Math 621 Homework 3—due Friday February 9

Spring 2018

This problem set covers material up through the extra lecture on Monday February 5.

1. (a) Find a smooth vector field X on $\mathbb{R}$ such that for *every* $t \neq 0$, the local flow ϕ_t fails to be defined on all of $\mathbb{R}$.

   (b) Let M be a smooth manifold, and let X be a vector field on M with compact support: that is, there is a compact set $K \subset M$ such that if $p \notin K$, then $X_p = 0$. Prove that in this case, ϕ_t is defined on all of M for all $t \in \mathbb{R}$, and that $\phi_t : M \to M$ is a diffeomorphism for all t. (It follows that ϕ_t is a one-parameter family of diffeomorphisms of M.)

2. (a) Let $\psi : M \to N$ be a diffeomorphism, and let X, Y be vector fields on M, with corresponding vector fields $\psi_* X, \psi_* Y$ on N. Prove that

$$[\psi_* X, \psi_* Y] = \psi_* [X, Y].$$

   (b) Let ψ be as in (a), and let $\phi_t : M \to M$ denote the time t flow of a vector field X on M. Express the time t flow of $\psi_* X$ in terms of ψ and ϕ_t.

   (c) Let X, Y be vector fields on M, and let ϕ_t, ψ_t denote time t flow for X, Y respectively. Prove that $[X, Y] = 0$ (i.e., the Lie bracket of X and Y is identically zero) if and only if ϕ_t and ψ_s commute for all $s, t \in \mathbb{R}$.

   Remark: For general vector fields X, Y, the map $\phi_t \circ \psi_t \circ \phi_t^{-1} \circ \psi_t^{-1}$ applied to a fixed point p gives a curve in M as t varies. At $t = 0$, the derivative of this curve is 0; but its second derivative at $t = 0$ is related to the value of $[X, Y]$ at p.

3. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra.

   (a) For $X \in \mathfrak{g}$ (viewed as a left invariant vector field), let $\gamma_X(t)$ be the integral curve for X with $\gamma_X(0) = e$. Prove that $\gamma_X(t)$ is defined for all $t \in \mathbb{R}$ and that $\gamma_X : \mathbb{R} \to G$ is a group homomorphism. (Compare do Carmo chapter 3 exercise 3, p. 80.)

   The element $\gamma_X(1) \in G$ is usually written as $\exp X$, and we have $\gamma_X(t) = \exp(tX)$ for all t (convince yourself that this is true if it isn't clear). The notation comes from the fact that if $G = GL(n, \mathbb{R})$ and $X \in \mathfrak{g} = M_{n \times n}(\mathbb{R})$, then $\exp X = I + X + X^2/2! + X^3/3! + \cdots$ is the usual exponential for matrices.

   (b) Let $\phi_t : G \to G$ denote time t flow for (the left invariant vector field) $X \in \mathfrak{g}$. Show that $\phi_t(g) = g \exp(tX) (= g\phi_t(e))$ for all $t \in \mathbb{R}$ and $g \in G$.

   (c) Assume that G is connected. In this case, G is generated as a group by the set $\{\exp(X) \mid X \in \mathfrak{g}\}$. (You don't have to prove this, but it's a worthwhile thing to think about.)

   Given this fact, prove that G is abelian if and only if $\mathfrak{g}$ is abelian (i.e., the Lie bracket on $\mathfrak{g}$ is identically zero). (Hint: use 2(c).)