Math 621 HW 3 — Outline of Solutions

Note Title 2/8/2018

1. (a) $X = x^2 \frac{\partial}{\partial x}$ works. The integral curve γ_{x_0} with $\gamma_{x_0}(0) = x_0$ is $\gamma(t) = \frac{x_0}{1 - t x_0}$ (note $\gamma'(t) = \frac{x_0^2}{(1-tx_0)^2} = (\gamma(t))^2$) and this is defined for $t \in (-\infty, \frac{1}{x_0})$ $(x_0 > 0)$ or $t \in (\frac{1}{x_0}, \infty)$ $(x_0 < 0)$. There's no $t \neq 0$ in all of these intervals.

(b) For all $p \in K$, choose V_p neighborhood of p and $\epsilon(p) > 0$ such that if $x \in V_p$, $\varphi_t(x)$ is defined for $|t| < \epsilon(p)$. Since K is compact, the open cover $\{V_p\}$ has a finite subcover. Thus there is some $\epsilon > 0$ such that if $x \in K$, $\varphi_t(x)$ is defined for $|t| < \epsilon$. Since $\varphi_t = \mathrm{id}$ outside of K, $\varphi_t : M \to M$ is defined for $|t| < \epsilon$. Then compose $\varphi_t = \varphi_{t_1} \circ \cdots \circ \varphi_{t_n}$ for $|t_i| < \epsilon$ to get φ_t defined for all $t \in \mathbb{R}$.

Since φ_t is smooth and $\varphi_t^{-1} = \varphi_{-t}$, φ_t is a diffeomorphism.

2. (a) For $p \in M$,

$$\psi_*[X,Y](f)(\psi(p)) = [X,Y](f\circ\psi)(p)$$
$$= X(Y(f\circ\psi))(p) - Y(X(f\circ\psi))(p)$$
$$= [\psi_* X, \psi_* Y](f)(\psi(p)).$$

(b) The answer is $\boxed{\psi\circ\varphi_t\circ\psi^{-1}}$.

We need $\frac{d}{dt}(\psi\circ\varphi_t\circ\psi^{-1})(p) = (\psi_*(X))_{\psi\circ\varphi_t\circ\psi^{-1}(p)}$ <u>for all</u> t.

Note $\frac{d}{dt}\Big|_{t=0} \psi\varphi_t\psi^{-1}(p) = \psi_*\left(\frac{d}{dt}\Big|_{t=0}\varphi_t(\psi^{-1}(p))\right)$
$$= \psi_*(X_{\psi^{-1}(p)})$$
$$= (\psi_* X)_p.$$

Then for arbitrary t_0,

$$\frac{d}{dt}\Big|_{t=t_0}\psi\varphi_t\psi^{-1}(p) = \frac{d}{dt}\Big|_{t=t_0}(\psi\varphi_{t-t_0}\psi^{-1})(\psi\varphi_{t_0}\psi^{-1}(p))$$
$$= \frac{d}{dt}\Big|_{t=0}(\psi\varphi_t\psi^{-1})(\psi\varphi_{t_0}\psi^{-1}(p))$$
$$= (\psi_* X)_{\psi\varphi_{t_0}\psi^{-1}(p)},$$

as desired.

(c) <u>$\Leftarrow$</u>: assume φ_t, ψ_s commute. From (b), the time s flow of $(\varphi_t)_* Y$ is $\varphi_t\circ\psi_s\circ\varphi_t^{-1} = \psi_s$. Since this is independent of t, $(\varphi_t)_* Y$ is independent of t, so $[X,Y] = \frac{d}{dt}\Big|_{t=0}(\varphi_{-t})_* Y = 0$.

<u>$\Rightarrow$</u>: assume $[X,Y]=0$. Then for any t_0,

$$\frac{d}{dt}\Big|_{t=t_0}((\varphi_{-t})_* Y)_p = \frac{d}{dt}\Big|_{t=t_0}(\varphi_{-t})_* Y_{\varphi_t(p)}$$
$$(t \mapsto t+t_0) \longrightarrow \quad = \frac{d}{dt}\Big|_{t=0}(\varphi_{-t_0})_*(\varphi_{-t})_* Y_{\varphi_t(\varphi_{t_0}(p))}$$
$$= (\varphi_{-t_0})_*\frac{d}{dt}\Big|_{t=0}(\varphi_{-t})_* Y_{\varphi_t(\varphi_{t_0}(p))}$$
$$= (\varphi_{-t_0})_*[X,Y]_{\varphi_{t_0}(p)}$$
$$= 0.$$

Thus $(\varphi_{-t})_* Y$ is independent of t. It follows that $\varphi_t^{-1}\circ\psi_s\circ\varphi_t$, which by (b) is the time s flow of $(\varphi_{-t})_* Y$, is independent of t (for fixed s). At $t=0$, this is ψ_s; thus $\varphi_t^{-1}\circ\psi_s\circ\varphi_t = \psi_s$, as desired.

3. (a) If $\gamma_X(t)$ is defined on some fixed interval $(-\epsilon, \epsilon)$ around 0, then for any $g \in G$, $L_g \gamma_X(t)$ is defined on $(-\epsilon, \epsilon)$ and is an integral curve for $L_g^* X = X$. In particular, if $\gamma_X(t)$ is also defined at some t_0, then $L_{\gamma_X(t_0)} \gamma_X(t - t_0)$ defines $\gamma_X(t)$ for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

It's easy to conclude that $\gamma_X(t)$ is defined for all $t \in \mathbb{R}$.

By this same argument, for all t, t_0 we now have

$$\gamma_X(t_0)\gamma_X(t - t_0) = L_{\gamma_X(t_0)} \gamma_X(t - t_0) = \gamma_X(t).$$

(b) From (a), since $L_g \gamma_X(t)$ is an integral curve for X and $L_g \gamma_X(0) = g$, $L_g \gamma_X(t) = \varphi_t(g)$.

(c) $\Leftarrow$: If $\mathfrak{g}$ is abelian, then for any $X, Y \in \mathfrak{g}$, $[X, Y] = 0$, so from 1(c), $\varphi_t \psi_s = \psi_s \varphi_t \ \forall s, t \Rightarrow$ from 2(b), $\exp(tX)\exp(sY) = \exp(sY)\exp(tX)$

$\Rightarrow \exp X, \exp Y$ commute. Since $\{\exp X\}$ generates G, G is abelian.

$\Rightarrow$: If G is abelian, then the same argument shows that if $X, Y \in \mathfrak{g}$, then $\varphi_t \psi_s = \psi_s \varphi_t \ \forall s, t \Rightarrow [X, Y] = 0$, so $\mathfrak{g}$ is abelian.