Math 621 Homework 4—due Monday February 19

Spring 2018

As a reminder:

- $+$: we will have classes on **Monday 2/12** and **Monday 2/19**
- $-$: we will not have class *Wednesday 2/14* or *Friday 2/16* (or, further ahead, *Wednesday 2/28*).

I am out of town 2/14–2/16 and 2/26–3/1 so I also won't have my usual office hours during those periods.

1. (a) Let E be a rank n vector bundle over a smooth manifold M. Prove that the following are equivalent:
   - E is trivial, meaning that it's bundle isomorphic to the trivial rank n vector bundle over M
   - there are n sections $\sigma_1, \ldots, \sigma_n \in \Gamma(E)$ that are linearly independent, in the sense that for all $x \in M$, $\sigma_1(x), \ldots, \sigma_n(x)$ form a basis for the vector space E_x.

   (b) Prove that the Möbius strip is a line bundle over S^1, and that this line bundle is not trivial.

   (c) A manifold M is called *parallelizable* if TM is trivial. Prove that any Lie group is parallelizable.

   Remark: Let Σ be any compact orientable surface except for T^2 (for example, S^2). By the "hairy ball theorem" or more generally the Poincaré–Hopf Theorem, any smooth vector field on Σ has a zero. It follows that Σ is not parallelizable.

(More problems on the next page.)

The rest of this problem set concerns the Lie derivative. You may use the facts that if $\phi : M \to N$ is a smooth map, then

$$\phi^*(S \otimes T) = \phi^*(S) \otimes \phi^*(T)$$

for any tensors S, T on N, and

$$\phi^*(c_{ij}(S)) = c_{ij}(\phi^*(S))$$

for any $1 \leq i \leq p$, $1 \leq j \leq q$, and $S \in \Gamma(T^p_q(N))$, where $c_{ij} : \Gamma(T^p_q(M)) \to \Gamma(T^{p-1}_{q-1}(M))$ denotes contraction. (You should convince yourself that these identities hold, but don't submit written proofs.)

2. (a) Let X be a smooth vector field on M. Prove that $\mathcal{L}_X$ satisfies the following properties:

   i. For any $f \in C^\infty(M) = \Gamma(T^0_0(M))$,
   $$\mathcal{L}_X f = Xf.$$

   ii. For any $Y \in \mathrm{Vect}(M)$,
   $$\mathcal{L}_X Y = [X, Y].$$

   iii. For any tensors S, T on M,
   $$\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T).$$

   iv. For any (p, q)-tensor S on M with $p, q > 0$, and any contraction c_{ij},
   $$\mathcal{L}_X(c_{ij}(S)) = c_{ij}(\mathcal{L}_X S).$$

   (Some of these may be obvious.)

   (b) Let P_X be a linear operator on tensors, i.e., a linear map $\Gamma(T^p_q(M)) \to \Gamma(T^p_q(M))$ for all $p, q \geq 0$. Prove that if P_X satisfies properties (i) and (iii) from part (a), with P_X in place of $\mathcal{L}_X$, then P_X is a local operator: that is, if $U \subset M$ is an open set and S, T are tensors on M with $S = T$ on U, then $P_X S = P_X T$ on U.

3. (a) Prove that $\mathcal{L}_X$ is the unique linear operator on tensors satisfying properties (i) through (iv) from problem 2(a).

   (b) Let $S \in \Gamma(T^0_q(M))$, so that if $Y_1, \ldots, Y_q$ are in $\mathrm{Vect}(M)$ then $S(Y_1, \ldots, Y_q)$ is in $C^\infty(M)$. Let $X \in \mathrm{Vect}(M)$. Prove that the Lie derivative $\mathcal{L}_X S \in \Gamma(T^0_q(M))$ satisfies:

   $$\begin{aligned}(\mathcal{L}_X S)(Y_1, \ldots, Y_q) &= X\left(S(Y_1, \ldots, Y_q)\right) - S([X, Y_1], Y_2, \ldots, Y_q) \\ &\quad - S(Y_1, [X, Y_2], \ldots, Y_q) - \cdots - S(Y_1, Y_2, \ldots, [X, Y_q]).\end{aligned}$$