1. (a) Extend \( \delta \) to \( C^0(M) \) by \( \delta(f) = \delta(f-f(p)) \) where \( f, f(p) \in F_p \). 
(Note if \( \delta \) is a derivation then this must hold since \( \delta(\epsilon) = 0 \) for \( \epsilon \) constant.) 
This \( \delta \) is a derivation: if \( f, g \in C^0(M) \) then 
\[
\delta(fg) = \delta((fg-f(p) g(p)) = \delta((f-f(p)) g(p) + f(p) (g-g(p)) + (f-f(p)) g(p)) \\
= \delta(f g(p)) + \delta(f) \delta(g) + 0.
\]

(b) Define \( \{ \text{derivations} \} \to (F_p/F_p^2)^* \) by sending \( \delta \) to \( \delta|_{F_p} : F_p \to \mathbb{R} \). This is well-defined since if \( f, g \in F_p \) then \( \delta(fg) = 0 \) by Leibniz, so \( \delta|_{F_p^2} = 0 \). 
The map is surjective by (a) and injective by uniqueness in (a).

2. If we show that \( \delta \) is surjective, then it's an isomorphism by dimension counting.

Let \( v_m \in T_p M \) and \( v_n \in T_q N \), and let \( \gamma_m, \gamma_n \) be curves in \( M, N \) with \( \gamma_m(0) = p, \gamma_n(0) = q \) such that \( \gamma_m(0) = v_m, \gamma_n(0) = v_n \). 
Then we can define a curve \( \gamma = (\gamma_m, \gamma_n) \) in \( M \times N \), 
and \( \gamma_m = \pi_{M} \circ \gamma, \gamma_n = \pi_{N} \circ \gamma \). Then by definition if differential, 
\[
(\text{d} \pi_M)|_{(0, 0)} \left[ \gamma \right] = [\gamma_m] = v_m
\]
and \( (\text{d} \pi_N)|_{(0, 0)} \left[ \gamma \right] = [\gamma_n] = v_n \); so if we define \( v = \gamma'(0) \in \mathbb{R}^{\text{dim}(M \times N)} \) then \( \text{d} \pi_M(v) = v_m, \text{d} \pi_N(v) = v_n \). This proves surjectivity.

Another solution: let \( \bar{U}_1, \bar{F}_2 \circ \bar{U}_1, \bar{U}_2, \bar{F}_2 \circ \bar{U}_2 \) be charts for \( p \in M, q \in N \); then 
\( \bar{U}_1 \times \bar{U}_2 = \bar{F}_2(\bar{U}_1 \times \bar{U}_2) \) is a chart for \( (p, q) \in M \times N \). Then \( \bar{F}_2^{-1} \circ \pi_{M \times N} \circ F : \bar{U}_1 \times \bar{U}_2 \to \bar{U}_1 \) 
is projection, so in coordinates, \( \text{d} \pi_M = \begin{bmatrix} I & 0 \end{bmatrix} \). Similarly \( \text{d} \pi_N = \begin{bmatrix} 0 & I \end{bmatrix} \) 
and so \( (\text{d} \pi_M, \text{d} \pi_N) \) is the identity matrix in coordinates.
3. Let \((F_x, U_x, V_x), (F_p, U_p, V_p)\) be overlapping charts on \(\mathbb{M}\),
giving rise to charts \((\tilde{F}_x, U_x \times \mathbb{R}^n, \tilde{V}_x), (\tilde{F}_p, U_p \times \mathbb{R}^n, \tilde{V}_p)\) on \(TM\).

If we have coordinates \((x_1, \ldots, x_n)\) on \(U_x\), \((y_1, \ldots, y_n)\) on \(U_p\)

\[
\begin{pmatrix}
(x_1, \ldots, x_n, v_1, \ldots, v_n) \text{ on } U_x \times \mathbb{R}^n,
(y_1, \ldots, y_n, w_1, \ldots, w_n) \text{ on } U_p \times \mathbb{R}^n
\end{pmatrix}
\]

then

\[
\begin{pmatrix}
y_1, \ldots, y_n, w_1, \ldots, w_n
\end{pmatrix} =
\begin{pmatrix}
(F_{x_1}^{y_1} F_{x_2}^{y_2} \cdots F_{x_n}^{y_n})(x_1, \ldots, x_n),
\frac{d}{d F_{x_1}^{y_1} F_{x_2}^{y_2} \cdots F_{x_n}^{y_n}}(v_1, \ldots, v_n)
\end{pmatrix}
\]

and the Jacobian for this map is of the form

\[
\begin{bmatrix}
d(F_{x_1}^{y_1} F_{x_2}^{y_2} \cdots F_{x_n}^{y_n}) & 0 \\
& d(F_{x_1}^{y_1} F_{x_2}^{y_2} \cdots F_{x_n}^{y_n})
\end{bmatrix}
\]

which has determinant \((\det d(F_{x_1}^{y_1} F_{x_2}^{y_2} \cdots F_{x_n}^{y_n}))^2 > 0\).

Thus \(\{(\tilde{F}_x, U_x \times \mathbb{R}^n, \tilde{V}_x)\}\) is an essential atlas for \(TM\).

4. First note that the functions \(r^\alpha x_1\) and \(r^\alpha x_2\) extend continuously
to \((x_1, x_2) = (0, 0)\) iff \(\alpha > -1\), and extend smoothly iff \(\alpha > 0\).

If \(-1 < \alpha < 0\) then \(\frac{3}{\alpha} \left| \frac{x_1}{x_2} \right|^{\frac{1}{\alpha}} \frac{x_1}{x_2} \to \infty\). Thus the vector field

\[
X := r^\alpha \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right)
\]
on \(\mathbb{R}^2\) for \((x_1, x_2) \neq (0, 0)\) extends smoothly to \(\mathbb{R}^2\) iff \(\alpha > 0\).

On \(S^2\), this is the same as extending smoothly from \(S^2 - \{(0, 0)\}\) to the south pole.

To determine when \(X\) extends smoothly to the north pole, we need to

write \(X\) in the other coordinate chart \((y_1, y_2)\), \(y_1 = \frac{x_1}{x_2}, y_2 = \frac{x_2}{x_2}\):

\[
\frac{\partial}{\partial y_1} = \frac{\partial}{\partial x_1} \frac{x_2}{x_1} + \frac{\partial}{\partial x_2} \frac{x_1}{x_2} = \frac{1}{r^2} \left( \frac{x_2}{x_1} \frac{\partial}{\partial y_1} - 2x_1 x_2 \frac{\partial}{\partial y_2} \right)
\]

and similarly \(\frac{\partial}{\partial y_2} = \frac{1}{r^2} \left( -2x_1 x_2 \frac{\partial}{\partial y_1} + (x_2^2 - x_1^2) \frac{\partial}{\partial y_2} \right)\). Note \(r = \frac{1}{\sqrt{y_1^2 + y_2^2}}\).

\[
X := r^\alpha \left( x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) = r^\alpha \left( -\frac{x_1^2 - x_2^2}{r^2} \frac{\partial}{\partial y_1} + \frac{x_2 x_2 - x_1^2}{r^2} \frac{\partial}{\partial y_2} \right) = - (y_1^2 + y_2^2) \left( \frac{2}{y_1^2 + y_2^2} \frac{\partial}{\partial y_1} + \frac{2}{y_1^2 + y_2^2} \frac{\partial}{\partial y_2} \right).
\]

This extends smoothly to \((y_1, y_2) = (0, 0)\) (which is the north pole) iff \(\alpha \leq 0\).

The result follows.