# Math 621 Homework 2—due Friday February 2

Spring 2018

1. (Yet another way to view tangent vectors.) Let $p \in M$, and let $\mathcal{F}_p$ denote the $\mathbb{R}$-vector space of smooth functions $f : M \to \mathbb{R}$ such that $f(p) = 0$. Also, for the purposes of this problem, let a derivation at $p$ be an $\mathbb{R}$-linear map $\delta : C^\infty(M) \to \mathbb{R}$ satisfying $\delta(fg) = f(p)\delta(g) + \delta(f)g(p)$. (This is slightly different from our definition in class, which used germs.)

   (a) If $\delta : \mathcal{F}_p \to \mathbb{R}$ is an $\mathbb{R}$-linear map such that $\delta(fg) = 0$ for any $f, g \in \mathcal{F}_p$, show that $\delta$ extends to a unique derivation at $p$.

   (b) Let $\mathcal{F}_p^2$ denote the subspace of $\mathcal{F}_p$ generated by all products $fg$ for $f, g \in \mathcal{F}_p$. Show that the vector space of all derivations at $p$ is isomorphic to the dual vector space $(\mathcal{F}_p/\mathcal{F}_p^2)^*$. (Thus we can view $T_pM$ as $(\mathcal{F}_p/\mathcal{F}_p^2)^*$.)

2. Let $M$ and $N$ be smooth manifolds. The product $M \times N$ is then naturally a smooth manifold as well (see do Carmo chapter 0 exercise 1, p. 31). Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ denote projection. Prove that the map

$$\phi : T_{(p,q)}(M \times N) \to T_pM \oplus T_qN$$

   defined by $\phi(v) = ((d\pi_M)_{(p,q)}(v), (d\pi_N)_{(p,q)}(v))$ is an isomorphism.

3. do Carmo chapter 0 exercise 2, p. 32.

4. A vector field on $\mathbb{R}^2 \setminus \{0\}$ can be thought of as a vector field on $S^2 \setminus \{N, S\}$, where $N, S$ are the north and south poles, by the differential of the usual stereographic projection map $\mathbb{R}^2 \to S^2 \setminus \{N\}$. (Equivalently, view stereographic projection as a coordinate chart on $S^2 \setminus \{N\}$; then tangent vectors to points in $S^2 \setminus \{N, S\}$ are in exact correspondence with tangent vectors to the corresponding points in $\mathbb{R}^2 \setminus \{0\}$.)

   Let $x_1, x_2$ be the usual coordinates on $\mathbb{R}^2$, and for some fixed $\alpha \in \mathbb{R}$, consider the radial vector field $r^\alpha(x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2})$ on $\mathbb{R}^2 \setminus \{0\}$, where $r = \sqrt{x_1^2 + x_2^2}$. Prove that the corresponding vector field on $S^2 \setminus \{N, S\}$ can be extended to a smooth vector field on all of $S^2$ if and only if $\alpha = 0$.