Math 621 Homework 6—due Friday March 23

Spring 2018

1. (a) Let ∇ be a connection on M. Prove that the map $\mathrm{Vect}(M)\times\mathrm{Vect}(M) \to \mathrm{Vect}(M)$ defined by $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is a tensor in each of X and Y. This map is called the *torsion* of ∇.

   (b) Prove that the space of affine connections on M is an affine space modeled on the vector space $\Gamma(T^{1,2}M)$ of $(1,2)$-tensors on M. (That is, if we fix one affine connection, then all other affine connections are the sum of this and some $(1,2)$-tensor, and any such sum produces an affine connection.)

   (c) In class, we proved the uniqueness of the Levi-Civita connection on a Riemannian manifold. Prove that this connection exists, as follows. For fixed $X, Y \in \mathrm{Vect}(M)$, show that the map from $\mathrm{Vect}(M)$ to $C^\infty(M)$ given by

   $$Z \mapsto \frac{1}{2}\left(X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle + \langle [X,Y],Z\rangle\right)$$

   is a tensor. Use this to define $\nabla_X Y \in \mathrm{Vect}(M)$ such that the above map sends Z to $\langle \nabla_X Y, Z\rangle$. Show that ∇ satisfies the properties of a connection, is torsion-free, and is compatible with the metric.

2. (a) Read do Carmo chapter 2 exercise 3, p. 57. In the notation from do Carmo, show that $[\overline{X},\overline{Y}]$ agrees with $[X,Y]$ at all points of U. Use this to prove that $\nabla_X Y$ as defined in the do Carmo problem, as a vector field on U, is independent of the extensions $\overline{X},\overline{Y}$ of X,Y.

   (b) Now solve do Carmo chapter 2 exercise 3.

3. Let $\mathbb{H}^2 = \{y > 0\} \subset \mathbb{R}^2$ be the hyperbolic plane with metric $\frac{1}{y^2}(dx\otimes dx + dy \otimes dy)$.

   (a) Calculate the Christoffel symbols Γ_{ij}^k for $1 \le i,j,k \le 2$ (where x, y play the role of coordinates x^1, x^2; you can check your answer against p. 58 of the book).

   (b) Show that the following curves in $\mathbb{H}^2$ are geodesics: $\gamma(t) = (a, e^t)$ for fixed $a \in \mathbb{R}$; and

   $$\gamma(t) = \left(a + r\left(\frac{1-e^{2t}}{1+e^{2t}}\right), \frac{2re^t}{1+e^{2t}}\right)$$

   for fixed $a \in \mathbb{R}$ and $r > 0$. (It's easiest to do the calculation directly. As a side note, though, do Carmo pp. 73–74 has an approach that avoids calculation by using the fact that geodesics map to geodesics under isometries.)

   (c) The geodesics in (b) are defined for all t and trace out the vertical line in $\mathbb{H}^2$ with $x = a$ and the semicircle of radius r centered at $(a, 0)$, respectively. Explain why this gives an exhaustive list, up to reparametrization, of all (nonconstant) geodesics on $\mathbb{H}^2$.

(One more problem on the next page.)

4. (a) Do Carmo chapter 3 exercise 3(b), pp. 80–81. (You've already done 3(a).)

   (b) Do Carmo chapter 3 exercise 5(a), p. 81.

   (c) Consider S^n with the round metric as a submanifold of $\mathbb{R}^{n+1}$ with coordinates $x^1, \ldots, x^{n+1}$. Show that for any i, j ($1 \leq i, j \leq n+1$), the vector field $x^i \partial_j - x^j \partial_i$ is a Killing field on S^n. (Note this vector field is defined on $\mathbb{R}^{n+1}$; as a preliminary step in this problem, you should explain why it induces a vector field on S^n, in the sense of being tangent to S^n at every point in S^n.)