(x) \in V_1 \cap V_2$

if $\det d(f_2^{-1} \circ f_1)(x) \left\{\begin{matrix} > 0 \\ < 0 \end{matrix}\right\}$.

f_1, f_2 determine $\left\{\begin{matrix}\text{same}\\ \text{opp}\end{matrix}\right\}$ orientation if they determine $\left\{\begin{matrix}\text{same}\\ \text{opp}\end{matrix}\right\}$ orientation at all points: note if $V_1 \cap V_2$ connected then one of these must happen.

An atlas for M is <u>oriented</u> if all coord charts determine the same orientations.

An <u>orientation</u> for M is a choice of oriented atlas, mod saying that 2 atlases are equiv if their union is oriented (i.e. all charts determine same orientation).

<u>Note</u>: Switching orientation. $f_1 : U \to V \subset M$, $h : \mathbb{R}^n \to \mathbb{R}^n$
$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1}, -x_n)$

$\rightsquigarrow$ chart $f_2 = f_1 \circ h : h(U) \to V$ has opposite orientation to f_1:
$f_2^{-1} \circ f_1 : U \to h(U)$ is h, and $\det h = -1$.

<u>Ex</u> S^n. 2 coord charts $f_1, f_2 : \mathbb{R}^n \to S^n$.

Recall $f_2^{-1} \circ f_1 : \mathbb{R}^n - 0 \circlearrowleft$

$$(y_1, \dots, y_n) \mapsto \left(\frac{y_1}{y_1^2 + \dots + y_n^2}, \dots, \frac{y_n}{y_1^2 + \dots + y_n^2} \right).$$

exercise: $\det d(f_2^{-1} \circ f_1) = -\dfrac{1}{\|y\|^{2n}}$.

So an oriented atlas is f_1 and $f_2 \circ h$, or $f_1 \circ h$ and f_2.

<u>Ex</u> Möbius strip $M = [0,1] \times (0,1) \big/ (0,y) \sim (1, 1-y)$

Atlas: $f_1 : (0,1)^2 \to M$, $(x,y) \mapsto (x,y)$ $\qquad$ $f_2 : (0,1)^2 \to M$, $(x,y) \mapsto (x + \frac{1}{2}, y)$ $\qquad$ where $(x,y) \sim (x+1, 1-y)$ if $x < \frac{1}{2}$.

on $(0, \frac{1}{2}) \times (0,1)$, $\quad f_2^{-1} \circ f_1 (x,y) = (x + \frac{1}{2}, 1-y)$

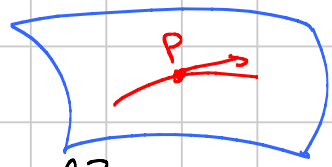
on $(\frac{1}{2}, 1) \times (0,1)$ $\quad f_2^{-1} \circ f_1 (x,y) = (x - \frac{1}{2}, y)$

So these charts agree on one part and disagree on the other.

Tangent Vectors

What's a tangent vector to a mfd M at a point $p \in M$?

If $M \subset \mathbb{R}^{n+k}$:



velocity vector of a curve passing through p.

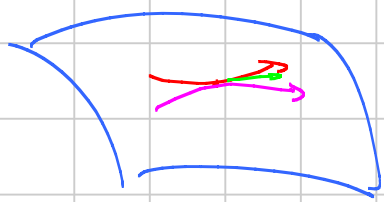
Intrinsically/in general?

APPROACH 1

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve (a smooth map) with $\gamma(0) = p$.

Any such curve gives rise to a tangent vector at p.

When are two curves the same?



these should give same tangent vector.

Use coord chart

$$\begin{tikzcd} & (-\epsilon,\epsilon) \arrow[dl, dashed, "f^{-1}\circ\gamma"'] \arrow[dr, "\gamma"] & \\ \mathbb{R}^n \supset U \arrow[rr, "f"] & & V \subset M \end{tikzcd}$$

Say $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to M$ with $\gamma_1(0) = \gamma_2(0) = p$ are <u>equivalent</u> if

$$\frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_1) = \frac{d}{dt}\Big|_{t=0} (f^{-1} \circ \gamma_2).$$

Need to check indep of coord chart.

$$\begin{tikzcd}
 & (-\epsilon,\epsilon) \arrow[dl, "f_1^{-1}\circ\gamma"'] \arrow[d, "\gamma"] \arrow[dr, "f_2^{-1}\circ\gamma"] & \\
\mathbb{R}^n \supset U_1 \arrow[r, "f_1"'] \arrow[rr, bend right, "f_2^{-1}\circ f_1"'] & V \ni p & U_2 \subset \mathbb{R}^n \arrow[l, "f_2"]
\end{tikzcd}$$

$$\frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma\right) = \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ f_1\circ f_1^{-1}\circ\gamma\right)$$

$$= \underbrace{d\left(f_2^{-1}\circ f_1\right)\left(f_1^{-1}(p)\right)}_{\text{nonsingular } n\times n \text{ matrix}} \cdot \underbrace{\frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma\right)}_{\text{vector in } \mathbb{R}^n}$$

(Chain Rule)

So $\begin{pmatrix} \frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma_1\right) \\ \frac{d}{dt}\Big|_{t=0}\left(f_1^{-1}\circ\gamma_2\right) \end{pmatrix} \xrightarrow{d(f_2^{-1}\circ f_1)(f_1^{-1}(p))} \begin{pmatrix} \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma_1\right) \\ \frac{d}{dt}\Big|_{t=0}\left(f_2^{-1}\circ\gamma_2\right) \end{pmatrix}$

and these are equal $\iff$ these are equal.

<u>Def</u> A <u>tangent vector</u> to M at p is an equivalence class of curves $\gamma: (-\epsilon,\epsilon)\to M$, $\gamma(0)=p$, with the above equiv. relation.

$T_pM = \{\text{tangent vectors to } M \text{ at } p\}$.

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Review: tangent vectors.

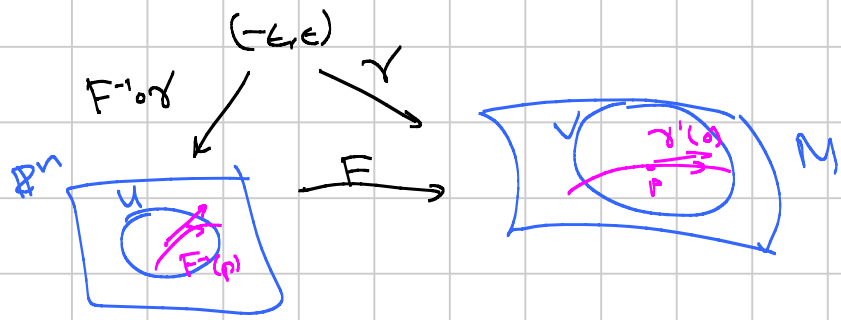
M = smooth mfd, $p \in M$, $T_pM = \{\text{tangent vectors to } p \text{ at } M\}$.

Def: $T_pM = \{\text{Curves in } M \text{ through } p\}/\sim$

$\qquad\qquad\qquad\qquad \gamma : (-\epsilon,\epsilon) \to M, \quad \gamma(0) = p$

Coord chart $\underset{\mathbb{R}^n}{U} \xrightarrow{F} \overset{p}{\underset{M}{V}}$, $\quad \gamma_1 \sim \gamma_2 \iff \frac{d}{dt}\big|_{t=0}(F^{-1}\circ\gamma_1) = \frac{d}{dt}\big|_{t=0}(F^{-1}\circ\gamma_2)$.

Notation: sometimes write $\gamma'(0)$ for $[\gamma] \in T_pM$.



Note we get a map $T_pM \longrightarrow \mathbb{R}^n$

$$[\gamma] \longmapsto \frac{d}{dt}\Big|_{t=0}(F^{-1}\circ\gamma)$$

Well-defined + injective by definition. It's also surjective:
for $w \in \mathbb{R}^n$, define $c : (-\epsilon,\epsilon) \to U$ by $c(t) = F^{-1}(p) + tw$;
if $\gamma = F\circ c$ then $[\gamma] \mapsto w$.

$\Rightarrow$ $T_pM \cong \mathbb{R}^n$; this gives T_pM the structure of a <u>vector space</u>.
Let $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$ denote the basis of T_pM corr. to the standard basis under this isom:

$$\begin{array}{ccc} T_pM & \xrightarrow{\cong} & \mathbb{R}^n \\ v_1\frac{\partial}{\partial x_1} + \dots + v_n\frac{\partial}{\partial x_n} & & (v_1, \dots, v_n) \end{array}.$$

(for now, just formal notation). <u>Note</u> depends on coord chart.

APPROACH 2 (Coord-free)

Write $C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}$. Any curve γ in M through p gives rise to

$$\begin{aligned} C^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto \tfrac{d}{dt}\big|_{t=0}(f\circ\gamma). \end{aligned}$$

Note:
- f only has to be defined in a nbd of p
- only depends on infinitesimal behavior of f near p.

Def M mfd, $p \in M$. The <u>germ of smooth functions</u> at p is

$$C^\infty_p(M) := \{f : V \to \mathbb{R} \mid V \ni p \text{ open in } M\}/\sim$$

$$(f_1 : V_1 \to \mathbb{R}) \sim (f_2 : V_2 \to \mathbb{R}) \iff \exists\, V_3 \subset V_1 \cap V_2 \text{ nbd of } p \text{ with } f_1|_{V_3} = f_2|_{V_3}.$$

Note: $C^\infty_p(M)$ is a <u>vector space</u> over $\mathbb{R}$ (add, scalar mult as usual).

Then γ gives a <u>linear</u> map

$$\begin{aligned} C^\infty_p(M) &\longrightarrow \mathbb{R} \\ [f] &\longmapsto \tfrac{d}{dt}\big|_{t=0}(f\circ\gamma). \end{aligned}$$

This map depends only on $[\gamma]$: choose coord chart $F : U \to V \subset M$

$$\begin{tikzcd} U \arrow[r, "F"] & V \arrow[r, "f"] & \mathbb{R} \\ (-\epsilon,\epsilon) \arrow[u, "F^{-1}\circ\gamma"] \arrow[ur, "\gamma"'] & & \end{tikzcd}$$

$$\begin{aligned} \tfrac{d}{dt}\big|_{t=0}(f\circ\gamma) &= \tfrac{d}{dt}\big|_{t=0}\left(f\circ F\circ F^{-1}\circ\gamma\right) \\ &= d(f\circ F)(F^{-1}(p)) \cdot \underbrace{\tfrac{d}{dt}\big|_{t=0}(F^{-1}\circ\gamma)}_{\text{only depends on } [\gamma]} \end{aligned}$$

So: for $v \in T_pM$, get the <u>directional derivative</u> $v(\cdot) : C^\infty_p(M) \to \mathbb{R}$ (just choose any γ with $[\gamma] = v$).

Def 2 $T'_pM = \{\text{linear maps } C^\infty_p(M) \to \mathbb{R} \text{ that are equal to } v(\cdot) \text{ for some } v \in T_pM\}$.

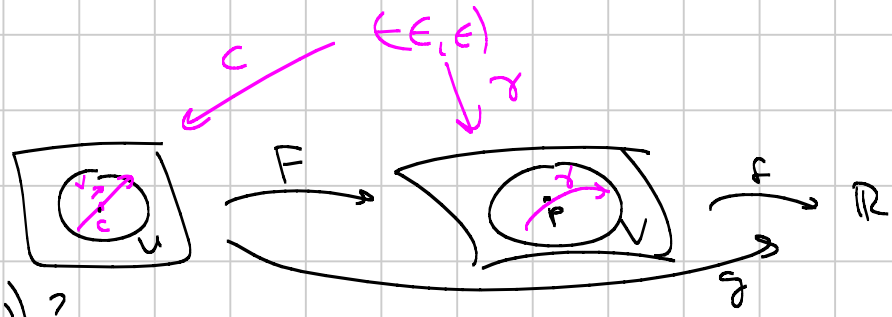
Note there's an obvious map

$$\Phi : T_pM \to T_p'M,$$

surjective by definition.

To prove $T_pM \cong T_p'M$, need Φ injective.

Let $U \xrightarrow{F} V$ be a chart, and let $(v_1, \dots, v_n) \in \mathbb{R}^n$. Define $c(t) = F^{-1}(p) + t(v_1, \dots, v_n)$ and $\gamma(t) = F(c(t))$. Notation from before: $\gamma'(0) = v_1 \frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n}$.



What's $\Phi(\gamma'(0))$?

Let $f \in C_p^\infty(M)$, and write $g = f \circ F : U \to \mathbb{R}$.

$$\frac{d}{dt}\Big|_{t=0}(f \circ \gamma) = \frac{d}{dt}\Big|_{t=0}(f \circ F \circ c) = \frac{d}{dt}\Big|_{t=0}(g \circ c) = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \cdots & \frac{\partial g}{\partial x_n} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \frac{\partial g}{\partial x_1} + \dots + v_n \frac{\partial g}{\partial x_n}$$

So diff values of $(v_1, \dots, v_n)$ give diff. maps $C_p^\infty(M) \to \mathbb{R}$ (plug in $g = x_1, \dots, g = x_n$).

With abuse of notation, identify f with g. Then the directional derivative $v(f) = v_1 \frac{\partial f}{\partial x_1} + \dots + v_n \frac{\partial f}{\partial x_n}$

which explains the notation $v = v_1 \frac{\partial}{\partial x_1} + \dots + v_n \frac{\partial}{\partial x_n}$.

<u>APPROACH 3</u> - most abstract.

Def A <u>derivation</u> at p is a linear map

$$\delta : C_p^\infty(M) \to \mathbb{R}$$

s.t. $\delta(fg) = f(p)\,\delta(g) + \delta(f)\,g(p)$.

note: $v \in T_pM \Rightarrow v(\cdot)$ is a derivation (product rule).

Prop $T_pM \cong \{\text{derivations at } p\}$. ← note: vector space

$v \longleftrightarrow (f \mapsto v(f))$

Pf Injective: easy exercise.

Surjective:

$$\begin{tikzcd} U \arrow[r, "F"] \arrow[dr, "x_i"'] & V \subset M \arrow[d, "x_i \circ F^{-1}"] \\ & \mathbb{R} \end{tikzcd}$$

Say $\delta(x_i \circ F^{-1}) = v_i \in \mathbb{R}$.

<u>Claim</u>: $\delta = v(\cdot)$ where $v = \sum v_i \frac{\partial}{\partial x_i}$.

Note: these agree on $x_i \circ F^{-1}$:

$$\sum v_j \frac{\partial}{\partial x_j}(x_i \circ F^{-1}) = \sum v_j \frac{\partial x_i}{x_j} = v_i.$$

Need: $\delta = v(\cdot)$ on $x_i \circ F^{-1}$ means $\delta(f) = v(f)$ for all f.

Say $f \in C^\infty_p(M)$, $F(0) = p$; write $g = f \circ F : \underset{\mathbb{R}^n}{\text{nbd}(0)} \subset \mathbb{R}$

Lemma Can write $g(x_1, \dots, x_n) = c + \sum x_i g_i(x_1, \dots, x_n)$

for some $c \in \mathbb{R}$ and smooth $g_i : \text{nbd}(0) \to \mathbb{R}$ with $g_i(\vec{0}) = \frac{\partial g}{\partial x_i}(\vec{0})$.

Pf: $g(\vec{x}) = g(\vec{0}) + \int_0^1 \frac{d}{dt} g(t\vec{x})\,dt = g(\vec{0}) + x_i \underbrace{\int_0^1 \frac{\partial g}{\partial x_i}(t\vec{x})\,dt}_{g_i(\vec{x})}$. $\square$

So: $\delta(f) = \delta(c) + \sum \delta\big((x_i \circ F^{-1})(g_i \circ F^{-1})\big)$

$$= 0 + \sum \Big(\underbrace{\delta(x_i \circ F^{-1})}_{v_i} \cdot g_i(0) + \underbrace{(x_i \circ F^{-1}(p))}_{0} \cdot \delta(g_i \circ F^{-1}) \Big)$$

$$= \sum v_i \frac{\partial g}{\partial x_i}(\vec{0})$$

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$$= v(f). \quad \square$$

Differentials of Smooth maps

Prop/Def $\varphi : M_1 \to M_2$ smooth, $p \in M_1$. There is a well-defined linear map

$$d\varphi_p : T_pM_1 \longrightarrow T_{\varphi(p)}M_2 \quad , \quad \text{the } \underline{\text{differential}} \text{ of } \varphi \text{ at } p.$$
$$[\gamma] \longmapsto [\varphi \circ \gamma]$$

Pf

$$\begin{tikzcd}
 & V_1 \subset M_1 \arrow[r, "\varphi"] & V_2 \subset M_2 \\
(-\epsilon, \epsilon) \arrow[ur, "\gamma"] \arrow[dr, "F_1^{-1}\circ\gamma"'] & & \\
 & U_1 \subset \mathbb{R}^n \arrow[uu, "F_1"] \arrow[r] & U_2 \subset \mathbb{R}^m \arrow[uu, "F_2"'] \\
 & (x_1, \dots, x_n) & (y_1, \dots, y_m)
\end{tikzcd}$$

$$F_2^{-1} \circ \varphi \circ F_1 (x_1, \dots, x_n) = (y_1(x_1, \dots, x_n), \dots, y_m(x_1, \dots, x_n))$$

$(F_1^{-1}\circ\gamma)(t) = (x_1(t), \dots, x_n(t)) \longrightarrow$ tangent vector ("$[\gamma]$") is

$$x_1'(0)\frac{\partial}{\partial x_1} + \dots + x_n'(0)\frac{\partial}{\partial x_n}.$$

$$\left(F_2^{-1}\circ(\varphi\circ\gamma)\right)(t) = \left(F_2^{-1}\circ\varphi\circ F_1\right)(F_1^{-1}\circ\gamma)(t)$$
$$= \left(y_1(x_1(t),\dots,x_n(t)), \dots, y_m(x_1(t),\dots,x_n(t))\right).$$

Derivative at $t=0$ is $\Big(\underbrace{\frac{d}{dt}\Big|_{t=0} y_1(x_1(t),\dots,x_n(t))}_{\left(\frac{\partial y_i}{\partial x_j}\right)\cdot\begin{pmatrix} x_1'(0)\\ \vdots \\ x_n'(0)\end{pmatrix}}\Big)\frac{\partial}{\partial y_1} + \dots$

$\left(\frac{\partial y_i}{\partial x_j}\right)$ $m\times n$ matrix

And this depends only on $x_1'(0), \dots, x_n'(0)$, not γ:

$$d\varphi_p\left(v_1\frac{\partial}{\partial x_1} + \dots + v_n\frac{\partial}{\partial x_n}\right) = \sum_i\left(\sum_j \frac{\partial y_i}{\partial x_j}v_j\right)\frac{\partial}{\partial y_i} \quad \left(= \left(\frac{\partial y_i}{\partial x_j}\right)\begin{pmatrix} v_1\\ \vdots \\ v_n\end{pmatrix}\right)$$

linear in $(v_1, \dots, v_n)$. $\square$

Summary: in coords, $d\varphi_p$ is the map $\underset{\cong T_pM_1}{\mathbb{R}^n} \longrightarrow \underset{\cong T_{\varphi(p)}M_2}{\mathbb{R}^m}$ given by the matrix $\left(\frac{\partial y_i}{\partial x_j}\right)$.

<u>Special Case</u>: change of coordinates, $\varphi = id$. $\quad T_pM \xrightarrow{id} T_pM$

$\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \qquad \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$

$$\begin{tikzcd} & M & \\ U_1 \arrow[ur, "f_1"] \arrow[rr] & & U_2 \arrow[ul, "f_2"'] \end{tikzcd}$$

$x_1 \dots x_n \qquad y_1 \dots y_n$

$$\boxed{\frac{\partial}{\partial x_j} = \sum \frac{\partial y_i}{\partial x_j}\frac{\partial}{\partial y_i}.}$$

<u>Thm</u> (Chain Rule)

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi\circ\varphi"'] & N \arrow[r, "\psi"] & P \end{tikzcd} \rightsquigarrow \begin{tikzcd} T_pM \arrow[r, "d\varphi_p"] \arrow[rr, bend right, "d(\psi\circ\varphi)_p"'] & T_{\varphi(p)}N \arrow[r, "d\psi_{\varphi(p)}"] & T_{\psi(\varphi(p))}P \end{tikzcd}$$

$p \in M$

$$\boxed{d(\psi\circ\varphi)_p = d\psi_{\varphi(p)} \circ d\varphi_p}.$$

Rmk $\varphi: M \to N$ diffeomorphism; then $M \xrightarrow{\varphi} N \xrightarrow{\psi} M$ $(\psi = \varphi^{-1})$, $\psi \circ \varphi = \mathrm{id}$

$d(\mathrm{id})_p = \mathrm{id}: T_pM \to T_pM$

So $d\psi_{\varphi(p)}$, $d\varphi_p$ are inverse maps

$\Rightarrow d\varphi_p : T_pM \to T_{\varphi(p)}N$ is an <u>isomorphism</u>.

Conversely:

Prop If $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is an isomorphism then φ is a local diffeomorphism at p.

Pf Inverse function theorem.

$$\varphi \text{ is } \left\{\begin{array}{l}\text{immersion}\\ \text{submersion}\\ \text{local diffeo}\end{array}\right\} \text{ at } p \iff d\varphi_p \text{ is } \left\{\begin{array}{l}\text{injective}\\ \text{surjective}\\ \text{isomorphism}\end{array}\right..$$

Tangent Bundle

M smooth n-mfd. Define $TM := \{(p,v) \mid p \in M, v \in T_pM\}$
$= \coprod_p T_pM$.

This isn't just a set but a smooth $(2n)$-manifold:

$\{(F_\alpha, U_\alpha, V_\alpha)\}$ = atlas for M.

$x_1, \dots, x_n$ coords on U_α. We saw: $p \in U_\alpha \Rightarrow T_pM$ has basis $\left\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right\}$.

Define

$$\tilde{F}_\alpha : \underbrace{U_\alpha \times \mathbb{R}^n}_{\subset \mathbb{R}^{2n}} \longrightarrow TM$$
$$(x_1, \dots, x_n, v_1, \dots, v_n) \longmapsto \left(F_\alpha(x_1, \dots, x_n), \sum v_i \frac{\partial}{\partial x_i}\right).$$

Since $\{V_\alpha\}$ cover M, $\{\tilde{F}_\alpha(U_\alpha \times \mathbb{R}^n)\}$ cover TM.

Need to check: transition functions smooth.

Suppose $V_\alpha \cap V_\beta \neq \emptyset$. Say $(p,v) \in \tilde{F}_\alpha(U_\alpha \times \mathbb{R}^n) \cap \tilde{F}_\beta(U_\beta \times \mathbb{R}^n)$.

Then
$$p = F_\alpha(x_1, \dots, x_n) \quad , \quad v = \sum v_i \frac{\partial}{\partial x_i}$$
$$= F_\beta(y_1, \dots, y_n) \qquad\qquad = \sum w_i \frac{\partial}{\partial y_i} .$$

$$(y_1, \dots, y_n) = (F_\beta^{-1} \circ F_\alpha)(x_1, \dots, x_n) \rightarrow (w_1, \dots, w_n) = d(F_\beta^{-1} \circ F_\alpha)(x_1, \dots, x_n)(v_1, \dots, v_n).$$

$$\rightarrow ((y_1, \dots, y_n), (w_1, \dots, w_n)) = ((F_\beta^{-1} \circ F_\alpha)(x_1, \dots, x_n), d(F_\beta^{-1} \circ F_\alpha)(x_1, \dots, x_n)(v_1, \dots, v_n)).$$

Since $F_\beta^{-1} \circ F_\alpha$ is smooth, so is $d(F_\beta^{-1} \circ F_\alpha)$

so $\tilde{F}_\beta^{-1} \circ \tilde{F}_\alpha$ is a smooth map. $\square$

Vector Fields

Def A <u>vector field</u> X on M is a map $M \longrightarrow TM$ mapping p to some $X(p) \in T_pM$ for all $p \in M$. (i.e. $M \xrightarrow{i_0} TM \xrightarrow{\pi} M$)

usually assume <u>smooth</u>: the map $M \rightarrow TM$ is smooth.

In local coords: can write as
$$X(x_1, \dots, x_n) = X_1(x_1, \dots, x_n)\frac{\partial}{\partial x_1} + \dots + X_n(x_1, \dots, x_n)\frac{\partial}{\partial x_n}.$$
X_i smooth functions.

Notation: write $\mathrm{Vect}(M) = \{\text{smooth vector fields on } M\}$ ← $\mathbb{R}$-vector spaces
$C^\infty(M) = \{\text{smooth functions } M \to \mathbb{R}\}$ ← ring.

tangent vector ⇝ derivation at a pt; vector field ⇝ global derivation

Def A <u>derivation</u> on M is a linear map $\delta : C^\infty(M) \to C^\infty(M)$ such that $f, g \in C^\infty(M) \Rightarrow$

$$\delta(f \cdot g) = f\,\delta(g) + \delta(f)\,g$$

i.e. $\delta(fg)(p) = f(p)\,\delta(g)(p) + \delta(f)(p)\,g(p)$.

$X_p \in T_pM$

$X(f)(p) = X_p(f)$

Prop Any (smooth) vector field X on M gives a derivation $X(\cdot)$ and this gives an isomorphism

$$\mathrm{Vect}(M) \xrightarrow{\cong} \{\text{Derivations on } M\}.$$

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Pf Let $p \in M$, $U \xrightarrow{F} V$ chart near p. For $i = 1, \dots, n$ $\exists$ smooth function $f_i : M \to \mathbb{R}$ s.t. near p, $f_i(x_1, \dots, x_n) = x_i$ (for this, need a bump function).

<u>Injective</u>: if $v = \sum v_i \frac{\partial}{\partial x_i}$ on U then $v(f_i) = v_i$.

<u>Surjective</u>: given δ, if $\delta(f_i)(p) = v_i$ then $\delta(\cdot) = v(\cdot)$ where $v = \sum v_i \frac{\partial}{\partial x_i}$.

So from δ we get a tangent vector at p $\forall p$. (Check: smooth, indep. of chart.) $\square$

Note: This needs the existence of smooth bump functions. The same <u>isn't</u> true for complex manifolds (where "smooth" = holomorphic).

Lie Bracket

$X, Y \in \mathrm{Vect}(M) \rightsquigarrow X(\cdot), Y(\cdot) : C^\infty(M) \to C^\infty(M)$.

Consider the map $f \mapsto X(Y(f)) : C^\infty(M) \to C^\infty(M)$.

This is $\mathbb{R}$-linear; is it a derivation?

$$X(Y(fg)) = X(fY(g) + Y(f)g)$$
$$= fX(Y(g)) + \underbrace{X(f)Y(g) + Y(f)X(g)} + X(Y(f))g$$

No; but

$$f \mapsto X(Y(f)) - Y(X(f)) \quad \underline{\text{is}}.$$

<u>Def</u> The Lie bracket $[X,Y] \in \mathrm{Vect}(M)$ is the vector field corresponding to the derivation $f \mapsto X(Y(f)) - Y(X(f))$.

In local coords: $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_i \frac{\partial}{\partial x_i}$

$$X(Y(f)) = \sum_{i,j} a_j \frac{\partial}{\partial x_j}\left(b_i \frac{\partial f}{\partial x_i}\right) = \sum a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_j b_i \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

$\Rightarrow$

$$[X,Y](f) = \sum_{i,j} a_j \frac{\partial b_i}{\partial x_j}\frac{\partial f}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i}$$

$$\Rightarrow \boxed{[X,Y] = \sum_i \left(\sum_j \left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\right)\frac{\partial}{\partial x_i}}.$$

Properties.

$X, Y \longmapsto [X,Y]$ is:

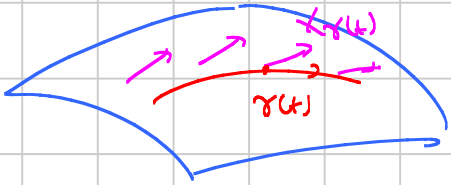
- $\mathbb{R}$-bilinear: $[aX_1 + bX_2, Y] = a[X_1,Y] + b[X_2,Y]$ for $a,b \in \mathbb{R}$.
- antisymmetric: $[Y,X] = -[X,Y]$
- a Lie bracket: satisfies the Jacobi identity
$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

Note: for $g \in C^\infty(M)$ it's not true that $[gX,Y] = g[X,Y]$:

$$\begin{aligned}[gX,Y]f &= gX(Y(f)) - Y((gX)(f)) \\ &= gX(Y(f)) - Y(gX(f)) \\ &= gX(Y(f)) - gY(X(f)) - Y(g)X(f)\end{aligned}$$

So $\boxed{[gX,Y] = g[X,Y] - Y(g)X}$.

Flow of a vector field



For $X \in \mathrm{Vect}(M)$, $\gamma : (a,b) \to M$ is an integral curve for X if $\gamma'(t) = X(\gamma(t)) \in T_{\gamma(t)}M \ \forall\, t \in (a,b)$.

Prop $p \in M$. $\exists\, (a,b)$ containing 0 and unique integral curve $\gamma : (a,b) \to M$ for X with $\gamma(0) = p$.

Pf Local existence/uniqueness for first order ODE. $\square$

Note maximal (a,b) may depend on p: but:

Prop $X \in \mathrm{Vect}(M)$. $\exists$ nbd V of p, interval $(a,b) \ni 0$ s.t. $\forall x \in V$, $\exists!$ integral curve $\gamma_x : (a,b) \to M$ for X with $\gamma(0) = x$, and the map

$$V \times (a,b) \longrightarrow M$$
$$(x, t) \longmapsto \gamma_x(t)$$

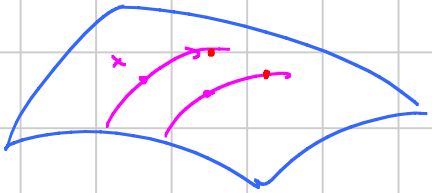
is smooth.

Pf Regularity for ODEs. $\square$

Note: doesn't have to hold on all of M for uniform (a,b).

For $t \in \mathbb{R}$, define local flow of X: $\varphi_t : \text{"}M\text{"} \longrightarrow M$

$$\varphi_t(x) = \gamma_x(t)$$

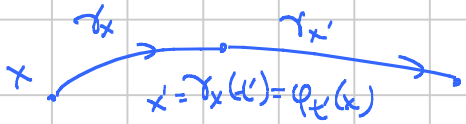


"time t flow under X"

only partially defined for $t \neq 0$.

Where defined: φ_t is a smooth map, and

$$\varphi_t \circ \varphi_{t'} = \varphi_{t+t'}.$$



$\gamma_x(t+t') = \gamma_{x'}(t)$ by uniqueness: these both satisfy $\frac{d}{dt}\gamma(t) = X_{\gamma(t)}$.

Also $\varphi_0 = \mathrm{id}$ so $\varphi_{-t} = (\varphi_t)^{-1}$.

If M is compact, φ_t is well-defined $\forall t$ and is a diffeomorphism.

Recall: given a vector field $X \in \mathrm{Vect}(M)$ $(p \in M \Rightarrow X_p \in T_pM)$,
integral curve $\gamma_x(t)$, $\gamma_x'(t) = X_{\gamma_x(t)}$, $\gamma_x(0) = x$.
$\Rightarrow$ time t flow of X, $\varphi_t : \text{"}M\text{"} \to M$, $\varphi_t(x) = \gamma_x(t)$.

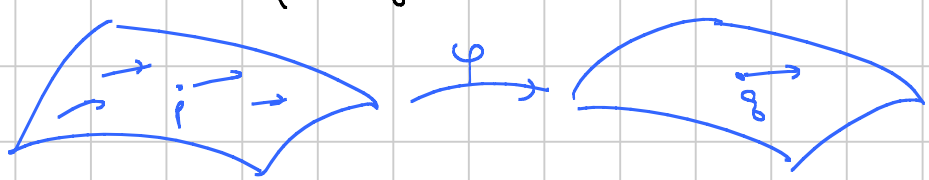
Lie Derivative of a vector field

$X, Y \in \mathrm{Vect}\, M \Rightarrow \mathcal{L}_X Y \in \mathrm{Vect}\, M$

Given diffeo $\varphi : M \to N$ and $X \in \mathrm{Vect}(M)$, define the <u>pushforward</u> $\varphi_* X \in \mathrm{Vect}(N)$ given by

$$(\varphi_* X)_q = (d\varphi)(X_p) \quad \text{where } \varphi(p) = q.$$

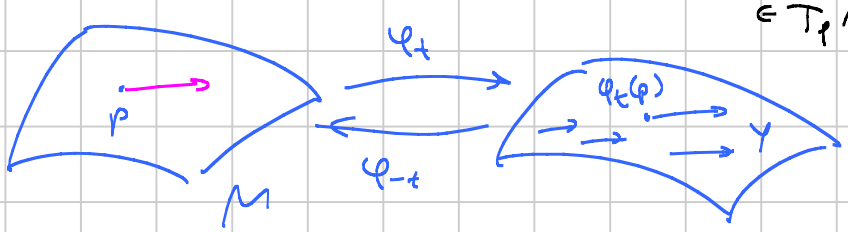
(henceforth we'll sometimes write $d\varphi : T_pM \to T_qN$ as φ_* as well)



For $X, Y \in \mathrm{Vect}\, M$, define $\varphi_t =$ flow of $X : M \to M$
$\Rightarrow (\varphi_{-t})_*(Y) \in \mathrm{Vect}(M)$.

<u>Def</u> $\mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_*(Y)$ i.e.

$$(\mathcal{L}_X Y)_p = \frac{d}{dt}\Big|_{t=0} \underbrace{(\varphi_{-t})_*(Y_{\varphi_t(p)})}_{\in T_pM}$$



<u>Prop</u> $\mathcal{L}_X Y = [X, Y]$.

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Lemma (similar to last time) $V = \text{nbd}(p)$, $h : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ smooth, $h(0,x) = 0 \ \forall x$. Then $\exists$ smooth $g : (-\epsilon,\epsilon)\times V \to \mathbb{R}$ with

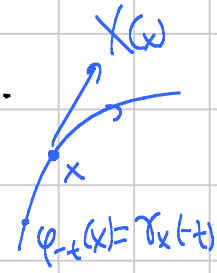
- $h(t,x) = t\,g(t,x)$
- $g(0,x) = \frac{\partial h}{\partial t}(0,x)$.

Pf $h(t,x) = \int_0^1 \frac{\partial}{\partial s} h(st,x)\,ds = t \underbrace{\int_0^1 \frac{\partial h}{\partial t}(st,x)\,ds}_{g(t,x)}$. $\square$

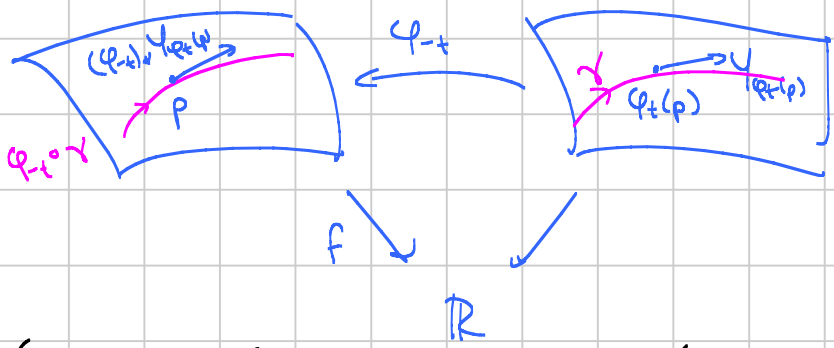
Pf of Prop $f \in C^\infty(M)$: want $[X,Y]f = (\mathcal{L}_X Y)(f)$.

Apply Lemma to $h(t,x) = f(\varphi_t(x)) - f(x)$:

$h(t,x) = t\,g(t,x)$, $\quad g(0,x) = \frac{\partial h}{\partial t}(0,x) = -Xf(x)$.



Then



define $\gamma(s)$ to be the integral curve for Y with $\gamma(0) = \varphi_t(p)$

$$\left((\varphi_{-t})_* Y_{\varphi_t(p)}\right)(f) = \frac{d}{ds}\Big|_{s=0} f(\varphi_{-t}\circ\gamma(s)) = \frac{d}{ds}\Big|_{s=0} (f\circ\varphi_{-t})(\gamma(s))$$

$$= Y_{\varphi_t(p)}(f\circ\varphi_{-t}) = Y_{\varphi_t(p)}(f + tg)$$

$$= (Yf)_{\varphi_t(p)} + t(Yg)_{\varphi_t(p)}$$

$$\Rightarrow (\mathcal{L}_X Y)(f)(p) = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* Y_{\varphi_t(p)}(f)$$

$$= \frac{d}{dt}\Big|_{t=0} (Yf)_{\varphi_t(p)} + (Yg)_p$$

$$= X(Yf)(p) - Y(Xf)(p)$$

$$= [X,Y]_p f. \qquad \square$$

Important example: Lie groups

Def A <u>Lie group</u> is a group G with the structure of a smooth mfd such that the maps:

- left mult $L_h : G \to G$, $g \mapsto hg$
- right mult $R_h : G \to G$, $g \mapsto gh$
- inverse $\mathrm{inv} : G \to G$, $g \mapsto g^{-1}$

are smooth.

ex: $\mathbb{R}^n$; quotients like $\mathbb{R}^n/\mathbb{Z}^n = T^n$;

matrix groups
- $GL(n,\mathbb{R})$ = open subset of $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
- $O(n) = \{M^T M = I\}$, $SO(n)$
- $GL(n,\mathbb{C})$, $U(n)$, $SU(n)$

$h \in G \Rightarrow L_h, R_h$ induce maps $TG \underset{(R_h)_* = dR_h}{\overset{(L_h)_* = dL_h}{\rightrightarrows}} TG$

Def $X \in \mathrm{Vect}\, G$ is <u>left/right invariant</u> if $(L_h)_* X = X$, $(R_h)_* X = X$ $\forall h \in G$.

X left invariant v.f. is <u>determined</u> by $X_e \in T_e G$: (e = identity in G)

$L_g : G \to G$, $e \mapsto g$ satisfies $(L_g)_*(X_e) = X_g$. $\forall g$

Conversely any $X_e \in T_e G$ gives rise to a left inv't v.f. X defined by

Check: $(L_h)_*(X_g) \overset{?}{=} X_{L_h g}$ ✓

$\overset{\shortparallel}{(L_h)_*(L_g)_* X_e} = (L_{hg})_* X_e = \overset{\shortparallel}{X_{hg}}$

So $\{\text{left inv't v.f.}\} \overset{1-1}{\longleftrightarrow} T_e G$.

Write $\boxed{\mathfrak{g} := T_e G}$ Lie algebra assoc to G.

Prop X, Y left inv't. Then $[X,Y]$ is as well.

Pf. From HW: $\varphi: M \to M$ diffeo $\Rightarrow [\varphi_* X, \varphi_* Y] = \varphi_*[X,Y]$.
Here $(L_h)_*[X,Y] = [(L_h)_* X, (L_h)_* Y] = [X,Y]$. $\square$

So bracket on vector fields induces $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.
This is $\mathbb{R}$-bilinear, antisymmetric, and satisfies Jacobi
$\Rightarrow$ gives $\mathfrak{g}$ the structure of a <u>Lie algebra</u>.

Next define $\psi_h = R_{h^{-1}} L_h : G \to G$: $\psi_h(g) = hgh^{-1}$.
Since $\psi_h(e) = e$, this give a map

$$(\psi_h)_* : T_eG \to T_eG$$
$$\text{!!}$$

adjoint representation $\quad \mathrm{Ad}\, h : \mathfrak{g} \to \mathfrak{g}$

<u>notes</u>:
- if X is left inv't then $(\psi_h)_* X = (R_{h^{-1}})_* (L_h)_* X = (R_{h^{-1}})_* X$
- $\mathrm{Ad}(h) : \mathfrak{g} \to \mathfrak{g}$ is a linear, Lie algebra map:
$$\mathrm{Ad}(h)[X,Y] = [\mathrm{Ad}(h)X, \mathrm{Ad}(h)Y]$$
- Ad is a representation: $\mathrm{Ad}(h_1 h_2) = \mathrm{Ad}(h_1)\mathrm{Ad}(h_2)$.

Prop $X, Y \in \mathfrak{g}$, φ_t = local flow of (left inv't v.f.) X.
Then
$$[X,Y] = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\varphi_t(e))\, Y.$$

Pf $[X,Y] = \mathcal{L}_X Y = \frac{d}{dt}\Big|_{t=0} (\varphi_{-t})_* Y$.

From HW: $\varphi_{-t}(g) = g\,\varphi_{-t}(e) = R_{\varphi_{-t}(e)}(g) \Rightarrow \varphi_{-t} = R_{\varphi_{-t}(e)}$

$$\Rightarrow [X,Y] = \frac{d}{dt}\Big|_{t=0} \underbrace{(R_{\varphi_{-t}(e)})_*}_{(\varphi_t(e))^{-1} \text{ from HW}} Y = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\varphi_t(e))\, Y. \quad \square$$

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Vector Bundles

Idea: generalize tangent bundle $TM = \coprod_{x\in M} T_xM \xrightarrow{\pi} M$.

Chart $U \xrightarrow{F} V$ for M gives a chart for TM:

$$\begin{tikzcd} & V\times\mathbb{R}^n \\ U\times\mathbb{R}^n \arrow[ur, "F\times\mathrm{id}", "\cong"'] \arrow[r, "\tilde{F}", "\cong"'] & \pi^{-1}(V) \subset TM \arrow[ul, dotted, "\cong"'] \end{tikzcd}$$

$$(x_1,\dots,x_n,v_1,\dots,v_n) \longmapsto \left(F(x_1,\dots,x_n),\ \sum v_i \tfrac{\partial}{\partial x_i}\right)$$

In particular we get $\{x\}\times\mathbb{R}^n \xleftarrow{\cong} T_xM$

If we have two charts V_1, V_2 and $x \in V_1 \cap V_2$ then we get maps

$$\begin{tikzcd} \{x\}\times\mathbb{R}^n \arrow[rr, bend left] & T_xM \arrow[l] \arrow[r] & \{x\}\times\mathbb{R}^n \end{tikzcd}$$

and the induced map $\mathbb{R}^n \to \mathbb{R}^n$ is <u>linear</u>: given by the Jacobian matrix $\left(\frac{\partial y_i}{\partial x_j}\right) =: \left(\frac{\partial y}{\partial x}\right)$ $\left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \leftarrow \sum_j v_j \frac{\partial}{\partial x_j} = \sum_{i,j} v_j \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} \longrightarrow \left(\frac{\partial y_i}{\partial x_j}\right)\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right)$

<u>Def</u> M smooth mfd. A <u>rank k (real) vector bundle</u> over M is a smooth mfd $E = \coprod_{x\in M} E_x$ where each E_x = rank k vector space $/\mathbb{R}$, such that:

1. the map $\pi: E \to M$ sending E_x to x is smooth
2. $\exists$ open cover $\{V_\alpha\}$ of M and diffeos

$$\varphi_\alpha : \pi^{-1}(V_\alpha) \xrightarrow{\cong} V_\alpha \times \mathbb{R}^k \qquad \text{"local trivialization" of } E$$

such that $\varphi_\alpha(E_x) = \{x\} \times \mathbb{R}^k$: i.e.

$$\begin{tikzcd} \pi^{-1}(V_\alpha) \arrow[rr, "\varphi_\alpha"] \arrow[dr, "\pi"'] & & V_\alpha \times \mathbb{R}^k \arrow[dl, "\pi"] \\ & V_\alpha & \end{tikzcd}$$

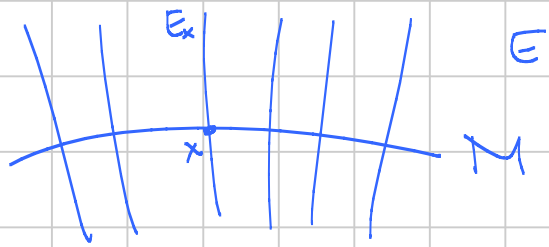
commutes

3. the transition functions

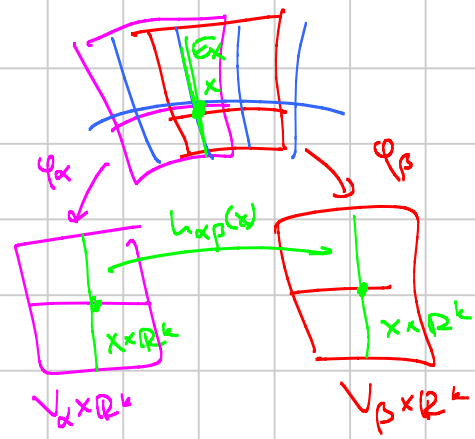
$$h_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : (V_\alpha \cap V_\beta) \times \mathbb{R}^k \to (V_\alpha \cap V_\beta) \times \mathbb{R}^k$$

are smooth and <u>linear</u> : that is, for $x \in V_\alpha \cap V_\beta$,

$$h_{\alpha\beta}(x) : \underset{\mathbb{R}^k}{\underset{\parallel}{\{x\} \times \mathbb{R}^k}} \to \underset{\mathbb{R}^k}{\underset{\parallel}{\{x\} \times \mathbb{R}^k}} \quad \text{is in } GL(k).$$



(NB different from a fiber bundle with fiber $\mathbb{R}^k$: then trans. fns. would be in $\mathrm{Diff}(\mathbb{R}^k)$)



<u>Rmks.</u> 1. Write $\mathbb{R}^k \to E \downarrow M$ or just $E \downarrow M$

2. if $k=1$ this is a <u>line bundle</u> over M.

3. <u>importantly</u>: can reconstruct E from the transition functions.

$$E = \coprod_\alpha (V_\alpha \times \mathbb{R}^k)/\sim \quad \text{where}$$

$$(x,v) \in V_\alpha \times \mathbb{R}^k \sim (y,w) \in V_\beta \times \mathbb{R}^k$$

if $x = y$ and $w = h_{\alpha\beta}(v)$.

<u>Ex.</u> 1. $M \times \mathbb{R}^k$ "trivial" vector bundle

2. TM: transition fns look like matrix $\left(\frac{\partial y}{\partial x}\right)$.

Def $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ $\begin{matrix} E' \\ \downarrow \\ M \end{matrix}$ vector bundles. A smooth map $\varphi: E \to E'$ is a <u>bundle map</u> if

$$\begin{tikzcd} E \arrow[rr, "\varphi"] \arrow[dr] & & E' \arrow[dl] \\ & M & \end{tikzcd}$$ commutes

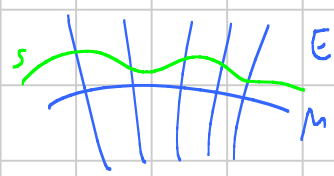
and $\forall x \in M$, $\varphi|_{E_x}: E_x \to E'_x$ is a <u>linear</u> map.

A <u>bundle isomorphism</u> is a bundle map that's invertible, with inverse = bundle map.
A vector bundle is <u>trivial</u> if it's isomorphic to $M \times \mathbb{R}^k$.

Most things associated to a vector bundle are "invariant" under isom. Eg:

Def A <u>section</u> of a vector bundle $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ is a smooth map $s: M \to E$ with $\pi \circ s = id$.

$\Gamma(E) :=$ vector <u>space</u> of sections of E.



$\to$ rmk: if $E \cong E'$ then $\Gamma(E) \cong \Gamma(E')$.

<u>Ex</u> · Sections of trivial $\mathbb{R}^k$ bundle $= \{$smooth maps $M \to \mathbb{R}^k\}$
· $\Gamma(TM) = \mathrm{Vect}(M)$.

<u>Operations on Vector Bundles</u>

<u>Dual</u>: $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ v.b. $\Rightarrow$ replace each E_x by E_x^* (note $\cong$ but not canonical)

$V_1 \xrightarrow{h_{\alpha\beta}} V_2$ dualizes to $$\begin{tikzcd} V_1^* \arrow[r, bend left, "(h_{\alpha\beta}^T)^{-1}"] & V_2^* \arrow[l, bend left, "(h_{\alpha\beta})^T"] \end{tikzcd}$$

Def $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ v.b., transition fns $h_{\alpha\beta}$ $\Rightarrow$ dual $\begin{matrix} E^* \\ \downarrow \\ M \end{matrix}$ v.b. with transition fns $(h_{\alpha\beta}^T)^{-1}$.

Ex: Cotangent bundle T^*M is the dual to TM.

$T_xM = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$ vector space

$T_x^*M = \langle dx_1, \dots, dx_n \rangle$ dual basis: $dx_i\left(\frac{\partial}{\partial x_j}\right) = \delta_{ij}$.

element of T_x^*M is a cotangent vector.

Two coord charts x_i, y_i $\Rightarrow$ recall $\frac{\partial}{\partial y_i} = \sum_j \frac{\partial x_j}{\partial y_i} \frac{\partial}{\partial x_j}$ (this swaps x, y from before)

$\Rightarrow dx_j\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial x_j}{\partial y_i} \Rightarrow dx_j = \sum_j \frac{\partial x_j}{\partial y_i} dy_i$

So the transition fn. $\{x\} \times \mathbb{R}^n \longleftarrow T_x^*M \longrightarrow \{x\} \times \mathbb{R}^n$ is give by the matrix

$(v_1, \dots, v_n) \to \sum v_i dx_i \qquad \sum v_i dy_i \leftarrow (v_1, \dots, v_n)$

$$\left(\frac{\partial x_j}{\partial y_i}\right) = \left(\frac{\partial x}{\partial y}\right)^T = \left(\left(\frac{\partial y}{\partial x}\right)^T\right)^{-1}.$$

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Other operations on vector spaces/bundles.

$\oplus$: $\begin{matrix} E \\ \downarrow \\ M \\ \text{rank } k \end{matrix} \quad \begin{matrix} F \\ \downarrow \\ M \\ \text{rank } l \end{matrix} \rightsquigarrow \begin{matrix} E \oplus F \\ \downarrow \\ M \\ \text{rank } k+l \end{matrix} \qquad (E \oplus F)_x = E_x \oplus F_x.$

transition fns. $h_{\alpha\beta}(x) \in GL(k)$ for E, $j_{\alpha\beta}(x) \in GL(l)$ for F

$\Rightarrow h_{\alpha\beta}(x) \oplus j_{\alpha\beta}(x) \in GL(k+l)$ for $E \oplus F$

$\otimes$: $\begin{matrix} E \otimes F \\ \downarrow \\ M \end{matrix}$ rank kl, $(E \otimes F)_x = E_x \otimes F_x$

transition fns $h_{\alpha\beta}(x) \otimes j_{\alpha\beta}(x) \in GL(kl)$ for $E \otimes F$.

Sym^m: $E \rightsquigarrow \mathrm{Sym}^m E$. $\quad (\mathrm{Sym}^m E)_x = \otimes^m E_x / I$

$I = \langle \cdots \otimes v \otimes w \otimes \cdots - \cdots \otimes w \otimes v \otimes \cdots \rangle$

rank $\binom{k+m-1}{m}$ vector bundle

Λ^m: $E \rightsquigarrow \Lambda^m E \qquad (\Lambda^m E)_x = \otimes^m E_x / I'$

$I' = \langle \cdots \otimes v \otimes w \otimes \cdots + \cdots \otimes w \otimes v \otimes \cdots, \cdots \otimes v \otimes v \otimes \cdots \rangle$

In this quotient, usually write $\otimes$ as $\wedge$. So $v\wedge w = -w\wedge v$, $v\wedge v = 0$.

$\Lambda^m E$ has rank $\binom{k}{m}$.

$$\underset{\text{trivial } \mathbb{R}\text{-bundle}}{\Lambda^0 E} \quad \underset{E}{\Lambda^1 E} \quad \Lambda^2 E \quad \cdots \quad \Lambda^k E$$

rank 1: line bundles.

Bundles from TM: consider the bundle

$$\underbrace{TM\otimes\cdots\otimes TM}_{p}\otimes\underbrace{T^*M\otimes\cdots\otimes T^*M}_{q} =: T^p_q M.$$

A section of this bundle is called a <u>(p,q)-tensor</u>.

- $(1,0)$ tensor: <u>vector field</u> in $\mathrm{Vect}(M) = \Gamma(TM)$
- $(0,1)$ tensor: <u>1-form</u>, in $\Omega^1(M) := \Gamma(T^*M)$.
- $(0,2)$ tensor: section of $T^*M\otimes T^*M$.

  At x, this is an elt of $T_x^*M\otimes T_x^*M$, i.e. a bilinear map

  $T_xM\otimes T_xM \to \mathbb{R}$ $\qquad (V^*\otimes V^* = (V\otimes V)^*)$

  important future example: <u>Riemannian metric</u>.
- $(0,m)$ tensor: <u>differential forms</u> are important examples. (we'll treat soon).

<u>Operations on tensors</u>

<u>Contraction</u>:

$$C_{ij}: \Gamma(T^p_q M) \to \Gamma(T^{p-1}_{q-1} M) \qquad 1\le i\le p,\ 1\le j\le q$$

This is defined fiberwise by the map

$$C_{ij}: V^{\otimes p}\otimes (V^*)^{\otimes q} \to V^{\otimes(p-1)}\otimes (V^*)^{\otimes(q-1)}$$

$$v_1\otimes\cdots\otimes v_p\otimes w_1\otimes\cdots\otimes w_q \longmapsto w_j(v_i)\ v_1\otimes\cdots\hat{v}_i\cdots\otimes w_1\otimes\cdots\hat{w}_j\cdots$$

<u>ex</u>: $p=q=1$. $\quad C: V\otimes V^* \to \mathbb{R}$

This is the <u>trace</u> $\quad \mathrm{tr}: \mathrm{End}(V)\to\mathbb{R}$.

Pullback $\quad M \xrightarrow{\varphi} N \quad \rightsquigarrow \quad T_xM \xrightarrow{\varphi_*} T_yN$

$x \qquad \varphi(x)=y \qquad\qquad$ (linear map)

Dualize $\rightsquigarrow \quad T_y^*N \xrightarrow{\varphi^*} T_x^*M$

$\alpha \longmapsto \varphi^*(\alpha) \qquad\qquad (\varphi^*\alpha)(v) = \alpha(\varphi_* v)$

This yields a map $\quad \varphi^* : \underset{\Omega^1 N}{\Gamma(T^*N)} \longrightarrow \underset{\Omega^1 M}{\Gamma(T^*M)}$

More generally, φ^* gives a map $\underset{(0,k)\text{ tensors on } N}{\Gamma(T^0_k N)} \longrightarrow \underset{(0,k)\text{ tensors on } M}{\Gamma(T^0_k M)}$.

$\alpha \in \Gamma(\otimes^k T^*N)$ means α eats k vectors $w_1, \dots, w_k \in T_yN$: $\alpha(w_1, \dots, w_k) \in \mathbb{R}$.

$$\varphi^*\alpha(\underbrace{v_1, \dots, v_k}_{\in T_xM}) = \alpha(\varphi_* v_1, \dots, \varphi_* v_k)$$

Note this is only a map of $(0,k)$ tensors: recall $\varphi : M \to N$ does not give a map $\mathrm{Vect}(M) \to \mathrm{Vect}(N)$ (or vice versa).

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But: if φ is a diffeo, we can define "pullback"

$$\varphi^* : \Gamma(TN) \longrightarrow \Gamma(TM)$$
$$X \longmapsto (\varphi^{-1})_* X$$

We can extend this to a pullback for any tensors.

$$\varphi^* : \Gamma(T^p_q N) \longrightarrow \Gamma(T^p_q M)$$
$$v_1 \otimes \cdots \otimes w_1 \otimes \cdots \longmapsto \varphi^*(v_1) \otimes \cdots \otimes \varphi^*(w_1) \otimes \cdots$$

In particular, suppose $X \in \mathrm{Vect}(M) \rightsquigarrow \varphi_t$ = time t flow of X.

Def The Lie derivative associated to X is the linear map

$$\mathcal{L}_X : \Gamma(T^p_q M) \to \Gamma(T^p_q M)$$
$$\mathcal{L}_X(S) = \frac{d}{dt}\Big|_{t=0} \varphi_t^*(S).$$

Ex: $Y \in \mathrm{Vect}(M) \Rightarrow \mathcal{L}_X Y = [X, Y]$. $\qquad$ (note here $\varphi_t^* = (\varphi_{-t})_*$)

Local Operators and Tensors

recall section $s \in \Gamma(E)$ is a map $x \in M \mapsto s_x \in E_x$.

Def A <u>local operator</u> is a linear map $P: \Gamma(E) \to \Gamma(F)$, E, F vector bundles over M, such that $\forall x \in M$ and $\forall U =$ nbd of x, if $s, s' \in \Gamma(E)$ satisfy $s_y = s'_y$ $\forall y \in U$, then $(Ps)_x = (Ps')_x$.

"at a point, the operator depends only on the section near that point"

<u>ex</u>: $X \in \mathrm{Vect}(M)$, $\Gamma(T^p_q M) \to \Gamma(T^p_q M)$ is local.
$$s \mapsto \mathcal{L}_X s$$

Special case of local operator: $P: \Gamma(E) \to \Gamma(F)$ such that if $s_x = s'_x$ then $(Ps)_x = (Ps')_x$.

Then P induces a map $E_x \to F_x$ ($\Leftrightarrow$ elt of $\mathrm{Hom}(E_x, F_x) = E_x^* \otimes F_x$) $\forall x$

$\Rightarrow$ section of $E^* \otimes F$. Call P a <u>tensor</u>. Why?

In particular, suppose $E = T^p_q M$, $F = T^r_\ell M$. Such a map P is a section of $(\otimes^p TM \otimes \otimes^q T^*M)^* \otimes (\otimes^r TM \otimes \otimes^\ell T^*M)$
$$\cong \otimes^{q+r} TM \otimes \otimes^{p+\ell} T^*M = T^{q+r}_{p+\ell} M.$$

So P itself is a tensor.

<u>ex</u>: • $\mathrm{tr}: \Gamma(T^1_1 M) \to \mathbb{R}$ (or contractions in general $\Gamma(T^p_q M) \to \Gamma(T^{p-1}_{q-1} M)$)
$(\mathrm{tr}\, s)_x \in \mathbb{R}$ only depends on s_x : tensor.

• $\mathcal{L}_X: T^p_q M \to T^p_q M$ <u>not a tensor</u>.
e.g. $\mathcal{L}_X: \mathrm{Vect}(M) \to \mathrm{Vect}(M)$ $(\mathcal{L}_X Y)_p$ depends on more than Y_p.

Useful characterization of tensors: note $C^\infty(M)$ acts on $\Gamma(E)$ by pointwise scalar multiplication: $(fs)_x = f(x)\, s_x \in E_x$.

<u>Prop</u> $\begin{matrix} E \\ \downarrow \\ M \end{matrix}$ $\begin{matrix} F \\ \downarrow \\ M \end{matrix}$ vector bundle, $P: \Gamma(E) \to \Gamma(F)$ local operator. TFAE:

1. P is a tensor: if $s_x = s'_x$ then $(Ps)_x = (Ps)_{x'}$.
2. P is $C^\infty(M)$-linear: $P(fs) = f\,P(s)$ $\forall$ $f \in C^\infty(M)$.

<u>Pf</u> <u>1 ⇒ 2</u>: Given $f \in C^\infty(M)$, $s \in \Gamma(E)$, define section $s' \in \Gamma(E)$ by $s'_y = f(x)\, s_y$. Then $s'_x = (fs)_x$ so

$$(P(fs))_x = (P(s'))_x = \underset{\uparrow \text{ constant}}{f(x)}\,(P(s))_x .$$

<u>2 ⇒ 1</u>: for $x \in M$, E is "locally trivial": $\exists$ nbd U of x st. $\pi^{-1}(U) = U \times \mathbb{R}^N$ $\quad (E \xrightarrow{\pi} M)$.

Over U, $\exists$ sections $s_1, \dots, s_N$ of E such that $s_1, \dots, s_N$ generate E pointwise. Now suppose $s, s' \in \Gamma(E)$ with $s_x = s'_x$.

Write $\quad s - s' = \sum_{i=1}^N f_i s_i \quad , \; f_i \in C^\infty(U) \; , \; f_i(x) = 0 \; \forall\, i$.

Then $\quad P(s-s') = P(\sum f_i s_i) = \sum f_i P(s_i)$

So $\quad P(s-s')_x = \sum f_i(x)\, P(s_i)_x = 0$. $\square$

<u>Ex</u>. $\mathcal{L}_X : \mathrm{Vect}(M) \to \mathrm{Vect}(M)$.

$$\mathcal{L}_X(fY) = [X, fY] \underset{\uparrow}{=} f[X,Y] + X(f)\,Y \neq f[X,Y] = f\,\mathcal{L}_X Y$$

$$[X, fY]g = X(fY(g)) - fY(X(g)) = X(f)Y(g) + f\,X(Y(g)) - f\,Y(X(g))$$
$$= (X(f)Y + f[X,Y])(g)$$

not a tensor!

Ex Given $\alpha \in \Omega^1 M := \Gamma(T^0_1 M)$, note this gives a map $\alpha(\cdot) : \mathrm{Vect}(X) \to C^\infty(X)$ (a tensor). Now define a map

$$d\alpha : \mathrm{Vect}\, M \otimes \mathrm{Vect}\, M \longrightarrow C^\infty(M)$$

by

$$d\alpha(X, Y) = X\alpha(Y) - Y\alpha(X) - \alpha([X,Y]).$$

This is a tensor in each input:

$$d\alpha(X, fY) = X\alpha(fY) - fY\alpha(X) - \alpha([X, fY])$$

$$= \underbrace{X(f\alpha(Y))}_{\cancel{X(f)\alpha(Y)} + f X\alpha(Y)} - f\, Y\alpha(X) - \underbrace{\alpha(f[X,Y] + (Xf)Y)}_{f\alpha[X,Y] + \cancel{(Xf)\alpha(Y)}}$$

$$= f\, d\alpha(X, Y).$$

So in fact $(d\alpha(X,Y))_x$ depends only on X_x and Y_x.

Over x this gives a map $T_xX \otimes T_xX \to \mathbb{R}$, so $d\alpha \in \Gamma(T^0_2 M)$.

$\leftrightarrow$ elt of $(T^*_x X)^{\otimes 2}$

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Differential Forms

First: some linear algebra.

$V = $ v.s$/\mathbb{R}$. A <u>k-multilinear form</u> on V is a map

$$\varphi : \overbrace{V \otimes \cdots \otimes V}^{k} \longrightarrow \mathbb{R}$$

that is linear in each input.

$$v_1 \otimes \cdots \otimes v_k \longmapsto \varphi(v_1, \ldots, v_k)$$

(k=2: <u>bilinear</u>)

Note $\{k\text{-multilinear forms}\} \cong (\underbrace{V \otimes \cdots \otimes V}_{k})^* \cong \underbrace{V^* \otimes \cdots \otimes V^*}_{k}$

under this isom, if $\varphi_1, \ldots, \varphi_k \in V^*$ then $(\varphi_1 \otimes \cdots \otimes \varphi_k)(v_1, \ldots, v_k) = \varphi_1(v_1) \cdots \varphi_k(v_k)$.

Some forms are antisymmetric: $\varphi(\cdots v_i, v_{i+1}, \cdots) = -\varphi(\cdots v_{i+1}, v_i \cdots)$

so $\varphi(\cdots v_i \cdots v_j \cdots) = -\varphi(\cdots v_j \cdots v_i \cdots)$.

Claim: $\{\text{antisymm } k\text{-multilinear forms}\} \cong \Lambda^k V^*$.

Recall $W = \text{v.s.} \rightsquigarrow \Lambda^k W = W^{\otimes k}/I$, I generated by $\cdots \otimes w_1 \otimes w_2 \otimes \cdots + \cdots \otimes w_2 \otimes w_1 \otimes \cdots$

Consider the map

$$\mathrm{Alt}: W^{\otimes k} \longrightarrow W^{\otimes k}$$

$$\mathrm{Alt}(w_1 \otimes \cdots \otimes w_k) = \sum_{\sigma \in S_k} (\mathrm{sgn}\,\sigma)\, w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(k)}.$$

eg $\mathrm{Alt}(w_1 \otimes w_2) = w_1 \otimes w_2 - w_2 \otimes w_1$.

exercise: $I = \ker \mathrm{Alt}$ so $\Lambda^k W \cong \mathrm{Im}\,\mathrm{Alt}$:

we can think of elts of $\Lambda^k W$ as particular elts of $W^{\otimes k}$.

<u>In particular</u>: if $W = V^*$ then an element of $\Lambda^k W$ is an elt of image

$$\underset{(\otimes^k V)^*}{\overset{\shortparallel}{\otimes^k V^*}} \xrightarrow{\mathrm{Alt}} \underset{(\otimes^k V)^*}{\overset{\shortparallel}{\otimes^k V^*}} : \text{ a } k\text{-multilinear map}.$$

$k=2$: $\varphi_1, \varphi_2 \in V^* \rightsquigarrow \mathrm{Alt}(\varphi_1 \otimes \varphi_2) = (\varphi_1 \otimes \varphi_2 - \varphi_2 \otimes \varphi_1)$.

note this is antisymmetric: $(\cdot\,\cdot)(v_1, v_2) = -(\cdots)(v_2, v_1)$.

In general: $\mathrm{Im}\,\mathrm{Alt} = \{\text{antisymmetric } k\text{-multilinear maps}\}$

so $\Lambda^k V^* \cong$ ↗.

<u>Wedge product</u>

For $\omega \in \Lambda^k W \subset \otimes^k W$, $\eta \in \Lambda^l W \subset \otimes^l W$, define

$$\omega \wedge \eta = \frac{1}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta) \in \Lambda^{k+l} W \subset \otimes^{k+l} W.$$

(note: weird factor is set up so that $\varphi_1 \wedge \cdots \wedge \varphi_k = \mathrm{Alt}(\varphi_1 \otimes \cdots \otimes \varphi_k)$ for $\varphi_i \in W$)

<u>Properties</u>:
- $\wedge$ is bilinear, associative
- $\wedge$ is graded commutative: $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$.

• $W = V^*$, $\varphi_1, \dots, \varphi_k \in V^* \Rightarrow \varphi_1 \wedge \cdots \wedge \varphi_k$ is the antisymmetric multilinear form

$$(\varphi_1 \wedge \cdots \wedge \varphi_k)(v_1, \dots, v_k) = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)\, \varphi_{\sigma(1)}(v_1) \cdots \varphi_{\sigma(k)}(v_k).$$

If $\begin{smallmatrix} E \\ \downarrow \\ M \end{smallmatrix}$ is a vector bundle then we can define $\begin{smallmatrix} \Lambda^k E \\ \downarrow \\ M \end{smallmatrix}$.

<u>Def</u> $\Omega^k(M) := \Gamma(\Lambda^k T^* M)$ <u>space of k-forms on M</u> ($\mathbb{R}$-v.s.; $C^\infty(M)$-module)

Locally a k-form looks like

$$\sum_{1 \le i_1 < \cdots < i_k \le n} \underbrace{a_{i_1 \cdots i_k}(x_1, \dots, x_n)}_{\in C^\infty} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$\Omega^k(M)$ is nonzero for $0 \le k \le n$: $\Omega^0(M) = C^\infty(M)$

$\Omega^1(M) = \Gamma(T^* M)$

$\vdots$

an element is locally $\leftarrow \Omega^n(M)$

$a(x_1, \dots, x_n)\, dx_1 \wedge \cdots \wedge dx_n$;

<u>volume form</u> if $a \neq 0 \ \forall x_1, \dots, x_n$.

<u>Def</u> $\Omega^*(M) := \bigoplus_{k=0}^{n} \Omega^k(M)$; then $\wedge$ gives $\Omega^*(M)$ the structure of a <u>graded-commutative ring</u>.

A k-form at a point is an antisymm. k-linear form on tangent vectors.

$\omega \in \Omega^k(M)$, $v_1, \dots, v_k \in T_x M \Rightarrow \omega(v_1, \dots, v_k) \in \mathbb{R}$.

A k-form acts on k <u>vector fields</u> to give a function

$\operatorname{Vect}(M)^{\otimes k} \to C^\infty(M)$, tensor in each input.

<u>ex</u>: Coords $x_1, \dots$, $\omega = dx_1 \wedge dx_2$.

$$\omega\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = (dx_1 \otimes dx_2 - dx_2 \otimes dx_1)\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) = 1 - 0 = 1$$

$\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}$ $\qquad = 0 - 1 = -1$

Prop (see HW) $\varphi^*(\omega\wedge\eta) = \varphi^*\omega \wedge \varphi^*\eta$

$$\mathcal{L}_X(\omega\wedge\eta) = (\mathcal{L}_X\omega)\wedge\eta + \omega\wedge(\mathcal{L}_X\eta).$$

Exterior derivative

$f \in C^\infty(M) \rightsquigarrow df \in \Omega^1(M)$ defined by $df(X) = X(f)$:

in coordinates, $df = \sum \frac{\partial f}{\partial x_i} dx_i$.

(note in particular if $f(x_1,\dots,x_n) = x_i$ then $df = dx_i$: explains "dx_i" notation)

Thm $\exists!$ operator $d: \Omega^k M \to \Omega^{k+1} M$ determined by:

1. for $f \in C^\infty(M)$, $df(X) = X(f)$
2. $d(df) = 0$
3. for $\omega \in \Omega^k M$, $\eta \in \Omega^\ell M$, $d(\omega\wedge\eta) = (d\omega)\wedge\eta + (-1)^k \omega\wedge(d\eta)$.

Furthermore, d is local and if $\omega = \sum a_{i_1\cdots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}$ then

(*) $$d\omega = \sum (da_{i_1\cdots i_k}) \wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k}.$$

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Rmks. 1. if d satisfies 1, 2, 3 then (*) must hold:

$$d(a_{i_1\cdots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}) = (da_{i_1\cdots i_k})\wedge dx_{i_1}\wedge\cdots\wedge dx_{i_k} + a_{i_1\cdots i_k}\underbrace{d(dx_{i_1}\wedge\cdots\wedge dx_{i_k})}_{0}.$$

2. for $M \subset \mathbb{R}^n$, just check that (*) satisfies 1, 2, 3.

To extend to all M:

Lemma $\varphi: U_1 \to U_2$, $U_1, U_2 \subset \mathbb{R}^n$. Then $\varphi^* \circ d = d \circ \varphi^*$:

$$\begin{tikzcd} \Omega^k(U_2) \arrow[r, "\varphi^*"] \arrow[d, "d"'] & \Omega^k(U_1) \arrow[d, "d"] \\ \Omega^{k+1}(U_2) \arrow[r, "\varphi^*"] & \Omega^{k+1}(U_1) \end{tikzcd}$$

Commutes.

Pf $k=0$: want $\varphi^* df = d(f\circ\varphi)$ i.e. $\varphi^*(df)(X) = d(f\circ\varphi)(X)$:

$$\varphi^*(df)(X) = (df)(\varphi_* X) = (\varphi_* X)(f) = \tfrac{d}{dt}\big|_{t=0}\,(f\circ(\varphi\circ\gamma))$$
$$d(f\circ\varphi)(X) = X(f\circ\varphi) = \tfrac{d}{dt}\big|_{t=0}\,((f\circ\varphi)\circ\gamma). \quad \checkmark$$

in general: use $\varphi^*(\omega\wedge\eta) = \varphi^*\omega\wedge\varphi^*\eta$ and induction on k. $\square$

Pf of Thm Atlas $\{(F_i, U_i, V_i)\}$ for M, $\omega\in\Omega^k M$.

On U_i, ω is given by $\omega_i\in\Omega^k U_i$: i.e., $\omega_i = F_i^*\omega$.

$$\begin{tikzcd} & M & \\ U_i \arrow[ur, "F_i"] \arrow[rr, "F_j^{-1}\circ F_i"'] & & U_j \arrow[ul, "F_j"'] \end{tikzcd}$$

The collection $\{\omega_i\}$ agrees on overlaps: i.e.,
$$(F_j^{-1}\circ F_i)^*\omega_j = \omega_i \quad . \qquad \big((F_j^{-1}\circ F_i)^* = F_i^*(F_j^{-1})^*\big).$$
Chain Rule

Conversely, a collection $\{\omega_i\}$ that agrees on overlaps gives $\omega\in\Omega^k M$.

But then $\quad (F_j^{-1}\circ F_i)^* d\omega_j = d\,(F_j^{-1}\circ F_i)^*\omega_j = d\omega_i$

So $\{d\omega_i\}$ agrees on overlaps and gives a well defined $(k+1)$-form $d\omega$. $\square$

Prop
- $d^2 = 0$ $\longrightarrow$ check locally
- $d\varphi^* = \varphi^* d$ for any smooth map $\varphi: M\to N$
- $\mathcal{L}_X d = d\,\mathcal{L}_X$. $\longrightarrow$ differentiate previous result

Cartan's (magic) formula

For $X\in \mathrm{Vect}(M)$, define the <u>interior product</u>
$$i_X : \Omega^k(M)\to\Omega^{k-1}(M) \qquad (\text{sometimes written } X\lrcorner)$$
by
$$i_X\omega = C_{11}(X\otimes\omega)$$
viewed as a $(0,k)$ tensor

i.e.
$$i_X\omega(X_1,\dots,X_{k-1}) = \omega(X, X_1,\dots,X_{k-1}).$$

Thm (Cartan's magic formula) On $\Omega^*(M)$, $\mathcal{L}_X = i_X d + d i_X$.

Pf HW.

<u>Coord-free formula for d.</u>

Prop $\omega \in \Omega^k(M)$, $X_0, \dots, X_k \in \mathrm{Vect}\, M$. Then

$$(d\omega)(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i X_i\left(\omega(X_0 \cdots \hat{X}_i \cdots X_k)\right) + \sum_{0 \le i < j \le k} (-1)^{i+j} \omega([X_i, X_j], X_0, \cdots \hat{X}_i \cdots \hat{X}_j \cdots, X_k).$$

ϕ

Pf HW.

<u>ex</u>: $k=1$: $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$:
we saw this already.

<u>Quick application of Cartan</u>:

Def $\omega \in \Omega^n M$ is a <u>volume form</u> if $\forall x \in M$, $\omega_x \in \Lambda^n T^* M \cong \mathbb{R}$ is nonzero.

Given ω = vol form, any elt of $\Omega^n(M)$ can be written(!) as $f\omega$, $f \in C^\infty(M)$.

$X \in \mathrm{Vect}(M) \Rightarrow d(i_X \omega) \in \Omega^n(M)$.

Def The <u>divergence of X</u>, $\mathrm{div}\, X \in C^\infty(M)$, is defined by $d(i_X\omega) = (\mathrm{div}\, X)\omega$.

(check: in $\mathbb{R}^n$ this is the usual divergence.)

Prop $\mathrm{div}\, X = 0 \Leftrightarrow X$ is volume-preserving: $\mathcal{L}_X \omega = 0$.