Math 621 Homework 7—due Friday March 30

Spring 2018

1. Let ∇ be an affine connection on M, and let $X \in \mathrm{Vect}(M)$.

   (a) Prove that ∇ extends to a unique linear operator

   $$\nabla_X : \Gamma(T^p_q(M)) \to \Gamma(T^p_q(M))$$

   for all $p, q \geq 0$ (where ∇_X is initially given for $(p, q) = (1, 0)$), satisfying the following properties:

   i. for any tensors S, T on M,

   $$\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T);$$

   ii. for any (p, q)-tensor S on M with $p, q > 0$, and any contraction c_{ij},

   $$\nabla_X(c_{ij}(S)) = c_{ij}(\nabla_X(S)).$$

   (b) If S is a $(0, q)$-tensor on M, and $X, X_1, \ldots, X_q \in \mathrm{Vect}(M)$, prove that:

   $$(\nabla_X S)(X_1, \ldots, X_q) = X(S(X_1, \ldots, X_q)) - \sum_{i=1}^{q} S(X_1, \ldots, X_{i-1}, \nabla_X X_i, \ldots, X_q).$$

   Then give a similar formula for $\nabla_X S$ if S is a $(1, q)$-tensor.

   Remarks (not to be proven):

   - if we allow X to vary, then we obtain a map $\nabla : \Gamma(T^p_q(M)) \to \Gamma(T^p_{q+1}(M))$, called the "covariant derivative on tensors";
   - the condition for an affine connection ∇ to be compatible with a metric g (viewed as a $(0, 2)$-tensor) is precisely that $\nabla g = 0$.

2. (a) Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For $X, Y, Z \in \mathrm{Vect}(M)$, prove that:

   $$(\mathcal{L}_X g)(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y).$$

   (b–d) Do Carmo chapter 3 exercise 5(b,c,d), pp. 81–82. (For the $\Rightarrow$ direction of (d), you could follow the hint in the book, but it's easier just to use part (a) above.)

3. Do Carmo chapter 3 exercise 7, p. 83. Hint: for standard coordinates $x^1, \ldots, x^n$ on T_pM, define vector fields $E_i = (\exp_p)_*(\partial/\partial x^i)$. Show that $\nabla_{E_i} E_j(p) = 0$ for all i, j, and then use Gram–Schmidt to modify E_i to get orthonormality.