1. Let $S^n$ denote the $n$-manifold given as the unit sphere \( \{ x_0^2 + \cdots + x_n^2 = 1 \} \subset \mathbb{R}^{n+1} \), where $x_0, \ldots, x_n$ are the standard coordinates on $\mathbb{R}^{n+1}$. Define an $n$-form $\eta$ on $\mathbb{R}^{n+1} - \{0\}$ by

\[
\eta = \frac{1}{r} \sum_{i=0}^{n} (-1)^i x_i \, dx_0 \wedge \cdots \hat{dx_i} \cdots \wedge dx_n,
\]

where the term on the right denotes the wedge product of $dx_0$ through $dx_n$ with $dx_i$ omitted, and $r : \mathbb{R}^{n+1} \to \mathbb{R}$ is the distance to the origin. Define $\omega \in \Omega^n S^n$ by $\omega = i^* \eta$ where $i : S^n \to \mathbb{R}^{n+1} - \{0\}$ is the inclusion map.

(a) Show that $dr \wedge \eta = dx_0 \wedge \cdots \wedge dx_n$ as differential forms on $\mathbb{R}^{n+1} - \{0\}$.

(b) Explain why $i^*(x_0 dx_0 + x_1 dx_1 + \cdots + x_n dx_n) = 0$.

(c) Consider the coordinate chart on $S^n$ given by $\phi : S^n - \{(1,0,\ldots,0)\} \to V$, where $V = \mathbb{R}^n$ and $\phi$ is stereographic projection,

\[
\phi(x_0, \ldots, x_n) = \left( \frac{x_1}{1-x_0}, \ldots, \frac{x_n}{1-x_0} \right).
\]

By the definition of differential forms on manifolds, any differential form on $S^n$ is in particular a differential form on $V$. Show by direct computation that the differential form on $V$ associated to the differential form $i^*(x_0 dx_0 + x_1 dx_1 + \cdots + x_n dx_n)$ on $S^n$ is 0. (Hint: first invert the map $\phi$.)

(d) Similarly, the differential form $\omega$ on $S^n$ is in part a differential form $\theta$ on $V$. If $y_1, \ldots, y_n$ are the standard coordinates on $V$, then since $\theta$ is an $n$-form, we can write

\[
\theta = f dy_1 \wedge \cdots \wedge dy_n
\]

for some function $f : \mathbb{R}^n \to \mathbb{R}$. Find $f$. 

(One more problem on the next page.)
2. This problem fills in some of the details for the proof of the Poincaré Lemma for compactly supported cohomology. Let $M$ be a smooth manifold. **For this problem, you may assume $M = \mathbb{R}^n$.** Everything holds for general smooth manifolds, and you should think about why this is true, but I don’t want you to get bogged down in notation.

Notation is as in class: let $x_1, \ldots, x_n$ denote coordinates on $M$, and let $t$ denote the coordinate on $\mathbb{R}$. Let $e : \mathbb{R} \to \mathbb{R}$ be a compactly supported function with $\int_{-\infty}^{\infty} e(t) \, dt = 1$. Then define maps $\pi_*, e_*, H$ as follows:

$$\pi_* : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M)$$

$$f(x, t) \, dx_I \mapsto 0$$

$$f(x, t) \, dx_I \wedge dt \mapsto \left( \int_{-\infty}^{\infty} f(x, t) \, dt \right) \, dx_I,$$

$$e_* : \Omega^{k-1}_c(M) \to \Omega^k_c(M \times \mathbb{R})$$

$$f(x) \, dx_I \mapsto f(x) e(t) \, dx_I \wedge dt,$$

$$H : \Omega^k_c(M \times \mathbb{R}) \to \Omega^{k-1}_c(M \times \mathbb{R})$$

$$f(x, t) \, dx_I \mapsto 0$$

$$f(x, t) \, dx_I \wedge dt \mapsto \left( \left( \int_{-\infty}^{t} f(x, t) \, dt \right) - \left( \int_{-\infty}^{\infty} f(x, t) \, dt \right) \left( \int_{-\infty}^{t} e(t) \, dt \right) \right) \, dx_I.$$ 

(a) Prove that $d\pi_* = \pi_* d$ and $de_* = e_* d$.

(b) Prove that on $\Omega^k_c(M \times \mathbb{R})$,

$$\text{id} - e_* \pi_* = (-1)^k (Hd - dH).$$

(For full disclosure, this computation is done in Bott & Tu as well as in my lecture notes, but you should try to work it out yourself.)