Math 621 Homework 5—due Wednesday March 7

Spring 2018

As a reminder, we will not have class on Wednesday February 28. Consequently, this problem set isn't due for two weeks. However, we'll have covered all the material you need to solve these problems by February 23 at the latest, and I strongly encourage you to work on these as soon as possible, before we move onto other things in class.

1. Prove Cartan's magic formula:

$$\mathcal{L}_X = di_X + i_X d$$

where both sides act on $\Omega^*(M)$.

Hint: first check that the formula holds when applied to $C^\infty(M)$, and that it holds when applied to dx_i where x_i is a local coordinate.

2. (a) Use HW 4 #3(b) and Cartan's magic formula to prove the following coordinate-free formula for the exterior derivative:

$$\begin{aligned} d\omega(X_0, \ldots, X_k) &= \sum_i (-1)^i X_i \left(\omega(X_0, \ldots, \widehat{X_i}, \ldots, X_k) \right) \\ &\quad + \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_k), \end{aligned} \tag{1}$$

where $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathrm{Vect}(M)$.

(Note that for $k = 1$, this agrees with the definition of $d\omega$ that we saw in class when we were discussing local operators and tensors.)

(b) Suppose that we used equation (1) to define $d\omega$. Check that, by this definition, $d\omega$ is indeed a $(k+1)$-form; that is, check that it determines a well-defined section of the bundle $\Lambda^{k+1}T^*M$.

(c) If d is defined by (1), check that $d^2\omega = 0$ for $\omega \in \Omega^0(M)$ or $\omega \in \Omega^1(M)$. (The general proof that $d^2 = 0$ is similar but more involved.)

(More problems on the next page.)

3. (Poincaré models for hyperbolic n-space.) Let $D^n = \{x_1^2 + \cdots + x_n^2 < 1\} \subset \mathbb{R}^n$, and let $H^n = \{x_1 > 0\} \subset \mathbb{R}^n$.

   (a) Consider the map ϕ on D^n given by
   $$\phi(x) = p + \frac{2(x-p)}{\|x-p\|^2},$$
   where x is viewed as a vector in $\mathbb{R}^n$, p is the vector $(-1, 0, \ldots, 0)$, and $\|\cdot\|$ is the usual norm on vectors. Prove that ϕ is a diffeomorphism from D^n to H^n.

   (b) Let δ_{ij} denote the usual Kronecker delta function (1 if $i = j$, 0 otherwise). Define metrics g on D^n and h on H^n by
   $$\begin{aligned} g(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}) &= \frac{4\delta_{ij}}{(1-\|x\|^2)^2} \\ h(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j}) &= \frac{\delta_{ij}}{x_1^2}. \end{aligned}$$
   Prove that ϕ is an isometry between (D^n, g) and (H^n, h). Either of these Riemannian manifolds is commonly called "hyperbolic n-space".

4. Let G be a connected Lie group, and let $\langle\ ,\ \rangle_e$ be a symmetric, positive definite bilinear form on $\mathfrak{g}$, extending to a left invariant Riemannian metric $\langle\ ,\ \rangle$ on G. We showed in class that if $\langle\ ,\ \rangle$ is bi-invariant, then
   $$0 = \langle [X,Y], Z\rangle_e + \langle Y, [X,Z]\rangle_e$$
   for all $X, Y, Z \in \mathfrak{g}$. Prove the converse.

   Hint: one approach is to let $\gamma(t)$ be a path in G starting at e, and consider how $\langle \mathrm{Ad}(\gamma(t))Y, \mathrm{Ad}(\gamma(t))Z\rangle$ depends (or doesn't depend) on t, where $Y, Z \in \mathfrak{g}$. Also, if it helps, as in HW 3 you can use the fact that G is generated by $\{\exp(X) \mid X \in \mathfrak{g}\}$.

5. (a) Since $GL(n,\mathbb{R})$ is an open subset of the Euclidean space $M_{n\times n}(\mathbb{R}) \cong \mathbb{R}^{n^2}$, the Lie algebra of the Lie group $GL(n,\mathbb{R})$ is $\mathfrak{gl}(n,\mathbb{R}) = M_{n\times n}(\mathbb{R})$. We can think of the Lie groups $SO(n)$ and $SL(n,\mathbb{R})$ as being submanifolds of $GL(n,\mathbb{R})$. Under this identification, show that the Lie algebra $\mathfrak{so}(n)$ is the vector space of skew-symmetric $n \times n$ matrices, and find a similar identification for $\mathfrak{sl}(n,\mathbb{R})$.

   (b) For parts (b), (c), and (d), let G be any of the Lie groups $GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, or $SO(n)$. For $n\times n$ matrices $M \in G$ and $X \in \mathfrak{g}$, show that $Ad(M)X = M \cdot X \cdot M^{-1}$, where $\cdot$ is usual matrix multiplication.

   (c) Let $X, Y \in \mathfrak{g}$. Show that the Lie bracket $[X,Y] \in \mathfrak{g}$ is given by $[X,Y] = X \cdot Y - Y \cdot X$.

   (d) Show that the bilinear form $\langle\ ,\ \rangle_e$ on $\mathfrak{g}$ defined by $\langle X, Y\rangle_e = \mathrm{tr}(X^T \cdot Y)$ is symmetric and positive definite, where tr denotes trace.

   (e) Show that $SO(n)$ has a bi-invariant Riemannian metric. (Note: it turns out that $GL(n,\mathbb{R})$ and $SL(n,\mathbb{R})$ do not!)