Math 621 HW 10 — Outline of Solutions

Note Title 4/18/2018

1. Follow the hint. Note Y_α give a proper variation, and

$$Y_\alpha' = \begin{cases} J' - \alpha\theta' Z, & t < t_0 \\ -\alpha\theta' Z, & t > t_0. \end{cases}$$

Calculate by 2nd Variational formula:

$$\frac{1}{2}E''(0) = \int_0^a \left(|Y_\alpha'|^2 - R(\gamma', Y_\alpha, \gamma', Y_\alpha)\right) dt$$

$$= \int_0^{t_0} \Big(|J'|^2 - 2\alpha\theta'\langle J', Z\rangle + \alpha^2(\theta')^2|Z|^2 - R(\gamma', J, \gamma', J)$$
$$+ 2\alpha\theta R(\gamma', J, \gamma', Z) - \alpha^2 R(\gamma', Z, \gamma', Z)\Big)\, dt$$
$$+ \alpha^2 \int_{t_0}^a (\cdots)\, dt$$

Jacobi equation

$$= \int_0^{t_0} \left(\langle J', J'\rangle + \langle J'', J\rangle - 2\alpha\theta'\langle J', Z\rangle - 2\alpha\theta\langle J'', Z\rangle\right) dt + O(\alpha^2)$$

$$= \left(\langle J', J\rangle - 2\alpha\langle \theta J', Z\rangle\right)\Big|_0^{t_0} + O(\alpha^2)$$

$$= -2\alpha|J'(t_0)|^2 + O(\alpha^2)$$

< 0 for $\alpha > 0$ sufficiently small.

[Note: Y_α isn't a smooth vector field, but it's piecewise differentiable, which is enough. In the version of 2nd Variation in do Carmo, he replaces $\int_0^a |V'|^2 dt$ by $-\int_0^a \langle V, V''\rangle dt - \Sigma\langle V(t_i), V'(t_i^+) - V'(t_i^-)\rangle$ where the sum is a correction term where the derivative isn't continuous; but we don't need to worry about this if we use $\int_0^a |V'|^2 dt$.]

Since $E'(0) = 0$ by first variation, $E(\gamma_s) < E(\gamma)$ for $|s|$ small, $s \neq 0$, for this choice of α.

2. From HW 6 # 4, $\nabla_X X = 0 \;\forall\; X \in \mathfrak{g}$. It follows that for $X, Y \in \mathfrak{g}$, $\nabla_{X+Y}(X+Y) \Rightarrow \nabla_X Y + \nabla_Y X = 0 \Rightarrow \nabla_X Y = \frac{1}{2}[X,Y]$ and so

$$R(X,Y)X = \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X,Y]} X$$
$$= -\tfrac{1}{4}[X,[Y,X]] + \tfrac{1}{2}[[X,Y],X]$$
$$= \tfrac{1}{4}[[X,Y],X]$$
$$\Rightarrow R(X,Y,X,Y) = \tfrac{1}{4}\langle [[X,Y],X], Y\rangle = \tfrac{1}{4}\,|[X,Y]|^2.$$

If $e_1, \ldots, e_n$ is an orthonormal basis of $\mathfrak{g} = T_eG$, then

$$Ric_e(X) = \frac{1}{4(n-1)} \sum_i |[X, e_i]|^2 > 0$$

for any $X \in \mathfrak{g}$ with $X \neq 0$, since $\mathfrak{g}$ has trivial center.

Since S^{n-1} is compact, there exists $c > 0$ such that $Ric_e(X) \geq c$ for all $X \in \mathfrak{g}$, $|X| = 1$. Now apply Myers to conclude that G and its universal cover are compact.