Math 621 Homework 10—"due" Wednesday April 25, but not to be handed in

Spring 2018

I strongly recommend that you work through this problem set, even though I'm not requiring you to turn it in. Solutions to this problem set are posted on the course web page.

1. Let $\gamma : [0, a] \to M$ be a geodesic with endpoints p and q. Suppose that p has a conjugate point along γ strictly before q. Use the second variational formula to prove that there is a proper variation γ_s of γ such that for $s > 0$ sufficiently small, $E(\gamma_s) < E(\gamma)$: that is, γ is not a local minimum for length and energy.

   Hint: Let J be a nonvanishing Jacobi field along γ with $J(0) = J(t_0) = 0$ for some $t_0 \in (0, a)$, and note that $J'(t_0) \neq 0$. Let Z be the parallel vector field along γ with $Z(t_0) = J'(t_0)$; let $\theta : [0, a] \to \mathbb{R}$ be a function with $\theta(0) = \theta(a) = 0$ and $\theta(t_0) = 1$; and for $\alpha \in \mathbb{R}$, define a vector field along γ,

   $$Y_\alpha(t) = \begin{cases} J(t) - \alpha\theta(t)Z(t) & t \leq t_0 \\ -\alpha\theta(t)Z(t) & t \geq t_0. \end{cases}$$

   Let γ_s be a variation of γ with variational vector field Y_α, and let $E(s) = E(\gamma_s)$. Show that $\frac{1}{2}E''(0) = -2\alpha|J'(t_0)|^2 + O(\alpha^2)$, and deduce the result.

   Remark: In fact, one can show that the converse is also true: if there are no conjugate points to p along γ, then for any proper variation γ_s of γ, $E(\gamma_s) \geq E(\gamma)$ when s is sufficiently close to 0. You can prove this yourself if you're interested (use the Gauss Lemma and the fact that $\exp_p$ is a local diffeomorphism).

2. Let G be a connected Lie group whose Lie algebra $\mathfrak{g}$ has trivial center: that is, the only $X \in \mathfrak{g}$ for which $[X, Y] = 0$ for all $Y \in \mathfrak{g}$ is $X = 0$. Suppose that G has a biinvariant metric. Use Myers' Theorem to prove that G and its universal cover are compact.

   Remarks:

   (a) Since $\mathfrak{sl}(n, \mathbb{R})$ has trivial center and $SL(n, \mathbb{R})$ is noncompact, this gives another proof that $SL(n, \mathbb{R})$ has no biinvariant metric.

   (b) Any compact connected Lie group has a biinvariant metric (see do Carmo, Exercise 7, pp. 46–47 for details). As a corollary, one obtains an important result in Lie theory, "Weyl's Theorem": the universal cover of any compact connected semisimple Lie group is compact. (Here "semisimple" implies in particular that $\mathfrak{g}$ has trivial center.)