Math 621 Homework 9—due Wednesday April 18

Spring 2018

Please note the unusual due date. Here are some other key dates in the schedule for the rest of the semester:

- Monday 4/9: office hour is 11:00–12:00 rather the usual 2:00–3:00
- Wednesday 4/11: no class
- Thursday 4/12: no office hour; instead, office hour Friday 4/13 1:00–2:00
- Monday 4/23: make-up class, 3:05–4:20
- Wednesday 4/25: final HW due; take-home final exam handed out
- Wednesday 5/2: take-home final exam due.

1. (a) Say that a Riemannian manifold M has *constant pointwise sectional curvature* if for each p, the sectional curvature $K(\sigma)$ is independent of $\sigma \subset T_pM$ (but it may depend on p). Use the second Bianchi identity to prove *Schur's Lemma*: any connected manifold with dimension ≥ 3 and constant pointwise sectional curvature actually has constant sectional curvature (over all points).

   (b) If M is a Riemannian 3-manifold, prove that the Ricci tensor $Ric(X, Y)$ completely determines the Riemann curvature tensor.

2. Do Carmo chapter 5 exercise 1, p. 119: let M be a Riemannian manifold with sectional curvature identically zero. Show that, for every $p \in M$, the mapping $\exp_p : B_\epsilon(0) \subset T_pM \to B_\epsilon(p)$ is an isometry, where $B_\epsilon(p)$ is a normal ball at p. (One possible approach involves Jacobi fields.)

3. (a) Let $\mathbb{H}^2 = \{y > 0\} \subset \mathbb{R}^2$ be the hyperbolic plane with metric $\frac{1}{y^2}(dx \otimes dx + dy \otimes dy)$. For an arbitrary point in $\mathbb{H}^2$, show directly via Christoffel symbols that the (unique) sectional curvature is -1.

   (b) As we saw in HW 5, $D^2 = \{x^2 + y^2 < 1\} \subset \mathbb{R}^2$ with the metric $\frac{4}{(1-x^2-y^2)^2}(dx \otimes dx + dy \otimes dy)$ is isometric to $\mathbb{H}^2$. Explain why the geodesics through $(0, 0)$ in D^2 are straight lines (parametrized strangely, but they follow straight radial lines in D^2; the easiest explanation involves symmetry). Then, independent of (a), using Jacobi fields and the Taylor expansion for $|J(t)|^2$, show that the sectional curvature of D^2 at $(0, 0)$ is -1.

(One more problem on the next page.)

4. (Lorentzian model for hyperbolic space.) Define the *Lorentzian metric* $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$ by $\langle x, x \rangle = -x_0^2 + x_1^2 + \cdots + x_n^2$ where $x = (x_0, x_1, \ldots, x_n)$. This is not a Riemannian metric, but it is nondegenerate and symmetric. Define

$$\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = -1,\ x_0 > 0\} = \{x_0^2 = x_1^2 + \cdots + x_n^2 + 1,\ x_0 > 0\}.$$

   (a) Prove that $\langle \cdot, \cdot \rangle$ pulls back to a Riemannian metric on $\mathbb{H}^n$.

   (b) Show that $\mathbb{H}^n$ equipped with this metric is isometric to the Poincaré models of hyperbolic space that we saw in HW 5.

   (c) Let $SO^+(1, n)$ denote the identity component of the group of invertible linear maps on $\mathbb{R}^{n+1}$ that preserve $\langle \cdot, \cdot \rangle$. Show that this acts transitively on $\mathbb{H}^n$. Then show that the isotropy subgroup of $p = (1, 0, \ldots, 0)$ acts transitively on the set of 2-planes in $T_p\mathbb{H}^n$. Deduce from this and problem #3 that $\mathbb{H}^n$ has constant sectional curvature -1.