Math 621 Homework 8—due Friday April 6

Spring 2018

1. (a) A Riemannian manifold is *homogeneous* if for any $p, q \in M$, there is an isometry of M that takes p to q. Show that any homogeneous manifold is geodesically complete.

   (b) Let G be a connected Lie group with a left invariant metric. Show that the map $\exp_e : \mathfrak{g} \to G$ is surjective.

   (c) Prove that there is no biinvariant metric on $SL(2, \mathbb{R})$.

   (Hint for one possible solution using (b): You may use the fact that on $\mathfrak{sl}(2, \mathbb{R})$, the 1-parameter subgroups are of the form $\{\exp(tM)\}$, where $M \in \mathfrak{sl}(2, \mathbb{R})$ and the exponential map on matrices is given by

   $$\exp(M) = 1 + M + \frac{M^2}{2!} + \frac{M^3}{3!} + \cdots .$$

   Now use (b) and the fact that the geodesics of a biinvariant metric are 1-parameter subgroups.)

2. Let ∇ be the Levi-Civita connection on a Riemannian manifold. Please do the following without using coordinates.

   We can view the Riemann curvature tensor as either a $(1,3)$-tensor $(X, Y, Z) \mapsto R(X, Y)Z$, or a $(0,4)$-tensor $(X, Y, Z, W) \mapsto R(X, Y, Z, W)$. For vector fields $X, Y, Z, W, T \in$ *Vect*(M), the *second Bianchi identity* can be written in two ways:

   $$\nabla_X R(Y, Z)W + \nabla_Y R(Z, X)W + \nabla_Z R(X, Y)W = 0$$

   where e.g. $\nabla_X R(Y, Z)W$ denotes the covariant derivative of the $(1,3)$ tensor $(Y, Z, W) \mapsto R(Y, Z)W$ in the direction of X; and

   $$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0$$

   where e.g. $\nabla_X R$ denotes the covariant derivative of the $(0,4)$ tensor R.

   Prove both of these. (It's not too hard to deduce one from the other.)

(One more problem on the back.)

3. This problem involves sectional curvature, which we probably won't discuss in class until Wednesday April 4, but all you need to know about sectional curvature is the definition (see the book or my notes).

(a–c) do Carmo chapter 4 exercise 1, p. 103. (This problem has three parts but they're all short.)

(d) If M is a Riemannian 2-manifold, the unique sectional curvature at each point p is called the *Gaussian curvature* $K(p)$. Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function, and let M be the surface in $\mathbb{R}^3$ given by the graph of F, $\{z = F(x, y)\}$, with the metric induced from $\mathbb{R}^3$. Establish the formula

$$K = \frac{F_{xx}F_{yy} - F_{xy}^2}{(1 + F_x^2 + F_y^2)^2}$$

for the Gaussian curvature at a point $(x, y, F(x, y))$, where $F_x = \partial F/\partial x$, $F_{xx} = \partial^2 F/\partial x^2$, etc. (First check that the metric on M is given by the matrix $(g_{ij}) = \begin{pmatrix} 1+F_x^2 & F_xF_y \\ F_xF_y & 1+F_y^2 \end{pmatrix}$.)

(e) Let $a > 0$ be fixed. Check that the Gaussian curvature of $z = \sqrt{a^2 - x^2 - y^2}$ at $(0, 0, a)$ is $1/a^2$. Explain why this implies that the round sphere $x^2 + y^2 + z^2 = a^2$ in $\mathbb{R}^3$ has constant Gaussian curvature $1/a^2$.