Math 621 - Differential Geometry

$\frac{1}{2}$ differential topology: manifolds, vector fields, vector bundles, forms, differential forms

$\frac{1}{2}$ Riemannian geometry: metrics, connections, curvature, geodesics - topics mentioned in qual syllabus both intrinsic (coord-free) and with coord.

Needs: basic multivar calculus, topology (top space, subspace topology, covering space, fundamental group)
Analysis: differential, inverse/implicit function theorem
Along the lines of Math 532.

- Text: do Carmo, Riemannian Geometry

- My notes will be posted on course webpage math.duke.edu/~nye/math621
- Grading based on HW and take-home final.
- Office hrs TBD. New office! 216.

Questions:
- make up classes on mammals?
- manifold? tangent space? vector field?
- differential form?
- covering space? find go?
- tensor product of vector spaces?
**Smooth Manifolds**

Example: Surfaces in $\mathbb{R}^3$ or more generally submanifolds of $\mathbb{R}^n$.

Definition: $M \subset \mathbb{R}^{n+k}$ is a (smooth) submanifold of dimension $n$ if $\forall p \in M$, $\exists W$ open of $p$ in $\mathbb{R}^{n+k}$, an open set $V \subset \mathbb{R}^n$, and a map $f: V \to \mathbb{R}^{n+k}$ so that

- $f$ is smooth,
- $f(V) = M \cap W$,
- $f|_V$ is a bijection $V \to M \cap W$,
- $df(x)$ is injective $\forall x \in V$.

**Submanifolds:**

- $\mathbb{R}^n \subset \mathbb{R}^{n+k}$
- $S^2 \subset \mathbb{R}^3$: unit spherical coord. except at $N, S$.
  \[(\phi, \theta) \mapsto (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)\]

**Not Submanifolds:**

- $\sqrt{x^2 + y^2} \subset \mathbb{R}^2$
  \[t \mapsto (t^3, t^2)\]

- **Image of** $f: \mathbb{R} \to \mathbb{R}^2$
Proof of TFAE:

1. \( M \) is a smooth submanifold of dimension \( n \) in \( \mathbb{R}^{n+k} \)

2. \( p \in M \), find \( W \) if \( p \) in \( \mathbb{R}^{n+k} \) and a smooth map 
   \( \phi : W \rightarrow \mathbb{R}^k \) such that 
   \( W \cap M = \phi^{-1}(0) \) 
   \( \phi \) is a submersion: \( \phi(x) \) is surjective \( \forall x \in W \).

\[ 
\phi \quad \rightarrow \quad \mathbb{R}^k 
\]

Proof: implicit function theorem.

Example: \( S^n \subset \mathbb{R}^{n+1} \) \( \times \) \( \{ x^2 + y^2 + z^2 = c \} \subset \mathbb{R}^3 \)
\[ \phi = x^2 + y^2 + z^2 - c \]
\( c = 0 \) is implicit - hence surjective

\[ \{ x^2 - y^2 = 0 \} \subset \mathbb{R}^2 \]
\[ \phi = x^2 - y^2 \] does not work at \((0,0)\)

but this is \( \{ x - y = 0 \} \) which does work.

\[ \{ x^2 + y^2 = 0 \} \subset \mathbb{R}^2 \]
\[ \phi = x^2 + y^2 \] does not work at \((0,0)\)

but this is \( \{ x - y = 0 \} \) which does work.

Proof: \( M \subset \mathbb{R}^{n+k} \) submanifold, \( p \in M \), two coordinate charts 
\( f_1, f_2 : V_1, V_2 \rightarrow \mathbb{R}^{n+k} \) mapping onto \( M \cap W_1, M \cap W_2 \), 
\( W_1, W_2 \) neighborhood of \( p \).

Then 
\[ f_2 \circ f_1^{-1} : f_1^{-1}(W_1 \cap W_2) \rightarrow f_2^{-1}(W_1 \cap W_2) \]
\( f_2 \circ f_1^{-1} \) is a diffeomorphism: smooth, smooth inverse.

Thus allows us to dispense with the "ambient space" \( \mathbb{R}^{n+k} \).
Def: A smooth atlas of coordinate charts on $M$ is a collection $\mathcal{F}$ for consistency.

$\mathcal{F} = \{ (U_\alpha, \phi_\alpha) \mid \alpha \in I \}$

$such that:\n\cdot f_\alpha is a bijection$
$\cdot U_\alpha \text{ cover } M: \bigcup_{\alpha} U_\alpha = M$
$\cdot \forall \alpha, \beta \text{ with } U_\alpha \cap U_\beta \neq \emptyset,$

$\quad f_\beta \circ f_\alpha^{-1}: f_\alpha^{-1}(U_\alpha \cap U_\beta) \to f_\beta^{-1}(U_\alpha \cap U_\beta) \text{ is smooth}.$

Topology on $M$ is induced by atlas:

$U \subset M \text{ is open } \iff f_\alpha^{-1}(U) \text{ is open } \forall \alpha.$

Def: $M$ is a smooth manifold of dimension $n$ (n-manifold) if it has a smooth atlas, and with this topology, $M$ is Hausdorff and second countable.

\[ \text{No topology has a countable base.} \]
Rank: Sometimes useful to consider maximal atlases

Not properly contained in any other atlas: any chart chart
compatible with the atlas is in the atlas:
if $f: U \rightarrow M$ is a chart then $f: U \cap U_\alpha \rightarrow M$ is also a chart for $U \cap U_\alpha$, $f = f_\alpha \circ f$.

- intrinsic vs extrinsic: Whitney embedding theorem: any small $n$-manifold $M$ can be smoothly embedded in $\mathbb{R}^{2n}$.

Example of $n$-manifolds.
- $\mathbb{R}^n$
- any $n$-dim subset of $\mathbb{R}^{n+k}$
- $S^n \subset \mathbb{R}^{n+1}$: here's an atlas:
  $N = (0, ..., 0, 1)$
  $S = (0, ..., 0, -1)$

\[ \pi_N: S^n \setminus N \rightarrow \mathbb{R}^n \]
\[ \pi_S: S^n \setminus S \rightarrow \mathbb{R}^n \]
\[ x_1, ..., x_{n+1} \]

\[ \pi_N(x_1, ..., x_{n+1}) = \left( \frac{x_1}{1-x_{n+1}}, ..., \frac{x_n}{1-x_{n+1}} \right) \]
\[ \pi_S(x_1, ..., x_{n+1}) = \left( \frac{x_1}{1+x_{n+1}}, ..., \frac{x_n}{1+x_{n+1}} \right) \]

This map: $y_1, y_2 \mapsto (y_1', ..., y_n')$
\[ y_i' = \frac{x_i}{1-x_{n+1}}, \quad y_i' = \frac{x_i}{1+x_{n+1}} \quad \Rightarrow \quad y_i' = \frac{y_i}{y_i^2 + y_n^2} \]
This is a smooth map $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$.  

$\pi_N(U_1, U_2) = \mathbb{R}^{n-1} \longrightarrow \mathbb{R}^{n-1} = \pi_N(U_1, U_2)$
\[ \mathbb{R}^n / \text{Points are equiv. classes in } \mathbb{R}^{n+1} \setminus \{0\} \]

\[ (x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1}) \quad \lambda \neq 0. \]

While \([x_i, -x_i] \text{ for equiv class } i \leq i \leq m]: V_i = \{ [x_i, x_i] \mid x_i \neq 0 \} \subset \mathbb{R}^n, \]

\[ V_i \longrightarrow U_i = \mathbb{R}^n \text{ bijective, inverse } f_i: U_i \longrightarrow V_i \]

\[ (x_1, \ldots, x_{n+1}) \rightarrow \left( \frac{x_2}{x_1}, \ldots, \frac{x_{n+1}}{x_1}, \frac{1}{x_1}, \ldots, \frac{1}{x_{n+1}} \right) \]

\[ f_i^* f_j = f_j^* f_i \text{ is a map on } \{ y_j \neq 0 \}. \]

**Smooth Maps**

**Def:** \( M_1, M_2 \text{ smooth manifolds.} \)

A map \( \varphi: M_1 \rightarrow M_2 \) is **smooth at** \( p \in M_1 \) if there exist charts \( f_1: U_1 \rightarrow M_1, f_2: U_2 \rightarrow M_2 \)

\[ p \in f_1(U_1), \varphi(p) \in f_2(U_2), \varphi(f_1(U_1)) \subset f_2(U_2), \]

and \( f_2 \circ \varphi \circ f_1: U_1 \rightarrow U_2 \) is smooth at \( f_1(p) \).

\[ \varphi \]

\[ f_2 \circ \varphi \circ f_1 \]
A map is smooth if it is smooth at all points.

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\textbf{Remark: This is independent of coordinate charts.}

\textbf{Special cases of smooth maps:}

\textbf{Def.} \( f: M \to N \) is an immersion if at any \( p \in M \), there are coordinate charts \( f_1 \) near \( p \), \( f_2 \) near \( f(p) \) such that

\( f_2 \circ f \circ f_1^{-1} : U_1 \to U_2 \)

is an immersion at \( f_1(p) \): \( d(f_2 \circ f \circ f_1^{-1}) \) is injective.

A map is a submersion if \( d(f_2 \circ f \circ f_1^{-1}) \) is surjective.

\textbf{Ex.} Submersion: \( \mathbb{R}^n \to \mathbb{R}^m \) projection, \( n \geq m \)

\textbf{Def.} \( f \) is an embedding if it is an immersion and homeomorphism onto its image (with subspace topology); in particular, injective.

\textbf{Diffeomorphism:} Smooth, smooth inverse (in particular, immersion and submersion).

\textbf{Ex.} \( \mathbb{R} \to \mathbb{R} \): not immersion:

\[ \begin{array}{ccc}
\mathbb{R} & \to & \mathbb{R}^2 \\
\uparrow & & \downarrow \\
\mathbb{R} & \to & \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\mathbb{R} & \to & \mathbb{R}^2 \\
\end{array} \]

\textbf{Immersion, not injective:}

\textbf{Immersion, injective, not embedding:}

\[ \begin{array}{ccc}
\mathbb{R} & \to & \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\mathbb{R} & \to & \mathbb{R}^2 \\
\downarrow & & \downarrow \\
\mathbb{R} & \to & \mathbb{R}^2 \\
\end{array} \]

\textbf{Immersion, injective, not embedding:}
Rules.
1. Any immersion is locally an embedding.
   \[ \varphi: M_1 \to M_2 \text{ immersion}, \quad p \in M_1 \implies \exists \text{ neighborhood } V \ni p \text{ with } \varphi|_V: V \to M_2 \text{ embedding.} \]
2. A map that's both an immersion and a submersion is a local diffeomorphism: restricted to a small neighborhood of any point, it's a diffeomorphism (e.g., covering maps).
3. If \( \varphi: M_1 \to M_2 \) is an embedding, then \( \varphi(M_1) \) is a submanifold of \( M_2 \).

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**Quotients by group actions**

**Def.** A group \( G \) acts on a manifold \( M \) if each \( g \in G \) gives a diffeomorphism \( \varphi_g: M \to M \) s.t.

\[ \varphi_g \circ \varphi_h = \varphi_{gh}, \quad (\Rightarrow \varphi_e = \text{id}). \]

\( G \) acts properly discontinuously if \( \forall p \in M \exists \text{ neighborhood } V \ni p \text{ s.t. } \bigcap_{g \neq 1} \varphi_g(V) = \emptyset. \)

\( G \) acting on \( M \) \( \implies \) quotient space \( M/G \) of orbits under \( G \) (\( p \sim \varphi_g(p) \)).

**Ex.**
1. \( \mathbb{R}^x \) acts on \( \mathbb{R}^{n+1} \{0\} \) by scalar multiplication, not properly discontinuously:
   \[ (\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^x = \mathbb{R}P^n. \]
2. \( \mathbb{Z}/2 = \{ \pm 1 \} \) acts on \( S^n \subset \mathbb{R}^{n+1} \) by antipodal map:
   \[ S^n/\{\pm 1\} = \mathbb{R}P^n. \]
3. Möbius Strip \( \mathbb{R} \times (0,1)/\{ (x,0) \sim (x+1,1-y) \} \).
Prop: If $G$ acts properly discontinuously on $M$, then $M/G$ is a smooth manifold and $\pi: M \to M/G$ is a smooth map (in fact, local diffeomorphism). (next page)

**Proof:** Actually need to either assume $M/G$ is Hausdorff or change the def of proper discontinuous. E.g., add:

For $p, q \in M$ not in the same $G$-orbit, there are open $U_1, \ldots, U_k$ of $p$, such that $U_i \cap \phi_g(U_i) = \emptyset$ for all $g \in G$.

Otherwise, point-set exercise:

$M = \mathbb{R}^2$, \hspace{1cm} $G = \mathbb{Z}$, \hspace{1cm} $n: (x, y) \mapsto (2^n x, 2^{-n} y)$

$p_1 = (1, 0), \hspace{1cm} p_2 = (0, 1)$. 
Proposition: If \( G \) acts properly discontinuously on \( M \), then \( M/G \) is a smooth manifold and \( \pi: M \to M/G \) is a smooth map (in fact, a local diffeomorphism).

First, construct charts on \( M/G \).

Let \( p \in M \). Choose a chart \( f_p: U_p \to M \), \( f_p(U_p) = V_p \ni p \), such that \( V_p \cap \varphi_g(U_p) = \emptyset \) for \( g \neq 1 \). (If \( V_p \) doesn't satisfy this, intersect it with \( V \) from def.)

Then \( \pi|U_p \) is injective so \( h_p = \pi \circ f_p: U_p \to M/G \) is injective.

Write \( \tilde{V}_p = \pi(U_p) = h_p(U_p) \).

Then \( h_p: U_p \to \tilde{V}_p \) is bijective.

Claim: \( (h_p, U_p, \tilde{V}_p) \) is an atlas for \( M/G \).

Need to check transition functions.

The following are bijections:

\[
f_p: U_p \to U_p \\
\pi_p: U_p \to \tilde{V}_p \\
h_p: U_p \to \tilde{V}_p \\
f_q: U_q \to U_q \\
\pi_q: U_q \to \tilde{V}_q \\
h_q: U_q \to \tilde{V}_q
\]
Near \( r \in h^{-1}(\Sigma_1 \cap \Sigma_2) \) we have
\[
 h_\gamma \circ h_p = f_\gamma \circ \pi^{-1}_\gamma \circ \pi_p \circ f_p .
\]

Write \( f_p = f_p(r) \), \( r_\gamma = \pi^{-1}_\gamma (\pi_p (r)) \) so \( \pi (r_\gamma) = \pi (r_p) \)

\[ \Rightarrow \exists g \text{ with } r_\gamma = \varphi_g (r_p) . \]
Near \( r_\gamma \), \( \pi = \pi_\gamma \circ \varphi_g = \pi_p \circ \pi_\gamma \circ \varphi_g = \pi_\gamma \circ \pi_p \circ \varphi_g . \)
So \( h_\gamma \circ h_p = f_\gamma \circ \varphi_g = f_p \) is smooth.  \( \square \)

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Orientations

\( M = n \)-manifold, atlas \( \{(f_\alpha : U_\alpha, V_\alpha)\} \).
\[
 f_1 : U_1 \to V_1 \subset M \quad \text{If } V_1 \cap V_2 \neq \emptyset \text{ then "transition function" } \quad f_2 \circ f_1 : f_1^{-1}(V_1 \cap V_2) \to f_2^{-1}(V_1 \cap V_2)
\]
is smooth with smooth inverse.
So for all \( x \in f_1^{-1}(V_1 \cap V_2) \),
\[
 d(f_2 \circ f_1)(x) : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is a nonsingular linear map.}
\]
\[
 \det (d(f_2 \circ f_1)(x)) \neq 0 .
\]

Say \( f_1, f_2 \) determine the \{same\} orientation at \( p = f_1(x) \in V_1 \cap V_2 \)
if \( \det d(f_2 \circ f_1)(x) \{ \geq 0 \} \).
\( f_1, f_2 \) determine \{opposite\} orientation if they determine \{same\} orientation at all points: note if \( V_1 \cap V_2 \) connected then one of these must happen.
An atlas for $M$ is oriented if all coordinate charts determine the same orientation.

An orientation for $M$ is a choice of oriented atlas, mod saying that 2 atlases are equivalent if their union is oriented (i.e., all charts determine same orientation).

Note: Switching orientation $f_i: U \to V \subset M$, $h: \mathbb{R}^n \to \mathbb{R}^n$ $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$

$\rightarrow$ chart $f_2 = f_1^* h: h(U) \to V$ has opposite orientation to $f_1$:
$f_2^* f_1: U \to h(U)$ is $h_1$ and $\text{det } h = -1$.

$\mathbb{S}^n$. 2 coord charts $f_1, f_2: \mathbb{R}^n \to \mathbb{S}^n$.

Recall $f_2^{-1} f_1: \mathbb{R}^n \to \mathbb{R}^n$ $(y_1, \ldots, y_n) \mapsto \left( \frac{y_1}{\sqrt{y_1^2 + \ldots + y_n^2}}, \ldots, \frac{y_n}{\sqrt{y_1^2 + \ldots + y_n^2}} \right)$.

Exercise: $\text{det } d(f_2 \circ f_1) = -\frac{1}{y_1^2 + \ldots + y_n^2}$.

So an oriented atlas is $f_1$ and $f_2 \circ h$, or $f_1 \circ h$ and $f_2$.

$\mathbb{N} \times \text{Unit Strip}$

$M = \{ (0, 1) \times (0, 1) \}/(0, y) \sim (1, 1-y)$

Atlas: $f_1: (0, 1)^2 \to M$ $(x, y) \mapsto (x, y)$

$f_2: (0, 1)^2 \to M$ $(x, y) \mapsto (x + \frac{1}{2}, 1-y)$ where $(x, y) \sim (x+1, 1-y)$ if $x < \frac{1}{2}$.

on $(0, \frac{1}{2}) \times (0, 1)$, $f_2 \circ f_1(x, y) = (x + \frac{1}{2}, 1-y)$

on $(\frac{1}{2}, 1) \times (0, 1)$, $f_2 \circ f_1(x, y) = (x - \frac{1}{2}, 1-y)$

So these charts agree on one part and disagree on the other.
**Tangent Vectors**

What's a tangent vector to a manifold $M$ at a point $p \in M$?

If $M \subset \mathbb{R}^n$:

Velocity vector of a curve passing through $p$.

Intrinsically, in general?

**Approach 1**

Let $\gamma : (-\epsilon, \epsilon) \to M$ be a curve (a smooth map) with $\gamma(0) = p$.

Any such curve gives rise to a tangent vector at $p$.

When are two curves the same?

These should give same tangent vector.

Use coord chart $f = \phi : U \subset \mathbb{R}^n \to V \subset M$.

Say $\gamma_1, \gamma_2 : (-\epsilon, \epsilon) \to M$ with $\gamma_1(0) = \gamma_2(0) = p$ are equivalent if

$$\left. \frac{d}{dt} \right|_{t=0} (f^{-1} \circ \gamma_1) = \left. \frac{d}{dt} \right|_{t=0} (f^{-1} \circ \gamma_2).$$

Need to check indep of coord chart.
\[
\begin{align*}
\frac{d}{dt} (f_i^{-1} \circ \gamma) &= \frac{d}{dt} (f_i^{-1} \circ \gamma) \\
\text{and they are equal} &\implies \text{there are equal.}
\end{align*}
\]

**Def.** A tangent vector to \( M \) at \( p \) is an equivalence class of curves \( \gamma: (-\varepsilon, \varepsilon) \to M, \gamma(0) = p \), with the above equivalent relation.\\
\[
T_p M = \{ \text{tangent vectors to } M \text{ at } p \}.
\]
Review: tangent vectors.

\( M = \text{smooth manifold}, \ p \in M, \ T_p M = \{ \text{tangent vectors to } p \ at \ M \}. \)

**Def:** \( T_p M = \{ \text{Curves in } M \text{ through } p \}/\sim \)

\( \gamma : (-\epsilon, \epsilon) \to M, \ \gamma(0) = p \)

Coord chart \( U \overset{\phi}{\leftrightarrow} \mathbb{R}^n, \ \gamma_1 \sim \gamma_2 \Leftrightarrow \frac{d}{dt} \big|_{t=0} \phi(F^{-1}(\gamma_1)) = \frac{d}{dt} \big|_{t=0} \phi(F^{-1}(\gamma_2)). \)

Notation: sometimes write \( \gamma'(0) \) for \([\gamma] \in T_p M.\)

\( F^{-1}(\gamma) \)

Note we get a map \( T_p M \to \mathbb{R}^n \)

\( [\gamma] \mapsto \frac{d}{dt} \big|_{t=0} \phi(F^{-1}(\gamma)) \)

Well-defined + injective by definition. It's also surjective:

for \( w \in \mathbb{R}^n \), define \( c : (-\epsilon, \epsilon) \to U \) by \( c(t) = F^{-1}(p) + tw \);

if \( F \circ c \) then \( [\gamma] \to w. \)

\( \Rightarrow T_p M \cong \mathbb{R}^n \) - this gives \( T_p M \) the structure of a vector space.

Let \( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \) denote the basis of \( T_p M \) corr. to the standard basis

under this isom: \( T_p M \overset{\cong}{\to} \mathbb{R}^n \)

\( v_1 \frac{\partial}{\partial x_1} + \ldots + v_n \frac{\partial}{\partial x_n} \quad (v_1, \ldots, v_n) \)

(for now, just formal notation). \( \text{Note depends on coord chart.} \)
**APPROACH 2**

While $C^\infty(M)$ = \{smooth functions $M \to \mathbb{R}$\}. Any curve $\gamma : I \to M$ through $p$

\[ C^\infty(M) \to \mathbb{R} \]

\[ f \to \frac{d}{dt}|_{t=0}(f \circ \gamma) \]

**Note:**

- $f$ only has to be defined in a nbhd of $p$
- only depends on infinitesimal behavior of $f$ near $p$.

**Def.** $M$ is called a manifold. The space of smooth functions at $p$ is $C^\infty_p(M) := \{f : V \to \mathbb{R} \mid V$ open in $M, f \text{ smooth on } V \}$.

\[ (f : V \to \mathbb{R}) \sim (g : W \to \mathbb{R}) \iff \exists \ V_0 \subseteq V, \ W_0 \subseteq W \text{ nbhd of } p \text{ with } f|_{V_0} = g|_{W_0} \]

**Note:** $C^\infty_p(M)$ is a vector space over $\mathbb{R}$ (add, scale maps as usual).

Then $\gamma$ gives a linear map $C^\infty_p(M) \to \mathbb{R}$

\[ [f] \to \left. \frac{d}{dt} \right|_{t=0}(f \circ \gamma) \]

This map depends only on $[\gamma]$: choose coord chart $F : U \to \mathbb{R}^m$

\[ F^{-1} \quad \text{and} \quad \mathbb{R}^m \]

\[ \frac{d}{dt}|_{t=0}(f \circ F^{-1} \gamma) = \left. \frac{d}{dt} \right|_{t=0}(f \circ F^{-1} \gamma) = \frac{d}{dt}|_{t=0}(f \circ \gamma) \cdot \left. \frac{d}{dt} \right|_{t=0}(F^{-1}(F \gamma)) \]

only depends on $[\gamma]$

**Stud.** for $v \in T_pM$, get the directional derivative $v(\cdot) : C^\infty_p(M) \to \mathbb{R}$

(just choose any $\gamma$ with $[\gamma] = \gamma$).

**Def.** $T_pM = \{\text{linear maps } C^\infty_p(M) \to \mathbb{R} \text{ that are equal to } v(\cdot) \text{ for some } v \in T_pM \}$.
Note there's an obvious map
\[ \Phi : T_p M \to T_{\Phi(p)} M, \]
surjective by definition.

To prove \( T_p M \cong T_{\Phi(p)} M \), need \( \Phi \) injective.

Let \( U \xrightarrow{F} V \) be a chart, and let \( (x^1, \ldots, x^n) \in \mathbb{R}^n \). Define
\[ C(t) = F^{-1}(t) + t(x^1, \ldots, x^n) \quad \text{and} \quad \Phi(t) = F(C(t)). \]
Notation from here onward:
\[ \Phi(t) = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}. \]

What's \( \Phi(\Phi(t)) \)?

Let \( f \in C^0_p(M) \), and write \( g = f \circ F : U \to \mathbb{R} \).

\[
\frac{d}{dt} \Phi(t) = \frac{d}{dt} F(C(t)) = \frac{d}{dt} G(t) = \left[ \frac{\partial G}{\partial x_1} \quad \cdots \quad \frac{\partial G}{\partial x_n} \right] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 \frac{\partial v}{\partial x_1} + \cdots + v_n \frac{\partial v}{\partial x_n}
\]

So different values of \( (x^1, \ldots, x^n) \) give different maps \( C^0(M) \to \mathbb{R} \) (plug in \( g = x_1, \ldots, g = x_n \)).

With abuse of notation, identify \( f \) with \( g \). Then the directional derivative
\[ \nabla(f) = v_1 \frac{\partial f}{\partial x_1} + \cdots + v_n \frac{\partial f}{\partial x_n} \]
which explains the notation
\[ v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}. \]

**Approach 3** - most abstract.

**Def.** A derivation at \( p \) is a linear map
\[ \delta : C^0_p(M) \to \mathbb{R} \]
\[ \text{st.} \quad \delta(fg) = f(p) \delta(g) + \delta(f) g(p). \]
Prove \( T_pM \cong \{ \text{derivations at } p \} \).

**Proof:**

- \( \forall v \in T_pM \), \( v \mapsto (f \mapsto v(f)) \) is a bijection.

**Claim:** \( \delta = v(\cdot) \) where \( v = \sum v_i \frac{\partial}{\partial x_i} \).

**Need:** \( \delta = v(\cdot) \) on \( x_i \circ F^{-1} \).

**Lemma:** Can write \( g(x_1, \ldots, x_n) = c + \sum x_i g_i(x_1, \ldots, x_n) \) for some \( c \in \mathbb{R} \) and smooth \( g_i : \mathbb{R}^n \rightarrow \mathbb{R} \) with \( g_i(\mathbf{0}) = \frac{\partial g(\mathbf{0})}{\partial x_i} \).

**Proof:** \( g(x) = g(\mathbf{0}) + \int_0^1 \frac{d}{dt} g(tx) \, dt = g(\mathbf{0}) + \sum x_i \int_0^1 \frac{\partial g(tx)}{\partial x_i} (tx) \, dt \).

So: \( \delta(f) = \delta(c) + \sum \delta(x_i \circ F^{-1})(g_i \circ F^{-1}) \)

\[ = 0 + \sum \delta(x_i \circ F^{-1}).g_i(\mathbf{0}) + (x_i \circ F^{-1}(p)).\delta(g_i \circ F^{-1}) \]

\[ = \sum v_i \frac{\partial g(\mathbf{0})}{\partial x_i} \]

\[ = v(f). \quad \square \]
**Differentials of Smooth Maps**

**Prop/Def** \( \varphi : M_1 \to M_2 \) smooth, \( p \in M_1 \). There is a well-defined linear map

\[
\text{d} \varphi_p : T_p M_1 \to T_{\varphi(p)} M_2, \quad \text{the differential of } \varphi \text{ at } p.
\]

**Def**

\[
\begin{align*}
&(-\varepsilon, \varepsilon) \\
&\overset{F_1}{\longrightarrow} \\
&\overset{F_2 \circ \varphi}{\longrightarrow} \\
&\overset{(-\varepsilon, \varepsilon)}{\longrightarrow} \\
&\overset{U_1 \subset \mathbb{R}^n}{\longrightarrow} \\
&\overset{\varphi}{\longrightarrow} \\
&\overset{U_2 \subset \mathbb{R}^m}{\longrightarrow} \\
&\overset{(x_1, \ldots, x_n)}{\longrightarrow} \\
&\overset{y_1, \ldots, y_m}{\longrightarrow}
\end{align*}
\]

\[
F_2 \circ \varphi \circ F_1 (x_1, \ldots, x_n) = (y_1(x_1, \ldots, x_n), \ldots, y_m(x_1, \ldots, x_n))
\]
\((F_1^{-1} \circ \varphi)(t) = (x_1(t), \ldots, x_n(t)) \rightarrow \text{tangent vector} \left( \frac{\partial}{\partial t} \right) \) is:
\[
\begin{align*}
\frac{\partial}{\partial t} \bigg|_{t=0} & \mathbf{y}_1(x_1(0), \ldots, x_n(0)) \frac{\partial}{\partial x_1} + \cdots + \mathbf{y}_n(x_1(0), \ldots, x_n(0)) \frac{\partial}{\partial x_n}.
\end{align*}
\]

\((F_2^{-1} \circ \varphi \circ F_1)(F_1^{-1} \circ \varphi)(t) = (y_1(x_1(t), \ldots, x_n(t)), \ldots, y_n(x_1(t), \ldots, x_n(t))).
\]

Derivative at \(t=0\) is:
\[
\left( \frac{\partial}{\partial t} \bigg|_{t=0} \begin{pmatrix} y_1(x_1(t), \ldots, x_n(t)) \\ \vdots \\ y_n(x_1(t), \ldots, x_n(t)) \end{pmatrix} \right) \frac{2}{\partial x_i} + \cdots
\]
\[
\begin{pmatrix} \frac{\partial y_i}{\partial x_1} \\ \vdots \\ \frac{\partial y_n}{\partial x_1} \end{pmatrix} \begin{pmatrix} x_1(0) \\ \vdots \\ x_n(0) \end{pmatrix}
\]
\[
\text{with matrix}
\]

And this depends only on \(x_1(0), \ldots, x_n(0)\), not \(\varphi\):
\[
\text{det}_\varphi \left( \begin{pmatrix} \frac{\partial y_1}{\partial x_1} + \cdots + \frac{\partial y_n}{\partial x_1} \\ \vdots \\ \frac{\partial y_1}{\partial x_i} \\ \vdots \\ \frac{\partial y_n}{\partial x_i} \end{pmatrix} \right) = \prod \left( \sum \frac{\partial y_i}{\partial x_j} \frac{\partial y_j}{\partial x_i} \right) = \left( \frac{\partial y_i}{\partial x_j} \right) \frac{\partial y_j}{\partial x_i}
\]
\[
\text{linear in } (v_1, \ldots, v_n).
\]

Summary: in coordinates, \(\text{det}_\varphi\) is the map
\[
\mathbb{R}^n \to \mathbb{R}
\]
\[
\begin{pmatrix} T_{\varphi_1} M^1 \\ \vdots \\ T_{\varphi_n} M^n \end{pmatrix}
\]
given by the matrix
\[
\left( \frac{\partial y_i}{\partial x_j} \right).
\]

Special Case: change of coordinates, \(\varphi = id\):
\[
\begin{pmatrix} T_{\varphi_1} M^1 \\ \vdots \\ T_{\varphi_n} M^n \end{pmatrix}
\]
\[
\begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}
\]
\[
= \begin{pmatrix} \frac{\partial y_1}{\partial x_1} \\ \vdots \\ \frac{\partial y_n}{\partial x_n} \end{pmatrix}
\]

Thus (Chain Rule):
\[
\frac{\partial}{\partial \varphi} \left( \begin{pmatrix} \frac{\partial y_1}{\partial \varphi_1} \\ \vdots \\ \frac{\partial y_n}{\partial \varphi_n} \end{pmatrix} \right) = \frac{\partial}{\partial \varphi} \left( \begin{pmatrix} \frac{\partial y_1}{\partial \varphi_1} \\ \vdots \\ \frac{\partial y_n}{\partial \varphi_n} \end{pmatrix} \right) \cdot \frac{\partial \varphi}{\partial \varphi}.
\]
If \( \varphi : M \rightarrow N \) is a diffeomorphism, then \( M \xrightarrow{\varphi} N \xrightarrow{\varphi^{-1}} M \) is an isomorphism.

\[ d((\text{id})_{\varphi})_{p} = \text{id} : T_{p}M \rightarrow T_{p}M \]

so \( d\varphi_{p} \), \( d\varphi^{-1}_{p} \) are inverse maps.

\[ \Rightarrow d\varphi_{p} : T_{p}M \rightarrow T_{\varphi(p)}N \text{ is an isomorphism.} \]

Conversely:

Prop: If \( d\varphi_{p} : T_{p}M \rightarrow T_{\varphi(p)}N \) is an isomorphism then \( \varphi \) is a local diffeomorphism at \( p \).

Pf: Inverse Function Theorem.

\[ \varphi \text{ is invertible at } p \iff d\varphi_{p} \text{ is injective.} \]

\[ \varphi \text{ is injective at } p \iff d\varphi_{p} \text{ is surjective.} \]

\[ \varphi \text{ is local diffeo} \iff \text{injective and surjective.} \]

---

**Tangent Bundle**

\( M \) a smooth \( n \)-manifold. Define \( TM := \{ (p,v) \mid p \in M, v \in T_{p}M \} = \bigsqcup_{p \in M} T_{p}M \).

This isn't just a set, but a smooth \((2n)\)-manifold:

\{ (F_{p}, u_{1}, v_{2}) \} = \text{at} (u,v) \text{ for } M.

\( x_{1}, ..., x_{n} \) coordinates on \( U_{\alpha} \), \( u_{1}, ..., u_{n} \) are \( \frac{\partial}{\partial x_{1}}, ..., \frac{\partial}{\partial x_{n}} \).

Define \( F_{p} : U_{\alpha} \times \mathbb{R}^{n} \rightarrow TM \)

\( (x_{1}, ..., x_{n}, u_{1}, ..., u_{n}) \mapsto (F_{p}(x_{1}, ..., x_{n}), \sum_{i=1}^{n} u_{i} \frac{\partial}{\partial x_{i}}). \)
Since \( \{ V_\alpha \} \) cover \( M \), \( \{ \tilde{F}_\alpha (U_\alpha \times \mathbb{R}^n) \} \) cover \( TM \).

Need to check transition functions smooth.

Suppose \( V_\alpha \cap V_\beta \neq \emptyset \). Say \((p, u, v) \in \tilde{F}_\alpha (U_\alpha \times \mathbb{R}^n) \cap \tilde{F}_\beta (U_\beta \times \mathbb{R}^n) \).

The \( p = F_\alpha (x_1, ..., x_n) \), \( v = \sum u_\alpha \frac{\partial}{\partial x_i} \), \( v = \sum \omega_i \frac{\partial}{\partial y_j} \).

\( (y_1, ..., y_n) = (F_\beta \circ F_\alpha)(x_1, ..., x_n) \). \( (\omega_1, ..., \omega_n) = \text{d}(F_\beta \circ F_\alpha)(x_1, ..., x_n) (\nu_1, ..., \nu_n) \).

\( \Rightarrow (y_1, ..., y_n), (\nu_1, ..., \nu_n) = (\text{d}F_\beta \circ \text{d}F_\alpha)(x_1, ..., x_n), \text{d}(F_\beta \circ F_\alpha)(x_1, ..., x_n) (\nu_1, ..., \nu_n) \).

Since \( F_\beta^{-1} \circ F_\alpha \) is smooth, so is \( \text{d}(F_\beta \circ F_\alpha) \).

So \( F_\beta^{-1} \circ F_\alpha \) is a smooth map.

**Vector Fields**

**Def** A vector field \( X \) on \( M \) is a map \( M \rightarrow TM \) i.e. mapping \( p \) to some \( X(p) \in T_p M \) for all \( p \in M \). (i.e. \( M \rightarrow TM \).

Usually assume smooth: the map \( M \rightarrow TM \) is smooth.

In local coords: can write as
\[
X(x_1, ..., x_n) = X_1(x_1, ..., x_n) \frac{\partial}{\partial x_1} + ... + X_n(x_1, ..., x_n) \frac{\partial}{\partial x_n}.
\]

\( X_i \) smooth functions.
Notation: write \( \text{Vect}(M) = \{ \text{smooth vector fields on } M \} \)

\( C^0(M) = \{ \text{smooth functions } M \to \mathbb{R} \} \) = \( \text{vector space} \)

Tangent vector \( \to \) derivation at a pt; \( \text{vector field} \to \) global derivation

Def. A derivation \( \delta : C^0(M) \to C^0(M) \)

such that \( f, g \in C^0(M) \)

\( \delta(f \cdot g) = f \cdot \delta(g) + \delta(f) \cdot g \)

i.e., \( \delta (f \cdot g) \cdot x = f(p) \cdot \delta(g) \cdot x + \delta(f) \cdot p \cdot g(x) \)

\( X_f(x) = x_{f(x)} \)

Prop. Any (smooth) vector field \( X \) on \( M \) gives a derivation \( X(\cdot) \)

and this gives an isomorphism

\( \text{Vect}(M) \xrightarrow{\cong} \{ \text{derivations on } M \} \)

Pf. Let \( p \in M \), \( U \supseteq V \) chart near \( p \). For \( i = 1, \ldots, n \) smooth function \( f_i : M \to \mathbb{R} \) s.t. near \( p \), \( f_i(x_1, \ldots, x_n) = x_i \) (for this, need a bump function).

Injective: If \( V = \sum_i v_i \frac{\partial}{\partial x_i} \) on \( U \) then \( \delta(v_i) = \delta(x_i) = 1 \cdot v_i \).

Surjective: Given \( \delta \), if \( \delta(f_i) \cdot p = v_i \) then \( \delta(\cdot) = v(\cdot) \) where \( v = \sum_i v_i \frac{\partial}{\partial x_i} \).

So from \( \delta \) we get a tangent vector at \( p \) \& \( p \). (Check: smooth, indep of chart.) \( \square \)

\( \delta \) Note: This needs the existence of smooth bump functions. The same isn't true for complex manifolds (where "smooth" = holomorphic).
Lie Bracket

\[ X, Y \in \text{Vect}(M) \mapsto X(Y(f)) : \mathcal{C}^0(M) \to \mathcal{C}^0(M). \]

Consider the map \( f \mapsto X(Y(f)) : \mathcal{C}^0(M) \to \mathcal{C}^0(M). \)

This is \( \mathbb{R} \)-linear? Is it a derivation?

\[
X(Y(fg)) = X(fY(g) + Y(f)g) \\
= fX(Y(g)) + X(f)Y(g) + Y(f)X(g) + X(Y(f))g
\]

No, but

\( f \mapsto X(Y(f)) - Y(X(f)) \) is.

**Def.** The Lie bracket \([X, Y] \in \text{Vect}(M)\) is the vector field corresponding to the derivation \( f \mapsto X(Y(f)) - Y(X(f)) \).

In local charts, \( X = \sum a_i \frac{\partial}{\partial x_i}, Y = \sum b_i \frac{\partial}{\partial x_i} \):

\[
X(Y(f)) = \sum_{ij} a_j \frac{\partial}{\partial x_j} \left( b_i \frac{\partial f}{\partial x_i} \right) = \sum a_j \frac{\partial b_i}{\partial x_j} \frac{\partial f}{\partial x_i} + a_j b_i \frac{\partial^2 f}{\partial x_i \partial x_j}.
\]

\[ [X, Y](f) = \sum_{ij} a_j \frac{\partial b_i}{\partial x_j} \frac{\partial f}{\partial x_i} - b_j \frac{\partial a_i}{\partial x_j} \frac{\partial f}{\partial x_i}. \]

\[ [X, Y] = \left( \sum_i \left( \sum_j \left( a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j} \right) \right) \frac{\partial}{\partial x_i} \right). \]
Properties:
\[ X, Y \mapsto [X, Y] \] is:
\[ \text{1) bilinear: } [aX_1 + bX_2, Y] = a[X_1, Y] + b[X_2, Y] \]
for \( a, b \in \mathbb{R} \).
\[ \text{2) anti-symmetric: } [Y, X] = -[X, Y] \]
\[ \text{3) a Lie bracket satisfies the Jacobi identity: } \]
\[ [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \]

Note: for \( g \in C^\infty(\mathcal{M}) \) it's not true that \( [gX, Y] = g[X, Y] \):
\[
[gX, Y] f = gX(Y(f)) - Y(gX(f)) = gX(Y(f)) - Y(gX(f)) = gX(Y(f)) - gY(X(f)) - Y(g)f(f)
\]
so \( [gX, Y] = g[X, Y] - Y(g)f \).

Flow of a vector field

For \( X \in \text{Vect}(\mathcal{M}) \), \( \gamma : (a, b) \to \mathcal{M} \) is an integral curve for \( X \) if
\[ \gamma'(t) = X(\gamma(t)) \in T_{\gamma(t)} \mathcal{M} \text{ for all } t \in (a, b). \]

Prop: \( p \in \mathcal{M} \). \( \exists (a, b) \) containing \( 0 \) and unique integral curve \( \gamma : (a, b) \to \mathcal{M} \) for \( X \) with \( \gamma(0) = p \).

Proof: local existence/uniqueness for first order ODE. \( \square \)
Note maximal $(a, b)$ may depend on $p$, but:

Prop. $X \in \text{Vect}(x)$. \exists nd $V$ of $p$, interval $(a, b) \ni 0$ st. $\forall x \in V$, \exists! \text{ integral curve } \gamma_x : (a, b) \to M \text{ for } X \text{ with } \gamma_x(0) = x$,

and the map $\gamma_x : (a, b) \to M$ is smooth.

PF. Regularity for ODEs. \hfill \square

Note: doesn't have to hold on all of $M$ for uniform $(a, b)$.

For $t \in \mathbb{R}$, define local flow $\xi_t X$ : $\phi_t : " M" \to M$

\[ \phi_t(x) = \gamma_x(t) \]

"Time $t$ flow under $X"$

only partially defined for $t \neq 0$.

Where defined: $\phi_t$ is a smooth map, and

\[ \phi_t \circ \phi_{t'} = \phi_{t+t'} \]

Also $\phi_0 = \text{id}$ so $\phi_{-t} = (\phi_t)^{-1}$.

If $M$ is compact, $\phi_t$ is well-defined $\forall t$ and is a diffeomorphism.
Recall: given a vector field $X \in \text{Vect}(M)$ ($p \in M \Rightarrow X_p \in T_pM$), integral curve $\gamma_t(x)$, $\gamma_t'(0) = X_{\gamma_t(0)}$, $\gamma_0(x) = x$.

$\Rightarrow$ time $t$ flow of $X$, $\varphi^t: M \rightarrow M$, $\varphi^t(x) = \gamma_t(x)$.

**Lie Derivative of a vector field**

$X, Y \in \text{Vect} M \Rightarrow L_X Y \in \text{Vect} M$

Given diffeo $\varphi: M \rightarrow N$ and $X \in \text{Vect}(M)$, define the pullback $\varphi^*X \in \text{Vect}(N)$ given by

$$(\varphi^*X)_p = (d\varphi)_p(X_p) \quad \text{where} \quad \varphi(p) = q.$$

Therefore, we'll sometimes write $d\varphi: T_pM \rightarrow T_qN$

For $X, Y \in \text{Vect} M$, define $\varphi^t = \text{flow of } X: M \rightarrow M$

$$(\varphi^{-t})_*(Y) \in \text{Vect}(M).$$

**Def**

$L_X Y = \frac{d}{dt}_{|t=0} (\varphi^{-t})_*(Y)$ is

$$(L_X Y)_p = \frac{d}{dt}_{|t=0} (\varphi^{-t})_* (Y_{\varphi^{-t}(p)}).$$

**Prop**

$L_X Y = [X, Y]$. 

$\triangleright$
Lemma (Similar to last thm) \( V = \pi S a(\rho) \) \( h : (\varepsilon, \varepsilon \varepsilon) \times V \to \mathbb{R} \) smooth,\\ \( h(0, x) = 0 \neq x \). The \( \exists \) smooth \( \varphi : (\varepsilon, \varepsilon) \times V \to \mathbb{R} \) with\\ \( h(t, x) = t \varphi(t, x) \)\\ \( \varphi(0, x) = \frac{\partial h}{\partial x}(0, x) \).

\[ \varphi(t, x) = \int_0^t \frac{\partial h}{\partial x}(s, x) ds. \]

\( \square \)

\[ \text{Pf of Prop} \ f \in C^0(M) : \text{want } [X, Y] f = \mathcal{L}_X Y f. \]

Apply Lemma to \( h(t, x) = f(\varphi(t, x)) - f(x) \):

\[ h(t, x) = t \varphi(t, x), \quad \varphi(0, x) = \frac{\partial h}{\partial x}(0, x) = -Xf(x). \]

Then

\[ (\varphi_{-t})_* \gamma_{\varphi(t, x)} f = \left. \frac{d}{ds} \right|_{s=0} f(\varphi_{-t} \cdot \gamma(s)) = \left. \frac{d}{ds} \right|_{s=0} (f \circ \varphi_{-t} \cdot \gamma(s)) \]

\[ = \gamma_{\varphi(t, x)} (f \circ \varphi_{-t}) = \gamma_{\varphi(t, x)} (f + t \varphi_+ g) \]

\[ = (\gamma f)_{\varphi(t, x)} + t (\gamma g)_{\varphi(t, x)}. \]

So

\[ \mathcal{L}_X Y f = \left. \frac{d}{ds} \right|_{s=0} (\varphi_{-t} + Y_{\varphi(t, x)} f) \]

\[ = \left. \frac{d}{ds} \right|_{s=0} ((Y f)_{\varphi(t, x)} + (Y g)_p) \]

\[ = X(Y f)_{\varphi(t, x)} - Y (X f)(p). \]

\[ = \gamma f \gamma_{\varphi(t, x)}. \]

\( \square \)
Important example: Lie groups

Def. A Lie group is a group $G$ with the structure of a smooth manifold such that the maps:

- left mult $L_g : G \to G$, $g \mapsto hg$
- right mult $R_g : G \to G$, $g \mapsto gh$
- inverse $\iota : G \to G$, $g \mapsto g^{-1}$

are smooth.

Ex: $\mathbb{R}^n$; quotients like $\mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$.

Matrix groups:
- $\text{GL}(n, \mathbb{R}) = \text{open subset of } M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$
- $\text{O}(n) = \{ M \in M_{n\times n}(\mathbb{R}) \mid M^T M = I \}$, $\text{SO}(n)$
- $\text{GL}(n, \mathbb{C})$, $\text{U}(n)$, $\text{SU}(n)$

$h \in G \Rightarrow L_h, R_h$ induce maps $T_G \xrightarrow{L_h^*} T_G, T_G \xrightarrow{R_h^*} T_G$

Def. $X \in \text{Vect } G$ is left/right invariant if $(L_h)_* X = X$ and $(R_h)_* X = X$

$X$ left invariant v.f. is determined by $X_e \in T_e G$: $e = \text{identity } \in G$

$L_g : G \to G$ satisfies $(L_g)_*(X_e) = X_{g^{-1}e}$, $\forall e, g \\

Conversely, any $X_e \in T_e G$ gives rise to a left mut v.f. $X$ defined by

Check: $(L_h)_*(X_e) = X_{h^{-1}e}$

$(L_{\text{id}})_*(X_e) = (\text{id})_* X_e = X_e$

So $\{ \text{left inv. v.f.} \} \xrightarrow{\text{inv}} T_e G$.

Write $\mathfrak{g} = T_e G$ lie algebra assoc to $G$. 

Prop. \( X, Y \) left invt. Then \([X, Y]\) is as well.

PF. From \( \text{HW: } \phi : \mathfrak{g} \rightarrow \mathfrak{g} \) 
\( \phi \circ \phi (X, Y) = \phi (X, Y) \).
\( [X, Y] = \phi ([X, Y]) = \phi (X, Y) \).
Here \( \phi ([X, Y]) = \phi (X, Y) \).

So bracket as vector fields induces \( \mathcal{L} : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \).
This is \( \mathbb{R} \)-bilinear, antisymmetric, and satisfies Jacobi.

Next define \( \psi : \mathfrak{g}_1 \rightarrow \mathfrak{g}_2 \) 
\( \psi (g) = \log (h \cdot g) \).
Since \( \psi (e) = e \), this gives a map
\( (\psi)_0 : T_e \mathfrak{g} \rightarrow T_e \mathfrak{g} \).

\begin{itemize}
\item \( \text{adjoint representation } \quad \text{Ad} : \quad \phi \rightarrow \phi \)
\item \( \text{Note: } \text{if } X \text{ is left invt then } (\psi)_0 \cdot X = \phi (X, \cdot) = \phi (\cdot, X) \).
\item \( \text{Ad} : \quad \phi \rightarrow \phi \) is a linear, Lie algebra map:
\[ \text{Ad} : \quad [X, Y] = [\text{Ad}(X), \text{Ad}(Y)] \]
\item \( \text{Ad} \) is a representation:
\[ \text{Ad} (\phi (X, Y)) = \text{Ad} (\cdot) \text{Ad} (\phi (X, Y)) \]
\end{itemize}

Prop. \( X, Y \in \mathfrak{g} \), \( \phi_t \) = local flow of \( \text{(left invt v.f.)} X \).

\[ [X, Y] = \frac{d}{dt} \bigg|_{t=0} \text{Ad}(\phi_t (e)) Y. \]

PF. \[ [X, Y] = \frac{d}{dt} \bigg|_{t=0} \text{Ad}(\phi_t (e)) Y. \]
Vector Bundles

Idea: generalize tangent bundle $TM = \frac{1}{\text{dim}^n} T_xM \rightarrow M$.

Chart $U \rightarrow \mathbb{R}^n$ for $M$ gives a chart for $TM$.

If we have two charts $V_1, V_2$ and $x \in V_1 \cap V_2$ then we get maps

$$\pi: TM \rightarrow \mathbb{R}^n$$

And the induced map $\mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear, given by the Jacobian matrix

$$\left( \frac{\partial y_i}{\partial x_j} \right) = \left( \frac{\partial y_j}{\partial x_i} \right)^T$$

Def: $M$ smooth mfd. A rank $k$ (real) vector bundle over $M$ is a smooth mfd $E = \frac{1}{\text{dim}^n} E_x$ where each $E_x = \text{rank } k$ vector space over $\mathbb{R}$ such that:

1. The map $\pi: E \rightarrow M$ sending $E_x$ to $x$ is smooth.

2. $E$ has an open cover $\{V_a\}$ of $M$ and diffeomorphisms

$$\phi_a: \pi^{-1}(V_a) \cong V_a \times \mathbb{R}^k \quad \text{"local trivialization of $E"}.$$
Such that \( \varphi_a(Ex) = [x] \times \mathbb{R}^k \): i.e. \( \pi'(U_a) \cong U_a \times \mathbb{R}^k \) commutes

3. The transition functions

\[ h_{x|y} = \varphi_y \circ \varphi_x^{-1} : (V_a \cap V_b) \times \mathbb{R}^k \to (V_a \cap V_b) \times \mathbb{R}^k \]

are smooth and linear: that is, for \( x \in V_a \cap V_b \),

\[ h_{x|y}(x) : \mathbb{R}^k \to \mathbb{R}^k \text{ is in } GL(k). \]

(No different from a fiber bundle with fiber \( \mathbb{R}^k \); the transition functions would be in \( Diff(\mathbb{R}^k) \))

Remark. 1. Write \( \mathbb{R}^k \to G \) a just \( \mathbb{R}^k \to M \)

2. if \( k=1 \) this is a line bundle over \( M \).

3. importantly: can reconstruct \( E \) from the transition functions.

\[ E = \coprod_{x} (V_a \times \mathbb{R}^k) / \sim \]

where

\[ (x,v) \in V_a \times \mathbb{R}^k \sim (y,w) \in V_b \times \mathbb{R}^k \]

if \( x = y \) and \( w = h_{x|y}(v) \).

Example.

1. \( M \times \mathbb{R}^k \) "trivial" vector bundle

2. \( TM \): transition functions look like matrix \( \begin{pmatrix} \partial y_i \end{pmatrix} \).
Def \( E \rightarrow E' \) vector bundles. A map \( \phi : E \rightarrow E' \) is a bundle map if
\[
\phi \circ \pi = \pi',
\]
and for \( x \in M \), \( \phi|_x : E_x \rightarrow E'_x \) is a linear map.

A bundle isomorphism is a bundle map that's invertible, with inverse a bundle map. A vector bundle is trivial if it's isomorphic to \( M \times \mathbb{R}^k \).

Most things associated to a vector bundle are "invariant" under isom. Eg:

**Def** A section of a vector bundle \( E \rightarrow M \) is a smooth map \( s : M \rightarrow E \) with \( \pi \circ s = \text{id} \).

\( \Gamma(E) := \text{vector space of sections of } E \).

\( \rightarrow \) rule: if \( E \cong E' \) then \( \Gamma(E) \cong \Gamma(E') \).

\( \text{Ex. Section of trivial } \mathbb{R}^k \text{ bundle} = \Gamma(\text{smooth maps } M \rightarrow \mathbb{R}^k) \).

\( \Gamma(TM) = \text{Vect}(M) \).

**Operations on Vector Bundles**

Dual: \( E \rightarrow E^* \). \( \text{V.b.} \rightarrow \) replace each \( E \) by \( E^* \) (note \( \cong \) but not canonical)

\( V_1 \rightarrow V_2 \) dualizes to \( V_1^\ast \rightarrow V_2^\ast \).
**Def**: \( E \rightarrow \text{v.s. transition for } h_x \rightarrow \text{dual } E^\ast \text{ v.s. v.s. transition for } (h_x)^{-1} \).

**Ex**: Cokanget bundle \( T^*M \) is the dual to \( TM \).

\[ T^*_x M = \langle \frac{\partial}{\partial x^i}, \ldots, \frac{\partial}{\partial x^n} \rangle \text{ vector space} \]

\[ T^*_x M = \langle dx^1, \ldots, dx^n \rangle \text{ dual basis: } dx_i(\frac{\partial}{\partial x^j}) = \delta_{ij}. \]

Element of \( T^*_x M \) is a cotangent vector.

Two coord charts, \( x, y \rightarrow \text{ recall } \frac{\partial}{\partial y^j} = \frac{\partial x^i}{\partial y^j} \frac{\partial}{\partial x^i} \) (this swaps \( x, y \) from before)

\[ dx_j(\frac{\partial}{\partial y^i}) = \frac{\partial x_j}{\partial y^i} \rightarrow dx_j = \sum \frac{\partial x_j}{\partial y^i} dy^i \]

So the transition for \( \{(x^1, \ldots, x^n) \rightarrow T^*_x M \rightarrow \Sigma x^i x^j \rightarrow (y^1, \ldots, y^n) \} \)

\[ \left( \frac{\partial x^i}{\partial y^j} \right) = \left( \frac{\partial x_j}{\partial y^i} \right)^{-1} \]

**Other operations on vector spaces/bundles**.

\( \oplus \): \( \begin{array}{ccc} E & \oplus & F \\ \downarrow \scriptstyle{\text{rank } k} & \oplus & \downarrow \scriptstyle{\text{rank } k} \\ E_x \oplus F_x \end{array} \)

Transition for \( h_x(x) \in GL(k) \text{ for } E, j_x(x) \in GL(l) \text{ for } F \)

\[ h_x(x) \circledast j_x(x) \in GL(k+l) \text{ for } E \oplus F \]

\( \otimes \): \( \begin{array}{ccc} E \otimes F & \otimes & (E \otimes F)_x = E_x \otimes F_x \\ \downarrow \scriptstyle{\text{rank kl}} & \otimes & \downarrow \scriptstyle{\text{rank } (k+l)} \\ E_x \otimes F_x \end{array} \)

Transition for \( h_x(x) \otimes j_x(x) \in GL(kl) \text{ for } E \otimes F. \)

**Sym**: \( E \rightarrow \text{Sym}^n E \).

\[ (\text{Sym}^n E)_x = \otimes^m E_x / I \]

\[ I = \langle \ldots \otimes \omega \otimes \omega \ldots - \ldots \otimes \omega \otimes \omega \ldots \rangle \]

\( \text{rank } (\otimes^m - ) \text{ vector bundle} \)

\( \Lambda^m \): \( E \rightarrow \Lambda^m E \).

\[ (\Lambda^m E)_x = \otimes^m E_x / I' \]

\[ I' = \langle \ldots \otimes \omega \otimes \omega \ldots + \ldots \otimes \omega \otimes \omega \ldots \rangle \]
In this quotient, usually write $\otimes$ as $\wedge$. So $V \wedge W = \wedge \wedge V \wedge W$, $V \wedge V = 0$.

$\Lambda^m E$ has rank $(m)$, $\Lambda^0 E \cong NE \cong N E \cdots \cong N^k E$. 

Trivial $R$-bundle $E$

\[ R^k \cong \cdots \otimes R \otimes \cdots \otimes R \cong \bigotimes_{i=0}^{p} R = R^p. \]

Bundles from $TM$: Consider the bundle

$TM \otimes \cdots \otimes TM \otimes T^*M \otimes \cdots \otimes T^*M =: T^p M$. 

A section of this bundle is called a $(p, q)$-tensor.

- $(1, 0)$-tensor: vector field in $\text{Vect}(M) = \Gamma(TM)$
- $(0, 1)$-tensor: $1$-form, in $\Omega^1(M) = \Gamma(T^*M)$.
- $(0, 2)$-tensor: section of $T^*M \otimes T^*M$.

At $x$, this is an elt of $T^*_x M \otimes T^*_x M$, i.e. a bilinear map

$T^*_x M \otimes T^*_x M \to \mathbb{R}$

$(\nu \otimes \nu^* = (\nu \circ \nu)^*)$

Important future example: Riemannian metric.

- $(0, m)$-tensor: differential forms are important examples (will treat soon).

Operations on tensors

**Contraction**

$c_{ij} : \Gamma(T^p M) \to \Gamma(T^p_{s-1} M) \quad 1 \leq i \leq p, 1 \leq j \leq q$

This is defined pointwise by the map

$c_{ij} : V \otimes P \otimes \bigotimes_{j \neq i} \nu^* \to V \otimes (\nu_j) \otimes (\nu^*_i) \bigotimes_{j \neq i} \nu^*$

$v_1 \otimes \cdots \otimes \nu_j \otimes \cdots \otimes \nu_q \to \nu_j(v_i) \nu_i \otimes \cdots \otimes \nu_j \otimes \cdots \otimes \nu_q$

**Ex:** $p = q = 1$, $c : V \otimes V^* \to \mathbb{R}$

This is the trace $\text{tr} : \text{End}(V) \to \mathbb{R}$. 
Pullback \[ M \xrightarrow{\phi} N \rightarrow T_xM \xrightarrow{\phi^*_x} T_{\phi(x)}N \]

Dualize \[ T^*N \xrightarrow{\phi^*} T^*_xM \]

This yields a map \[ \phi^*: \Gamma(T^*N) \rightarrow \Gamma(T^*_xM) \]

More generally, \( \phi^* \) gives a map \[ \Gamma(T^*^N) \rightarrow \Gamma(T^*_xM) \]

\[ \alpha \in \Gamma(T^*^N) \text{ means } \alpha \text{ eats } k \text{ vectors } \omega_1, \ldots, \omega_k \in T_yN : \alpha(\omega_1, \ldots, \omega_k) \in \mathbb{R} \]

\[ \phi^*\alpha(\omega_1, \ldots, \omega_k) = \alpha(\phi^*\omega_1, \ldots, \phi^*\omega_k) \]

Note this is only a map of \( \mathbb{R}^k \) tensors: recall \( \phi: M \rightarrow N \) does not give a map \( \text{Vect}(M) \rightarrow \text{Vect}(N) \) (or vice versa).

But: if \( \phi \) is a diffeo, we can define "pullback" \[ \phi^*: \Gamma(TN) \rightarrow \Gamma(TM) \]

\[ X \rightarrow (\phi^*)_X \]

We can extend this to a pullback for any tensors.

\[ \phi^*: \Gamma(T^p_M) \rightarrow \Gamma(T^p_N) \]

\[ v_1 \otimes \ldots \otimes v_p \otimes w_1 \otimes \ldots \otimes w_p \mapsto \phi^*(v_1) \otimes \ldots \otimes \phi^*(w_p) \otimes \]

In particular, suppose \( X \in \text{Vect}(M) \) \( \Rightarrow \phi_t = \text{time } t \text{ flow of } X \).

**Def:** The Lie derivative associated to \( X \) is the linear map

\[ \mathcal{L}_X: \Gamma(T^p_M) \rightarrow \Gamma(T^p_M) \]

\[ \mathcal{L}_X(S) = \frac{d}{dt}|_{t=0} \phi^*_t(S) \cdot \]

**Ex:** \( Y \in \text{Vect}(M) \) \( \Rightarrow \mathcal{L}_XY = [X,Y] \) (Note here \( \phi^*_t = (\phi_t)^* \))
Local Operators and Tensors

**Def.** A local operator is a linear map \( P : \Gamma(E) \to \Gamma(F) \), \( E, F \) vector bundles over \( M \), such that \( \forall x \in M \) and \( \forall U = \text{nbhd of } x \), if \( s, s' \in \Gamma(E) \) satisfy \( s_y = s'_y \) \( \forall y \in U \), then \( (Ps)_x = (Ps')_x \).

"at a point, the operator depends only on the section near that point"

**Ex.** \( X : \text{Vect}(M) \to \Gamma(T^*_p M) \rightleftharpoons \Gamma(T^*_p M) \) is local \( \quad S \mapsto L_x S \)

Special case of local operator: \( P : \Gamma(E) \to \Gamma(F) \) such that if \( s_x = s'_x \) then \( (Ps)_x = (Ps')_x \).

Then \( P \) induces a map \( \text{Ex} \to F_x \) \( \left( \text{elt of } \text{Hom}(E_x, F_x) = E^*_x \otimes F_x \right) \forall x \) \to section of \( E^* \otimes F \). Call \( P \) a tensor. Why?

In particular, suppose \( E = T^*_p M, \ F = T^*_p M \). Such a map \( P \) is a section of \( \left( \bigwedge^p T^*_M \otimes \bigwedge^q T^*_M \right)^* \otimes \left( \bigwedge^r T^*_M \otimes \bigwedge^s T^*_M \right) = \bigwedge^p \bigwedge^q \bigwedge^r \bigwedge^s T^*_M = T^{p+q \cdot r+q}_M. \)

So \( P \) itself is a tensor.

**Ex.** \( \bullet T^*_x : \Gamma(T^*_p M) \to R \) \( \left( \text{contract the } i \text{th contraction in general } \Gamma(T^*_p M) \right) \)

\((T^*_x S)_x \in R \) only depends on \( s_x \): tensor.

\( \bullet L^* : T^*_p M \to T^*_p M \) \( \left( \text{not a tensor.} \right) \)

eg. \( L^*_x : \text{Vect}(M) \to \text{Vect}(M) \) \( (L^*_x Y)_x \) depends on more than \( Y_p \).
Useful characterization of tensors: note $\text{C}^0(M)$ acts on $\Gamma(E)$ by pointwise scalar multiplication: $(fS)_x = f(x)S_x \in \Gamma_x E$.

Prop: For $\pi : E \to M$ a vector bundle, $P : \Gamma(E) \to \text{C}^0(M)$ a local operator. Then $E$ is:
1. A tensor if $S_x = S_x'$ then $(P S)_x = (P S')_x$.
2. A tensor if $\text{C}^0(M)$-linear: $P((fS)) = f P(S)$ for $f \in \text{C}^0(M)$.

PF
1 $\Rightarrow$ 2: Given $f \in \text{C}^0(M)$, $S \in \Gamma(E)$, define section $S' \in \Gamma(E)$ by $S'_x = f(x) S_y$. Then $S_x = (fS)_x$ so $(P(fS))_x = (P(S'))_x = f(x)(PS)_x$.

2 $\Rightarrow$ 1: For $x \in M$, $E$ is "locally trivial": exists $U$ of $x$ st. $\pi^{-1}(U) = U \times \mathbb{R}^N$ ($E \cong \pi^{-1}M$).

Over $U$, $E$ sections $s_1, \ldots, s_N \in \Gamma(E)$ such that $s_1, \ldots, s_N$ generate $E$ pointwise. Now suppose $S, S' \in \Gamma(E)$ with $S_x = S_x'$. Write $S - S' = \sum_{i} f_i \cdot s_i$, $f_i \in \text{C}^0(U)$, $f_i(x) = 0 \forall i$.
The $P(S - S') = P(S) - P(S') = \sum f_i P(s_i)$
so $P(S - S')_x = \sum f_i(x)(PS)_x = 0$. □

Ex. $\mathbb{L}_x : \text{Vect}(M) \to \text{Vect}(M)$.
$\mathbb{L}_x(fy) = \left[ x, fy \right] = x(fy) - f(y(x)) = f$.
$\left[ x, fy \right] g = x(fy(g)) - fy(x(g)) = x(f(y(g)) + f(x(y(g))) - f(y(x(g))) = \left( x(g) f + f[xy](g) \right) g$

Not a tensor!
Ex. Given $\alpha \in \Omega^1(M) = \Gamma(T^*M)$, note this gives a map

$\alpha(\cdot) : \text{Vect}(M) \to \text{Co}^0(M) \text{ (a tensor)}. \text{ Now define a map}$

$\text{d} \alpha : \text{Vect} M \otimes \text{Vect} M \to \text{Co}^0(M)$

by $\text{d} \alpha (X, Y) = X \alpha(Y) - Y \alpha(X) - \alpha([X, Y])$.

This is a tensor in each input:

$\text{d} \alpha(X, fY) = X \alpha(fY) - fY \alpha(X) - \alpha([X, fY])$

$= X(f \alpha(Y)) - f \alpha(X) - \alpha((f[X, Y]) + (Xf)Y)$

$= f \alpha \alpha(X) + f \alpha(Y)$

$= f \alpha \alpha(X, Y)$.

So in fact $(\text{d} \alpha(X, Y))_x$ depends only on $X_x$ and $Y_x$.

Over $x$ this gives a map $T_xX \otimes T_xX \to \mathbb{R}$, so $\text{d} \alpha \in \Gamma(T^2M)$. 

Note $\text{d} \alpha \in \Gamma(T^2M)$.

Differential Forms

First: some linear algebra.

$V = \text{VT}/\mathbb{R}$. A $k$-multilinear form on $V$ is a map

$\varphi : \bigwedge^k V \to \mathbb{R}$

that is linear in each input.

$\text{Hom}(\bigwedge^k V, \mathbb{R}) \cong \bigwedge^k V^* \to \bigwedge^k V^*$

Under this isomorphism $\varphi_1, ..., \varphi_k \in V^*$, then $\varphi_1 \otimes \cdots \otimes \varphi_k (v_1, ..., v_k) = \varphi_1(v_1) \cdots \varphi_k(v_k)$.

Some forms are antisymmetric: $\varphi(-v_i, v_i, ...) = -\varphi(...v_i, v_i, ...)$

So $\varphi(-v_i, -v_j, ...) = -\varphi(...v_j, v_i, ...)$. 
Claim: \[ \{\text{antisymmetric } \bigwedge^k W\} \cong \bigwedge^k V^*. \]

Recall \( \bigwedge^k W = W^k / I \), I generated \(-\omega w_1 \omega \cdots - \omega w_k \omega \cdots \).

Consider the map
\[
\text{Alt}: \bigwedge^k W \to W^k
\]
\[
\text{Alt}(\omega w_1 \cdots \omega w_k) = \sum_{\sigma \in S_k} (-1)^{\sigma} w_{\sigma(w_1)} \cdots w_{\sigma(w_k)}.
\]

E.g., \( \text{Alt}(\omega w_1 \omega w_2) = w_1 w_2 - w_2 w_1. \)

Exercise: \( I = \ker \text{Alt} \) so \( \bigwedge^k W \cong \text{Im} \text{Alt} \).

We can think of \( \text{Im} \text{Alt} \) of \( \bigwedge^k W \) as particular elt of \( W^k \).

In particular, if \( W = V^* \) then an element of \( \bigwedge^k W \) is an elt of \( \bigwedge^k V^* \): a \( \bigwedge^k \) multilinear map.

\[
\bigwedge^k V^* \xrightarrow{\text{Alt}} \bigwedge^k V^*: \text{a } k\text{-multilinear map}
\]

\( k=2 \):
\( \phi_1, \phi_2 \in V^* \Rightarrow \text{Alt}(\phi_1 \phi_2) = (\phi_1 \phi_2 - \phi_2 \phi_1). \)

Note this is antisymmetric: \( (\bigwedge^2 V^*) \cong (V^*)^2 = \bigwedge^2 V^*. \)

In general: \( \text{Im} \text{Alt} = \{\text{antisymmetric } \bigwedge^k \text{multilinear maps}\} \) so \( \bigwedge^k V^* \cong \bigwedge^k \bigwedge^2 V^* \).

Wedge product
For \( \omega \in \bigwedge^k W \otimes W, \eta \in \bigwedge^\ell W \otimes W \), define
\[
\omega \wedge \eta = \frac{1}{k! \ell!} \text{Alt}(\omega \otimes \eta) \in \bigwedge^{k+\ell} W \otimes W.
\]

Note: weird factor is set up so that \( \phi_1 \cdots \phi_k \wedge \phi_{k+1} \cdots \phi_{k+\ell} = \text{Alt}(\phi_1 \otimes \cdots \otimes \phi_k \otimes \phi_{k+1} \otimes \cdots \otimes \phi_{k+\ell}) \) for \( \phi_i \in W \).

Properties:
- \( \wedge \) is bilinear, antisymmetric
- \( \wedge \) is graded commutative: \( \omega \wedge \eta = (-1)^{k \ell} \eta \wedge \omega \).
\( \{ \Omega \in \mathbb{V}^* \text{ such that } \Omega(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} (\text{sgn } \sigma) \Omega(v_{\sigma(1)}(v), \ldots, v_{\sigma(k)}(v)) \} \) is the multilinear form \( \wedge_k \mathbb{V} \).

If \( E \) is a vector bundle over \( M \), then we can define \( \Lambda^k E \).

**Def**: \( \Omega^k(M) := \Gamma(\Lambda^k T^*M) \) is the space of \( k \)-forms on \( M \) (\( R \)-valued \( C^\infty \)-sections).

Locally, a \( k \)-form looks like
\[
\sum_{1 \leq i_1 < \cdots < i_k \leq n} a_{i_1, \ldots, i_k}(x_1, \ldots, x_n) \, dx_{i_1} \wedge \cdots \wedge dx_{i_k} 
\in C^\infty(M)
\]

\( \Omega^k(M) \) is unital for \( 0 \leq k \leq n \):
\[
\Omega^0(M) = C^\infty(M)
\]
\( \Omega^k(M) = \Gamma(T^*M) \):

An element is locally a \( (x_1, \ldots, x_k) \) \( dx_{i_1} \wedge \cdots \wedge dx_{i_k} \) (volume form if \( a \neq 0 \) \( \forall x_1, \ldots, x_n \).

**Def**: \( \Omega^*(M) := \bigoplus \Omega^k(M) \); then \( \wedge \) gives \( \Omega^*(M) \) the structure of a graded-commutative ring.

A \( k \)-form at a point is an antilinear \( k \)-linear form on tangent vectors.
\( \omega \in \Omega^k(M), \ v_1, \ldots, v_k \in T_x M \Rightarrow \omega(v_1, \ldots, v_k) \in \mathbb{R} \).

A \( k \)-form acts on \( k \) vector fields to give a function
\[
\text{Vect}(M)^k \to C^\infty(M)
\]

\( \text{tension in each input} \).

\( \text{Coord } : x_1, \ldots, \omega = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \).
\[
\omega \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = \left( \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} \right) \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right) = 1 - 0 = 1
\]
\[
= 0 - 1 = -1
\]
Prop (see HW) \[ \phi^*(\omega \wedge \eta) = \phi^* \omega \wedge \phi^* \eta \]
\[ L_\phi(\omega \wedge \eta) = (L_\phi \omega) \wedge \eta + \omega \wedge (L_\phi \eta) \]

**Exterior derivative**

\[ f \in C^0(M) \mapsto df \in \Omega^1(M) \text{ defined by } df(x) = X(f) ; \]

in coordinates, \[ df = \sum \frac{\partial f}{\partial x^i} \, dx_i \]

(note in particular if \( f(x_1, \ldots, x_n) = x_i \) then \( df = dx_i \); explain “\( dx_i \)” notation)

Then \( \exists! \) operator \( d: \Omega^k M \to \Omega^{k+1} M \) determined by:

1. for \( f \in C^0(M) \), \( df(x) = X(f) \)
2. \( d(df) = 0 \)
3. for \( \omega \in \Omega^k M, \eta \in \Omega^l M \), \( d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge (d\eta) \).

Furthermore, \( d \) is local and if \( \omega = \sum a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \) then \( d\omega = \sum (da_{i_1 \ldots i_k}) \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \).

**Rule.** 1. if \( d \) satisfies 1, 2, 3 then (4) must hold:

\[ d(a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (da_{i_1 \ldots i_k}) dx_{i_1} \wedge \cdots \wedge dx_{i_k} + a_{i_1 \ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k} \]

2. for \( M \subset \mathbb{R}^n \), just check that (4) satisfies 1, 2, 3.

To extend to all \( M \):

**Lemma** \( \phi: U_1 \to U_2, \ U_1, U_2 \subset \mathbb{R}^n \). Then \( \phi^* \circ d = d \circ \phi^* \):

\[ \begin{array}{ccc}
\Omega^k(U_1) & \xrightarrow{\phi^*} & \Omega^k(U_2) \\
\downarrow d & & \downarrow d \\
\Omega^{k+1}(U_1) & \xrightarrow{\phi^*} & \Omega^{k+1}(U_2)
\end{array} \]

Comment:
Prop \( k=0 \): want \( \varphi^* df = d(\varphi^* \varphi) \) i.e. \( \varphi^* (df)(X) = d(\varphi^* \varphi)(X) \):

\[
\varphi^* (df)(X) = \varphi^* (df)(\varphi_* X)(f) = \left. \frac{d}{dt} \right|_{t=0} (f \circ (\varphi \circ t))(\varphi_* X)(f) - \frac{d}{dt} \left|_{t=0} (\varphi \circ t)(\varphi_* X)(f) \right.
\]

\( \varphi^* (df)(X) = X(\varphi^* \varphi) = \left. \frac{d}{dt} \right|_{t=0} (\varphi \circ t)(\varphi^* \varphi)(X) \). \( \square \)

In general: \( \varphi^* (\omega \wedge \eta) = \varphi^* \omega \wedge \varphi^* \eta \) and induce on \( k \).

Proof of Thm. Atlas \( \{(F_i, U_i, \phi_i) \} \) for \( M \), \( w \in \Omega^k M \).

On \( U_i \), \( \omega \) is given by \( \omega_i \in \Omega^k U_i \): i.e. \( \omega_i = F_i^* \omega \).

\( \text{The collection } \{\omega_i\} \text{ agrees on overlaps: i.e., } \)

\[ (F_j^* - F_i^*)^* \omega_j = \omega_i \quad \text{CHAIN RULE} \]

Conversely, a collection \( \{\omega_i\} \) that agrees on overlaps gives \( \omega \in \Omega^k M \).

But then \( \left( \left( F_j^* - F_i^* \right)^* \omega \right)_j = d \left( F_j^* F_i^* \right)^* \omega_j = d \omega_i \).

So \( \{d \omega_i\} \) agrees on overlaps and gives a well defined \((k-1)\)-form \( d \omega \).

Prop. \( d^2 = 0 \)

- \( d \varphi^* = \varphi^* d \) for any smooth map \( \varphi : M \to N \)
- \( \iota_X d \omega = d (\iota_X \omega) \) \( \rightarrow \text{check locally} \)

- Differentiate previous result

Cartan's (magic) formula

For \( X \in \text{Vect}(M) \), define the interior product

\[ i_X : \Omega^k(M) \to \Omega^{k-1}(M) \quad \text{(sometimes written } X^\bigwedge \text{)} \]

\[ i_X \omega = C_i (X \lrcorner \omega) \]

i.e. \( i_X \omega (X_1, \ldots, X_{k-1}) = \omega (X, X_1, \ldots, X_{k-1}) \).
The (Cartan's magic formula) On $\Omega^k(M)$, \[ L_x = i_x d + d i_x. \]

**Proof**

**Coordinate-free formula for $d$.**

**Proposition:** Let $\omega \in \Omega^k(M)$, $x_0, \ldots, x_k \in \text{Vect } M$. Then

\[
(d\omega)(x_0, \ldots, x_k) = \sum_{i=0}^{k} (-1)^{i} x_i \left( \omega(x_0, \ldots, \hat{x_i}, \ldots, x_k) \right) + \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \hat{x_i}, \ldots, \hat{x_j}, \ldots, x_k).
\]

**Proof**

**Example:** $k=1$. $d\omega(x, y) = x\omega(y) - y\omega(x) - \omega([x, y])$.

We saw this already.

**Quick application of Cartan:**

**Definition:** $\omega \in \Omega^k(M)$ is a **volume form** if $\forall x \in M$, $\omega_x \in \Lambda^k T_x^* M \cong \mathbb{R}$ is nonzero.

**Given $\omega = \text{vol form.}$** Any elt of $\Omega^k(M)$ can be written as $\omega = f \omega$ for $f \in C^\infty(M)$.

**$X \in \text{Vect}(M) \Rightarrow d (i_x \omega) \in \Omega^k(M).$**

**Definition:** The **divergence** of $X$, $\text{div } X \in C^\infty(M)$, is defined by $d (i_x \omega) = (\text{div } X) \omega$.

(See: In $\mathbb{R}^n$, this is the usual divergence.)

**Proof:** $\text{div } X = 0 \iff X$ is volume-preserving: $L_x \omega = 0$. 